Geometric topology of moduli spaces of curves

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Happy Birthday,

Peter!
Introduction.

Let $\Sigma_g$ be a compact Riemann surface of genus $g$,

and $\mathcal{M}_g$ be the moduli space of compact Riemann surfaces $\Sigma_g$.

Fact. $\mathcal{M}_g$ is a noncompact quasi-projective variety.

(Noncompactness follows from degeneration of Riemann surfaces, and the quasi-projectivity was first proved by Baily, 1960, via Satake compactification of the Siegel modular variety.)
$\mathcal{M}_g$ has been studied extensively, for example in algebraic geometry and mathematical physics.

It admits a compactification, the Deligne-Mumford compactification, a projective variety, by adding stable Riemann surfaces.

It also admits another compactification, Satake compactification, induced from the Satake compactification of the Siegel modular variety (used in the earlier result of Baily).
Fact. \( \mathcal{M}_g \) is a complex orbifold and also admits many natural metrics, for example, the Teichmuller metric (a Finsler metric) and Weil-Petersson metric (a Kahler metric).

It also admits many other metrics. Recently, a lot of work has been done on metrics of \( \mathcal{M}_g \), for example, in a series of papers of Liu-Sun-Yau.

Conclusion. \( \mathcal{M}_g \) should be an interesting space (orbifold) in **Geometry, topology and Analysis**.
The **Goal** of this talk is to understand $\mathcal{M}_g$ from points of view of geometric topology and geometric analysis.

All results work for the more general moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus $g$ with $n$ punctures. For simplicity, we will concentrate on the moduli space $\mathcal{M}_g$ of the compact Riemann surfaces.

We usually assume that $g \geq 2$ so that each Riemann surface $\Sigma_g$ admits a canonical metric conformal to the complex structure, and $\Sigma_g$ is considered as a hyperbolic metric (so that hyperbolic geometry can be used.)
One effective way to study $\mathcal{M}_g$ is to realize $\mathcal{M}_g$ as a quotient of the Teichmuller space $\mathcal{T}_g$ by the mapping class group $\text{Mod}_g$.

$$\mathcal{M}_g = \text{Mod}_g \backslash \mathcal{T}_g$$

The action of $\text{Mod}_g$ on $\mathcal{T}_g$ is also crucial for many problems about $\text{Mod}_g$, for example, classification of its elements, its cohomological properties, large scale geometric properties.

We will also mention some properties of $\mathcal{T}_g$ and $\text{Mod}_g$ below.
Plan of the rest of the talk

1. Basic definitions

2. Simplicial volume of $\mathcal{M}_g$

3. Duality property of $\text{Mod}_g$ and its cohomological dimension

4. Spine of Teichmuller space

5. $L^p$-cohomology of $\mathcal{M}_g$.

6. Spectral theory of Weil-Petersson metric
1. Basic Definition

Let \( S_g \) be a compact oriented surface of genus \( g \).

A marked compact Riemann surface is a compact Riemann surface \( \Sigma_g \) together with a diffeomorphism \( \varphi : \Sigma_g \to S_g \).

Two marked Riemann surfaces \((\Sigma_g, \varphi), (\Sigma'_g, \varphi')\) are called equivalent if there exists a biholomorphic map \( h : \Sigma_g \to \Sigma'_g \) such that the two maps \( h \circ \varphi', \varphi : \Sigma_g \to S_g \) are isotropic (or homotopy equivalent in this case).

The Teichmüller space

\[
T_g = \{ (\Sigma_g, \varphi) \} / \sim.
\]
**Fact.** $T_g$ is a complex manifold diffeomorphic to $\mathbb{R}^{6g-6}$. In particular, it is contractible.

It is easy to see that is real analytically diffeomorphic to $\mathbb{R}^{6g-6}$ by Fenchel-Nielsen coordinates.

Homeomorphic to $\mathbb{R}^{6g-6}$ was proved by Teichmuller using optimal (least distorted quasi-conformal maps).

Complex structures was outlined by Teichmuller, proved by Rauch and Ahlfors, and a simpler proof by Bers later.
Let $\text{Diff}^+(S_g)$ be the group of all orientation preserving diffeomorphisms of $S_g$, and $\text{Diff}^0(S_g)$ the identity component (a normal subgroup).

