Morse theory and stable pairs

Richard A. Wentworth

UNIVERSITY OF MARYLAND

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Joint with

Georgios Daskalopoulos (Brown University)
Jonathan Weitsman (Northeastern University)
Graeme Wilkin (University of Colorado)
Outline

1. Introduction
2. Cohomology of symplectic quotients
3. Approach in a singular setting
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1 Introduction

2 Cohomology of symplectic quotients

3 Approach in a singular setting
Cohomology of Kähler and hyperKähler quotients

The goal is to compute the equivariant cohomology of symplectic (Kähler or hyperKähler) reductions.

By the Kempf-Ness, Guillemin-Sternberg theorem, examples arise in geometric invariant theory.

Kirwan, Atiyah-Bott: In the symplectic case there is a “perfect” Morse stratification.

HyperKähler case still unknown.
Infinite dimensional examples:

- Higgs bundles (Hitchin)
- Stable pairs (Bradlow)
- Quiver varieties (Nakajima)

These involve symplectic reduction in the presence of singularities.

Key points:

- this poses no (additional) analytic difficulties.
- Singularities can cause the Morse stratification to lose “perfection.”
- Computations of cohomology are (sometimes) still possible.
Application to representation varieties

- $M$ = a closed Riemann surface $g \geq 2$
- $\pi = \pi_1(M, \ast)$
- $G$ = a compact connected Lie group
- $G^\mathbb{C}$ = its complexification (e.g. $G = U(n)$, $G^\mathbb{C} = GL(n, \mathbb{C})$)
- Representation varieties:
  \[
  \text{Hom}(\pi, G)/G \sim \text{moduli of } G\text{-bundles}
  \]
  \[
  \text{Hom}(\pi, G^\mathbb{C})//G^\mathbb{C} \sim \text{moduli of } G\text{-Higgs bundles}
  \]
Theorem (Daskalopoulos-Weitsman-Wentworth-Wilkin '09)

The Poincaré polynomial is given by

\[
P_t^{SL(2,\mathbb{C})}(\text{Hom}(\pi, SL(2, \mathbb{C}))) = \frac{(1 + t^3)^{2g} - (1 + t)^{2g} t^{2g+2}}{(1 - t^2)(1 - t^4)}
\]

\[
- t^{4g-4} + \frac{t^{2g+2}(1 + t)^{2g}}{(1 - t^2)(1 - t^4)} + \frac{(1 - t)^{2g} t^{4g-4}}{4(1 + t^2)}
\]

\[
+ \frac{(1 + t)^{2g} t^{4g-4}}{2(1 - t^2)} \left( \frac{2g}{t + 1} + \frac{1}{t^2 - 1} - \frac{1}{2} + (3 - 2g) \right)
\]

\[
+ \frac{1}{2} (2^{2g} - 1) t^{4g-4} \left( (1 + t)^{2g-2} + (1 - t)^{2g-2} - 2 \right)
\]
Introduction

Cohomology of symplectic quotients

Approach in a singular setting

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1 Introduction

2 Cohomology of symplectic quotients

3 Approach in a singular setting
Symplectic reduction

- \((X, \omega)\) symplectic manifold (compact)
- \(G\) compact, connected Lie group, acting symplectically
- \(\mu : X \rightarrow g^*\) a moment map (\(d\mu^\xi(\cdot) = \omega(\xi^\# , \cdot)\))
- the Marsden-Weinstein quotient \(\mu^{-1}(0)/G\) is a symplectic variety
- What is the cohomology of \(\mu^{-1}(0)/G\)?
Example

- Take $S^2$ with the action of $S^1$ by rotation in the $xy$-plane.
- A moment map is given by the height:
  \[ \mu(x, y, z) = z + \text{const}. \]
  
- If $\mu = z$, then $\mu^{-1}(0)$ is the equator with a free action of $S^1$.
- If $\mu = z + 1$, then $\mu^{-1}(0)$ is a point with a trivial $S^1$ action.
- In both cases, $\mu^{-1}(0)/S^1$ is a point.
Equivariant cohomology

- $Z$ topological space with an action by $G$
- Classifying space: $EG \to BG$ is a contractible, principal $G$-bundle
- Equivariant cohomology: $H^*_G(Z) = H^*(Z \times_G EG)$
- If the action is free: $H^*_G(Z) = H^*(Z/G)$
- If the action is trivial: $H^*_G(Z) = H^*(Z) \otimes H^*(BG)$
Perfect equivariant Morse theory

