Properties of variance

1. If X is constant (i.e. $X(s) = c \ \forall s \in S$) then $\boxed{Var(X) = 0.}$

$\lceil Var(X) = \mathbb{E}(X-\mu)^2 = 0$ since $X = \mu = c \rfloor$

1. In general, $\boxed{Var(X+Y) \neq Var(X) + Var(Y)}$

(e.g. for $Y = -X$)

2. If X is constant, (translation invariance)

$$\boxed{Var(X+c) = Var(X)}$$

$\mathbb{E}(X+c) = \mathbb{E}X + c \Rightarrow$

$$Var(X+c) = Var\left[\left((X+c) - \mathbb{E}(X+c)\right)^2\right] = Var\left[(X - \mathbb{E}X)^2\right]$$

$$= Var(X)$$

3. $\boxed{Var(aX) = a^2 Var(X)}$ (homogeneity of order 2)

$$\mathbb{E}\left[\underbrace{(aX)^2}_{a^2X^2}\right] - \left(\mathbb{E}[aX]\right)^2$$

$$= a^2 \mathbb{E}[X^2] - \left(a\mathbb{E}(X)\right)^2 \quad \text{(linearity)}$$

$$= a^2\left(\mathbb{E}[X^2] - (\mathbb{E}(X))^2\right) = a^2 Var(X)$$

$$\Rightarrow \boxed{\sigma(aX) = a \cdot \sigma(X)}$$ (homogeneity of order 1)

Ex (The matching problem II)

$\boxed{\begin{array}{l}\text{N exams are returned to N students at random.}\\ \text{What is the expected \# of students who receive their own exam?}\end{array}}$

$$X = \sum_{i=1}^{N} X_i \quad \text{where} \quad X_i = \begin{cases} 1 & \text{if student } i \text{ gets own exam} \\ 0 & \text{otherwise} \end{cases}$$

probability $= 1/N$

probability $= 1 - 1/N$

$$E(X_i) = 1 \cdot \frac{1}{N} + 0 \cdot \left(1 - \frac{1}{N}\right) = \frac{1}{N}. \qquad (*)$$

$$\Rightarrow E[X] = \sum_{i=1}^{N} E[X_i] = N \cdot \frac{1}{N} = 1$$

Ex $\boxed{\text{Standard deviation of } X = ?}$

$$Var(X) = \underbrace{E(X^2)}_{= ?} - (\underbrace{EX}_{=1})^2$$

$$E(X^2) = E\left[\left(\sum_{i=1}^{N} X_i\right)^2\right] = \underbrace{\sum_{i=1}^{N}}_{N \text{ terms}} E(X_i^2) + \underbrace{\sum_{i \neq j}}_{N^2 - N \text{ terms}} E[X_i X_j]$$

$$\left(\sum_{i=1}^{N} X_i\right)\left(\sum_{j=1}^{N} X_j\right) = \sum_{i,j=1}^{N} X_i X_j = \sum_{i=1}^{N} X_i^2 + \sum_{i \neq j} X_i X_j$$

$$E[\underset{X_i}{X_i^2}] = 1/N \quad \text{by } (*)$$

$$E(X_i X_j) = 1 \cdot P\{X_i X_j = 1\} + 0 P\{X_i X_j = 0\} = P\{X_i = 1 \text{ and } X_j = 1\}$$

$$= P\{\text{both students } i, j \text{ get own exams}\} = \frac{1}{N(N-1)}$$

\# ways to select 2 exams from N exams, order matters

$$\Rightarrow E[X^2] = N \cdot \frac{1}{N} + (N^2 - N) \cdot \frac{1}{N(N-1)} = 2.$$

$$\Rightarrow Var(X) = 2 - 1 = 1. \qquad \Rightarrow \sigma(X) = \sqrt{1} = 1$$