The quotient group $\text{Diff}^+(S_g)/\text{Diff}^0(S_g)$ is called the **mapping class group**.

$\text{Mod}_g$ acts on $\mathcal{T}_g$ by changing the markings,

and the quotient

$$\text{Mod}_g \backslash \mathcal{T}_g = \mathcal{M}_g$$

(divide out all markings on the surface $S_g$)
**Fact.** $\text{Mod}_g$ acts holomorphically and properly on $\mathcal{T}_g$.

**Corollary.** $\mathcal{M}_g$ is an orbifold.

$\text{Mod}_g$ provides one effective way to understand $\mathcal{M}_g$.

In many situations, for example, in geometric group theory, $\text{Mod}_g$ is the **main interesting object**, and its action on $\mathcal{T}_g$ provides an effective way to understand its properties. (Hence $\mathcal{T}_g$ and $\mathcal{M}_g$ are secondary.)
Example and Analogy

When \( g = 1 \),

\[ \mathcal{T}_g = \mathbb{H}^2 = \{ x + iy \mid x \in \mathbb{R}, y > 0 \} \], the Poincare upper half plane,

\( \text{Mod}_g = SL(2, \mathbb{Z}) \).

\( \mathcal{M}_1 = SL(2, \mathbb{Z}) \backslash \mathbb{H}^2 \), the modular curve, a locally symmetric space.

This suggests that \( \mathcal{M}_g \) is an analogue of locally symmetric space \( \Gamma \backslash X \), where \( X = G/K \) is a symmetric space of noncompact type, and \( \Gamma \subset G \) is an arithmetic subgroup.

\( G \) is a semisimple Lie group, \( K \subset G \) is a maximal compact subgroup.
The correspondence is that

**Teichmüller space** $\mathcal{T}_g$ is an analogue of **symmetric space** $X = G/K$.

**Mapping class group** $\text{Mod}_g$ is an analogue of **arithmetic groups** $\Gamma$

and **the moduli space** $\mathcal{M}_g$ is an analogue of a **locally symmetric space** $\Gamma \backslash X$. 
This point of view has motivated many problems about \( \mathcal{M}_g \) and \( \text{Mod}_g \) and their solutions by considering problems about \( \Gamma\backslash X \) and arithmetic subgroups \( \Gamma \).

Many results on homological properties and geometric group theoretic properties of the mapping class group \( \text{Mod}_g \) are motivated by corresponding results of arithmetic subgroups of semisimple Lie groups.
**Simplicial volume** introduced by Gromov.

Let $M^n$ be a compact oriented manifold. Then it has a fundamental class $[M] \in H_n(M, \mathbb{Z})$. Denote the image of $[M]$ under the map

$$H_n(M, \mathbb{Z}) \to H_n(M, \mathbb{R})$$

also by $[M]$.

For every $n$-cycle $c = \sum_\sigma a_\sigma \sigma$, define its $\ell_1$-norm

$$|c|_1 = \sum_\sigma |a_\sigma|.$$
The simplicial volume of $M$ is defined by

$$|M| = \inf \{|c_1| \mid c \text{ is an } \mathbb{R}-\text{cycle representing } [M] \}.$$ 

If $M$ is compact, $|M|$ is finite. But it could be zero.
It follows from definition

**Fact:** If \( f : M \to N \) is a map of degree \( d \), then

\[ d|N| \leq |M|. \]

**Corollary.** If \( M \) admits a self-map of degree not equal to \( \pm 1 \), then \( |M| = 0 \).

**Example.** Spheres \( S^n \), tori \( \mathbb{Z}^n \setminus \mathbb{R}^n \) have such self-maps and hence have zero simplicial volume.
If $M$ is an oriented noncompact manifold, then there is a fundamental class $[M]^{lf}$ in the locally finite homology group $H_n^{lf}(M,\mathbb{Z})$ and also in $H_n^{lf}(M,\mathbb{R})$.

Using locally finite cycles, we can also define the simplicial volume $|M|$ of $M$ as before.

In this case $|M|$ could be $\infty$. Of course, it could also be zero.
If $M$ is an orbifold and admits a finite smooth cover, $N \to M$, then we can define

$$|M|_{orb} = |N|/d,$$

where $d$ is the degree of the covering $N \to M$.