- $S^1$ acts freely on the sphere $S^\infty$, so $BS^1 = \mathbb{C}P^\infty$
- Therefore $P_t(BS^1) = 1 + t^2 + t^4 + \cdots = \frac{1}{1 - t^2}$
- Example of the sphere:

$$P_t^{S^1}(S^2) \cong P_t(H^*(S^2) \otimes H^*(BS^1)) = \frac{1 + t^2}{1 - t^2}$$

- Sum over critical points of $f = \mu^2$:

$$P_t^{S^1}(S^2) = \sum_{p_j \text{ crit. pt.}} t^{\lambda_j} P_t^{S^1}(p_j)$$
Example

- \( f = \mu^2, \mu = z \): two critical points of index 2, plus the minimum:
  
  \[
  1 + \frac{t^2}{1 - t^2} + \frac{t^2}{1 - t^2} = \frac{1 + t^2}{1 - t^2}
  \]

- \( f = \mu^2, \mu = z + 1 \): two critical points, one of index 2 and one of index zero:
  
  \[
  \frac{1}{1 - t^2} + \frac{t^2}{1 - t^2} = \frac{1 + t^2}{1 - t^2}
  \]
For the general case, study the gradient flow $f = \|\mu\|^2$.

Critical sets $\eta_\beta$ are characterized in terms of isotropy in $G$.

Gradient flow $\leadsto$ smooth stratification $X = \bigcup_{\beta \in I} S_\beta$; with normal bundles $\nu_\beta$.

The corresponding long exact sequence splits

$$\cdots \longrightarrow H_G^*(S_\beta, \bigcup_{\alpha < \beta} S_\alpha) \longrightarrow H_G^*(S_\beta) \longrightarrow H_G^*(\bigcup_{\alpha < \beta} S_\alpha) \longrightarrow \cdots$$

Compute change at each step from $H_G^*(S_\beta, \bigcup_{\alpha < \beta} S_\alpha)$.
Two key steps

- **Morse-Bott Lemma:**

  \[ H^*_G(S_\beta, \cup_{\alpha<\beta} S_\alpha) \cong H^*_G(\nu_\beta, \nu_\beta \setminus \{0\}) \cong H^*_{\lambda_\beta}(\eta_\beta) \]

- **Atiyah-Bott Lemma:** criterion for multiplication by the equivariant Euler class to be injective.

\[ \cdots \longrightarrow H^p_G(S_\beta, \cup_{\alpha<\beta} S_\alpha) \longrightarrow H^p_G(S_\beta) \longrightarrow \cdots \]

\[ \downarrow \cong \downarrow \]

\[ H^p_G(\nu_\beta, \nu_\beta \setminus \{0\}) \longrightarrow H^p_G(\eta_\beta) \]
Perfect equivariant Morse theory

Theorem (Kirwan, Atiyah-Bott)

\[ P^G_t(\mu^{-1}(0)) = P^G_t(X) - \sum_{\beta} t^{\lambda_\beta} P^G_t(\eta_\beta) \]

Theorem (Kirwan surjectivity)

The map on cohomology

\[ H^*_G(X) \longrightarrow H^*_G(\mu^{-1}(0)) \]

induced from inclusion \( \mu^{-1}(0) \hookrightarrow X \) is surjective.
Vector bundles on Riemann surfaces

- $M$ a Riemann surface.
- $\mathcal{A} = \{\text{unitary connections } A \text{ on hermitian bundle } E \to M\}$
- $\mathcal{G} = \text{gauge group of unitary endomorphisms of } E$.
- $\mu : \mathcal{A} \to \text{Lie}(\mathcal{G})$ is given by $A \mapsto F_A$
- $\|\mu\|^2 = \text{Yang-Mills functional}$
Minimum is the space of projectively flat connections (i.e. representation variety)

\[ \sqrt{-1} * F_A = \text{const.} \cdot 1 \]

The flow converges and the Morse stratification is smooth (Daskalopoulos)

Higher critical sets correspond to split Yang-Mills connections, i.e. representations to smaller groups. For example, \( E = L_1 \oplus L_2, \, d = \deg L_1 > \deg L_2 \):

\[ \eta_d = \text{Jac}(M) \times \text{Jac}(M) \]

