S4:E6

Expectation of a function of a r.v.

- Recall: if r.v. X takes values x_i with prob. p_i, expectation ("mean"):

$$E[X] = \sum_i x_i p_i$$

- If $g: \mathbb{R} \to \mathbb{R}$ is $\forall$ function, $g(X) = g \circ X$ is a r.v. that takes values $g(x_i)$ with prob. p_i $\Rightarrow$

$$S \xrightarrow{X} \mathbb{R} \xrightarrow{g} \mathbb{R}$$

$$E[g(X)] = \sum_i g(x_i) p_i \quad , \quad \text{e.g. } E[X^2] = \sum_i x_i^2 p_i$$

VARIANCE

- How much X deviates from $\mu = \mathbb{E}(X)$?
- First attempt: $\mathbb{E}|X-\mu|$. Abs. value is inconvenient. The square is better:

$$\boxed{\begin{array}{l} \textbf{Def} \text{ The variance of a r.v. } X \text{ with expectation ("mean") } \mu = \mathbb{E}X \text{ is} \\ \qquad Var(X) := \mathbb{E}\left[(X-\mu)^2\right] \\ \text{Standard deviation is} \\ \qquad \sigma(X) = \sqrt{Var(X)} \end{array}}$$

Expand the square:

$$\begin{aligned} Var(X) &= \mathbb{E}\left[X^2 - 2X\mu + \mu^2\right] \\ &= \mathbb{E}\left[X^2\right] - 2\mu \underbrace{\mathbb{E}(X)}_{\mu} + \mu^2 \qquad \text{(linearity)} \\ &= \mathbb{E}(X^2) - \mu^2. \end{aligned}$$

$$\boxed{\textbf{Prop.} \quad Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2}$$

not the same #'s

$\underline{\underline{\text{Ex}}}$ Compare 2 investment options:

Stock A: 3% return with prob 0.8; 2% loss with prob. 0.2

Stock B: 5% return with prob. 0.6; 2% loss with prob. 0.4.

Expected profit: $\mathbb{E}(X_A) = 3 \cdot 0.8 - 2 \cdot 0.2 = 2\%$

$\mathbb{E}(X_B) = 5 \cdot 0.6 - 2 \cdot 0.4 = 2.2\%$ ← a little better.

However, compare the risks:

$\mathbb{E}(X_A^2) = 3^2 \cdot 0.8 + 2^2 \cdot 0.2 = 8$

$\mathbb{E}(X_B^2) = 5^2 \cdot 0.6 + 2^2 \cdot 0.4 = 16.6$

$\overset{\text{Prop}}{\Rightarrow}$ $Var(X_A) = 8 - 2^2 = 4$; $Var(X_B) = 16.6 - (2.2)^2 = 11.8$

St. deviation:

$\sigma(X_A) = \sqrt{4} = 2$; $\sigma(X_B) = \sqrt{11.8} \approx 3.4$

⇒ typical return for stock A: 2 ± 2 %

stock B: 2.2 ± 3.4 %

⇒ Stock A has almost same return with less risk