It is independent of smooth covers.

If follows from the fact that the simplicial volume behaves multiplicatively.

One can show $|M|_{orb} \geq |M|$.

It could happen that $|M|_{orb} > 0$ but $|M| = 0$.

(Note that once a space has a fundamental class, its simplicial volume can be defined. In particular, any algebraic variety and orbifold has a simplicial volume.)
Motivations for simplicial volume.

The simplicial volume $|M|$ is a homotopy invariant of $M$ (since any homotopy equivalence is of degree $1$).

When the Euler characteristic is zero, the simplicial volume gives a possible nonzero homotopy invariant.
It also gives a lower bound for the minimal volume of a manifold

\[ \min - \text{vol}(M) = \inf \{ \text{vol}(M, g) \mid g \text{ is complete, } -1 \leq K_g \leq 1 \}. \]

\[ \min - \text{vol}(M) \geq c(n) |M|, \ n = \dim M. \]
It is not easy to compute $|M|$. A natural problem is to decide whether it vanishes or not.

As mentioned before, if a compact manifold admits a nontrivial automorphism of degree greater than 1, then its simplicial volume is zero.

Example, the torus and the sphere.

If a noncompact manifold admits a nontrivial automorphism of degree greater than 1, then its simplicial volume is either infinity and 0.

For $\mathbb{R}^n$, it is equal to 0. (Due to Gromov).
Here is a brief summary of known results.

(1) (Thurston-Gromov) If $M$ is a complete hyperbolic manifold of finite area, then $|M| > 0$. (Also true if the curvature is negatively pinched.)

(2) (LaFont-Schmidt, conjectured by Gromov). If $M$ is a compact locally symmetric space of noncompact type, then $|M| > 0$.

(3) (Löh-Sauer) If $M$ is a locally symmetric space of $\mathbb{Q}$-rank at least 3 (hence noncompact), then $|M| = 0$.

For some locally symmetric spaces of $\mathbb{Q}$-rank 1, $|M| > 0$.

Not known for $\mathbb{Q}$-rank 2 locally symmetric spaces.
Given the dictionary between $\mathcal{M}_g$ and $\Gamma \backslash X$, the next result is natural.

**Theorem.** The simplicial volume of $\mathcal{M}_g$, $|\mathcal{M}_g|_{\text{orb}} = 0$ if and only if $g \geq 2$.

When $g = 1$, $\mathcal{M}_1$ admits finite smooth covers given by hyperbolic manifolds of finite area, and hence $|\mathcal{M}_g|_{\text{orb}} > 0$.

A simple known corollary:

**Corollary.** When $g \geq 2$, $\mathcal{M}_g$ does not admit a negatively pinched complete Riemannian metric.
What is needed for the proof?

The simplicial volume of a compact manifold only depends on the fundamental group of the manifold. If it is amenable, then it is zero.

For the vanishing result of noncompact manifolds, we use covers which are amenable and amenable at infinity. Such covers are constructed using good models of classifying spaces.

For a torsion-free finite index subgroup $\Gamma \subset \text{Mod}_g$, one model of the universal covering space $E\Gamma$ of the classifying space $B\Gamma$ of $\Gamma$ is given by a Borel-Serre partial compactification of $T_g$, and an analogue of Solomon-Tits theorem on the topology of Tits buildings for curve complex of the surface $S_g$ is needed to controlled the topology of the Borel-Serre partial compactification at infinity.
Cohomological properties

The Borel-Serre type compactification and the Solomon-Tits theorem for curve complex have also applications to $\text{Mod}_g$.

A discrete group $\Gamma$ is called a **Poincare duality group of dimension $n$** if for every $\mathbb{Z}\Gamma$-module $A$, there is an isomorphism

$$H^i(\Gamma, A) \cong H_{n-i}(\Gamma, A).$$

$\Gamma$ is called a **Generalized Poincare duality group of dimension $n$** if there is a $\mathbb{Z}\Gamma$-module $D$, called the **dualizing module**, such that for every $\mathbb{Z}\Gamma$-module $A$, there is an isomorphism

$$H^i(\Gamma, A) \cong H_{n-i}(\Gamma, D \otimes A).$$

If $D$ is trivial, then $\Gamma$ is a Poincare duality group.
Duality groups must be torsion-free. By passing to torsion-free finite index subgroups, we can define virtual (generalized) Poincare duality group.

