S4:E5

Number of cycles in a random permutation

- Recall: permutation = a rearrangement of n distinct objects. There are $n!$ permutations (S1:E2)

- $\forall$ permutation breaks down into cycles.

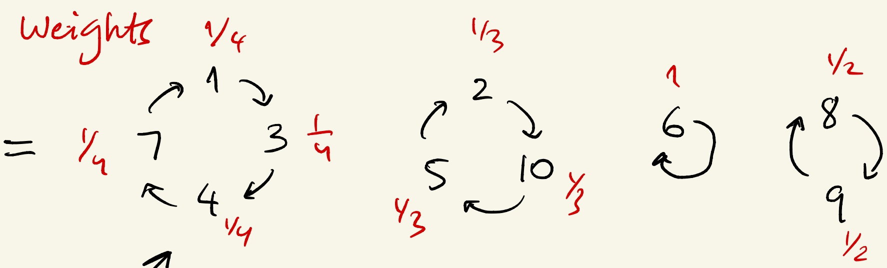
For example:

people

$$\pi: \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \downarrow & \downarrow & \cdots & & & & & & & \\ 3 & 10 & 4 & 7 & 2 & 6 & 1 & 9 & 8 & 5 \end{array}$$

jackets

$=$



Q! What is $E\,\#$(cycles) in a random permutation ?

drawn from the set of all $n!$ permutations with uniform probability.

- Intuition: 1^{st} cycle has length $\sim n$ (e.g. $n/2$), next $n/4$, ... $\Rightarrow \log n$.

- Thm The expected # of cycles in a random permutation of n elements is
$$\sum_{k=1}^{n} \frac{1}{k} \simeq \ln n + 0.58 \qquad \text{(harmonic series)}$$

Proof • To each element in a cycle of length k, assign weight $:= 1/k$.

Sum of all weights = # cycles.

E.g, here sum = $1+1+1+1=4$

• Formally, let

$Y_j :=$ length of the cycle the element j is in;

$$X = \sum_{j=1}^{n} \frac{1}{Y_j}$$

Linearity $\Rightarrow$ $$E[X] = \sum_{j=1}^{n} \underbrace{E\left[\frac{1}{Y_j}\right]}_{\text{all equal by symmetry}} = n \cdot \underbrace{E\left[\frac{1}{Y_1}\right]}_{= \, ?}$$

$$E\left[\frac{1}{Y_1}\right] = \sum_{k=1}^{n} \frac{1}{k} \underbrace{P\{Y_1 = k\}}_{=}$$

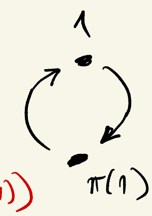
$P\{$element 1 is in a cycle of length $k\} = ?$

• $P\{1$ is in a cycle of length $1\} = \frac{1}{n}$ ← n choices of $\pi(1)$

choices of $\pi(1) \neq 1$

$$P\{1 \text{ is in a cycle of length } 2\} = \frac{n-1}{n(n-1)} = \frac{1}{n}$$

n choices of $\pi(1)$ ↗ ↖ n-1 choices of $\pi(\pi(1))$



. . .

$P\{1$ is in a cycle of length $k\} = \frac{1}{n}$ (DIY)

• $\Rightarrow$ $E\left[\frac{1}{Y_1}\right] = \sum_{k=1}^{n} \frac{1}{k} \cdot \frac{1}{n}$ $\Rightarrow$ $E[X] = \sum_{k=1}^{n} \frac{1}{k}$ QED