Expectation of a function of a r.v.

- Recall: If r.v. $X$ takes values $x_i$ with prob. $p_i$, expectation ("mean"):
  \[ E[X] = \sum_i x_i p_i \]

- If $g: \mathbb{R} \to \mathbb{R}$ is a function, $g(x) = g \circ X$ is a r.v. that takes values $g(x_i)$ with prob. $p_i$ \(\Rightarrow\) $S \xrightarrow{\chi} \mathbb{R} \xrightarrow{g} \mathbb{R}$
  \[ E[g(x)] = \sum_i g(x_i) p_i \text{, e.g. } E[X^2] = \sum_i x_i^2 p_i \]

VARIANCE

- How much $X$ deviates from $\mu = \mathbb{E}[X]$?

- First attempt: $E|X-\mu|$. Abs. value is inconvenient. The square is better:

**Def.** The variance of a r.v. $X$ with expectation ("mean") $\mu = \mathbb{E}[X]$ is
  \[ \text{Var}(X) := \mathbb{E}[(X-\mu)^2] \]

  Standard deviation is
  \[ \sigma(X) = \sqrt{\text{Var}(X)} \]

Expand the square:
  \[ \text{Var}(X) = \mathbb{E}\left[ x^2 - 2x\mu + \mu^2 \right] \]
  \[ = \mathbb{E}\left[ x^2 \right] - 2\mu \mathbb{E}[X] + \mu^2 \]
  \[ = \mathbb{E}\left[ x^2 \right] - \mu^2 \]

**Prop.** $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
Ex.

Compare 2 investment options:

Stock A: 3% return with prob 0.8; 2% loss with prob 0.2
Stock B: 5% return with prob 0.6; 2% loss with prob 0.4

Expected profit:
\[ E(X_A) = 3 \times 0.8 - 2 \times 0.2 = 2\% \]
\[ E(X_B) = 5 \times 0.6 - 2 \times 0.4 = 2.2\% \]

\( \approx \) a little better.

However, compare the risks:

\[ E(X_A^2) = 3^2 \times 0.8 + 2^2 \times 0.2 = 8 \]
\[ E(X_B^2) = 5^2 \times 0.6 + 2^2 \times 0.4 = 16.6 \]

\[ \Rightarrow \text{Var}(X_A) = 8 - 2^2 = 4; \quad \text{Var}(X_B) = 16.6 - (2.2)^2 = 11.8 \]

St. deviation:
\[ \sigma(X_A) = \sqrt{4} = 2; \quad \sigma(X_B) = \sqrt{11.8} \approx 3.4 \]

\( \Rightarrow \) typical return for stock A: 2\% ± 2 \%
stock B: 2.2\% ± 3.4 \%

\( \Rightarrow \) Stock A has almost same return with less risk.