If $\Gamma$ is a virtual generalized Poincare duality group of dimension $n$, then its virtual cohomological dimension is equal to $n$. 
Theorem. (Harer) $\text{Mod}_g$ is a virtual generalized Poincare duality group of dimension $4g-5$.

Theorem. (Ivanov-J) $\text{Mod}_g$ is a not virtual Poincare duality group.

Motivated by Borel-Serre results on arithmetic groups $\Gamma \subset G$:

$\Gamma$ is a virtual generalized Poincare duality group, and it is a virtual Poincare duality group if and only if $\Gamma \backslash X$ is compact.

The Borel-Serre partial compactification of $X = G/K$, and the Solomon-Tits theorem on the rational Tits building of $G$ are used.

By applying a duality theorem to the partial Borel-Serre compactification of $X$. 
The boundary components of the Borel-Serre partial compactification of $X$ are parametrized by \textbf{rational} parabolic subgroup of $G$, or equivalently by simplices of the rational Tits building of $G$.

A Borel-Serre partial compactification of Teichmuller space was introduced by William Harvey, and its boundary components are parametrized by simplices of the curve complex $\mathcal{C}(S_g)$ of a surface $S_g$.

Briefly, each vertex of $\mathcal{C}(S_g)$ corresponds to a homotopy class of essential simple closed curve in $S_g$.

Several vertices form the vertices of a simplex if they admit disjoint simple closed curves.

Disjoint simple closed curves (or geodesics) can be pinched simultaneously. Hence, simplices can parametrize boundary components.
Harer showed $\mathcal{C}(S_g)$ is homotopy equivalent to a bouquet of spheres.

By a similar proof of the Borel-Serre result, we can show that $\text{Mod}_g$ is a virtual generalized Poincare duality group.

The following question was open (or raised).

**Question.** How many spheres in the bouquet?

**Theorem.** (Ivanov-J) It contains infinitely many spheres, corresponding to the fact that $\text{Mod}_g$ is not a virtual Poincare duality group.

Solomon-Tits Theorem shows that rational Tits building is homotopy to a bouquet of *infinitely many spheres*. 

Universal spaces for proper actions of a discrete group $\Gamma$.

$E\Gamma$ is a space where $\Gamma$ acts properly, and for every finite subgroup $F \subset \Gamma$, the set of fixed point set $E\Gamma^F$ is nonempty and contractible.

$E\Gamma$ exists and is unique up to homotopy equivalence.

If $\Gamma$ is torsion-free, then $E\Gamma$ is equal to $E\Gamma$, the universal covering space of the classifying space $B\Gamma$ of $\Gamma$.

$\pi_1(B\Gamma) = \Gamma$, $\pi_i(B\Gamma) = \{1\}$ when $i \geq 2$.

Cohomology of $\Gamma$ can be computed by cohomology of $B\Gamma$.

**Fact.** If $\Gamma$ contains torsion elements, then there are non zero cohomology groups in arbitrarily large degree.
This implies that if $\Gamma$ contains torsion elements, then $\dim B\Gamma = \infty$ for every model. Hence $\dim E\Gamma = \infty$.

Find good models of $E\Gamma$, which could have a chance of finite dimension.

Want models of $E\Gamma$ such that $\Gamma \backslash E\Gamma$ is compact, or a finite CW-complex, called a cofinite model.

Also want to have small $\dim E\Gamma$.

The smallest dimension is the virtual cohomology dimension of $\Gamma$.

Such good models are important for proof of Baum-Connes conjecture, Novikov conjecture, ...
Proposition. $\mathcal{T}_g$ is a universal space for proper actions of $\text{Mod}_g$.

For any finite subgroup $F \subset \text{Mod}_g$, the nonemptyness of the fixed point set $(\mathcal{T}_g)^F$ is the positive solution of the Nielsen realization problem.

The contractibility of the fixed point set can be proved using earthquake paths.

Or we can use W-P metric and its geodesic convexity.
But $\Mod_g \backslash \mathcal{T}_g$ is noncompact. Hence $\mathcal{T}_g$ is not a cofinite model.