Morse-Bott lemma: Negative directions given by \( H^{0,1}(L_1^* \otimes L_2); \lambda_d = \text{dim is constant.} \)
Theorem (Atiyah-Bott, Daskalopoulos)

\[
P^\text{SU}(2)_t(\text{Hom}(\pi, \text{SU}(2))) = P(BG) - \sum_{d=0}^{\infty} t^{\lambda_d} P_t^{S^1}(\text{Jac}_d(M))
\]

\[
= \frac{(1 + t^3)^{2g} - t^{2g+2}(1 + t)^{2g}}{(1 - t^2)(1 - t^4)}
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Holomorphic pairs

- $\mathcal{B}^{pairs} = \{(A, \Phi) : \Phi \in \Omega^0(E), \bar{\partial} A \Phi = 0\}$
- Higher rank version of $\text{Sym}^d(C)$
- Moduli space of Bradlow pairs corresponds to solutions of the $\tau$-vortex equations:
  \[
  \sqrt{-1} * F_A + \Phi \Phi^* = \tau \cdot I
  \]
- This is the moment map for the action on $\mathcal{B}^{pairs} \subset A \times \Omega^0(E)$. 

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Higgs bundles

- $\mathcal{B}^{\text{higgs}} = \{(A, \Phi) : \Phi \in \Omega^{1,0}(\text{End } E), \bar{\partial}_A \Phi = 0\}$
- Dimensional reduction of anti-self dual equations.
- Moduli space corresponds to solutions of the Hitchin equations:
  \[ F_A + [\Phi, \Phi^*] = 0 \]
- Homeomorphic to the space of flat $GL(n, \mathbb{C})$ connections (Corlette-Donaldson).
- Hyperkähler structure.

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Morse theory and stable pairs
Singularities

- Singularities because of the jump in \( \dim \ker \bar{\partial}_A \).
- Kuranishi model: \( \{ \text{Slice} \} \hookrightarrow H^1(\text{deformation complex}) \)
- Negative directions: \( \nu_\beta \) is the intersection of negative directions with the image of the slice.
- Morse-Bott isomorphism: Need to define a deformation retraction.
Critical Higgs bundle

- $E = L_1 \oplus L_2$, $A = A_1 \oplus A_2$, $\Phi = \begin{pmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{pmatrix}$
- Negative directions $\nu$: $(a, \varphi)$ strictly lower triangular.

\[ a \in H^{0,1}(L_1^* \otimes L_2), \quad \varphi \in H^{1,0}(L_1^* \otimes L_2) \]

- $\deg(L_1^* \otimes L_2) < 0$
- $\deg(L_1^* \otimes L_2 \otimes K_M)$ is not necessarily negative
- Can still prove $H^*_G(X_d, X_{d-1}) \simeq H^*_G(\nu_d, \nu_d \setminus \{0\})$
Critical pair

- The set of pairs \((A, 0)\), where \(A\) is minimal Yang-Mills is a critical set of \(B_{pairs}\).
- \(\nu = H^0(E)\)
- If \(d > 4g - 4\), then \(H^0(E)\) is constant in dimension, and the Morse-Bott lemma holds.
- If \(d \leq 4g - 4\), \(H^0(E)\) jumps in dimension; describes Brill-Noether loci. Can still compute the contribution from this critical set.
Theorem (Daskalopoulos-Weitsman-Wentworth-Wilkin)

For the case of Higgs bundles, Kirwan surjectivity holds for $GL(2, \mathbb{C})$ but fails for $SL(2, \mathbb{C})$.

Theorem (Wentworth-Wilkin)

For stable pairs, Kirwan surjectivity holds, even though the Morse stratification fails to be perfect.
Theorem (MacDonald)

*The embedding* $\text{Sym}^d M \hookrightarrow \text{Sym}^{d+1} M$ *of symmetric products of Riemann surfaces induces a surjection in cohomology.*

Theorem (Wentworth-Wilkin)

*The same result holds for rank 2 semistable pairs.*

These are important in showing splitting of the long exact sequences.
Conclusion

- More examples, general construction?
- Proof of the Morse-Bott lemma in general? (Kuranishi model)
- Hyperkähler reduction using the sum of the squares of the moment map?
- Finite dimensional hyperkähler example where Kirwan surjectivity fails?