Borrow ideas from locally symmetric spaces $X = G/K$ is a universal space for $\Gamma$ since $X$ is a simply connected and nonpositively curved manifold.

Assume $\Gamma \backslash X$ is noncompact.

The Borel-Serre compacification of $X$ is a candidate.

**Theorem (J)** The Borel-Serre compacification of $X$ is indeed a cofinite model of $E\Gamma$.

Another method is to consider a truncated submanifold.
The analogue of Borel-Serre of $\mathcal{T}_g$ was outlined by Harvey.

As mentioned before, in this work, he introduced the curve complex, which turns out to be fundamental in many applications.

Later Ivanov gave a construction of the Borel-Serre partial compactification. But it is difficult to use.

Instead we use a subspace of $\mathcal{T}_g$.

For small $\varepsilon > 0$, define the thick part $\mathcal{T}_g(\varepsilon)$ consisting of hyperbolic surfaces without geodesics shorter than $\varepsilon$. 
**Fact.** $\mathcal{T}_g(\varepsilon)$ is a manifold with corners, stable under $\text{Mod}_g$ with a compact quotient.

**Theorem.** (J-Wolpert) There is a $\text{Mod}_g$-equivariant deformation retraction of $\mathcal{T}_g$ to $\mathcal{T}_g(\varepsilon)$.

Hence $\mathcal{T}_g(\varepsilon)$ is a **cofinite model** of $\mathcal{T}_g(\varepsilon)$. 
Problem. Find equivariant deformation retraction of $\mathcal{T}_g$ of dimension as small as possible.

These are called spines of $\mathcal{T}_g$.

The smallest possible dimension is $4g - 5$, vcd of $\text{Mod}_g$.

Thurston’s attempt to construct spines of $\mathcal{T}_g$ in 1985.

Remark. If $n > 0$, an explicit spine of $\mathcal{T}_{g,n}$ of the smallest possible dimension is known, due to Mumford, Thurston, Penner, Bowditch-Epstein.
Even though things should be easier for symmetric spaces, spines of symmetric spaces of the smallest possible dimension with respect to an arithmetic subgroup is not known in general, except for a few cases.

There are some work of Soule and Ash for linear symmetric spaces,

i.e., which are homothety section of symmetric cones such as $GL^+(n,\mathbb{R})/SO(n)$, the space of positive definite quadratic forms.
Define $R \subset \mathcal{T}_g$ to consist of hyperbolic surfaces where at least two shortest closed geodesics intersect.

This a closed real-analytic subset stable under $\text{Mod}_g$ with a compact quotient.

**Theorem** (J) There is an equivariant deformation retraction of $\mathcal{T}_g$ to $R$. In particular, $R$ is a cofinite model of $E \text{Mod}_g$.

$R$ is a spine of positive codimension. But $\mathcal{T}_g(\epsilon)$ is of codimension 0.

This is the first spine of positive codimension.

**Problem.** Get spines of smaller dimension.

Work in progress, at least get codimension 2.
**Borrow idea from symmetric space** $SL(n, \mathbb{R})/SO(n)$ of unimodular lattices of $\mathbb{R}^n$.

There is a well-rounded deformation retraction to well-rounded lattices.

Given a lattice $\Lambda \subset \mathbb{R}^n$, define

$$m(\Lambda) = \inf \{ \|v\| \mid v \in \Lambda - \{0\} \},$$

$$M(\Lambda) = \{ v \in \Lambda \mid \|v\| = m(\Lambda) \},$$ the set of shortest vectors.

If $M(\Lambda)$ spans $\Lambda$, $\Lambda$ is called a **well-rounded** lattice.

**Deformation procedure**: successive scaling up shortest vectors at the same time. i.e., scaling up the linear subspace spanned by $M(\Lambda)$ and scaling down the orthogonal complement to keep the lattice unimodular, until a well-rounded lattice is reached.
Procedure to deform $\mathcal{T}_g$ to the spine $R$ of positive codimension.

Simultaneously increase the length of all systoles (shortest geodesics) until they intersect.

Using gradient flow of the length functions.

In the paper of J-Wolpert, the deformation depends on a partition of unity and is not canonical. This procedure gives a canonical deformation retraction of $\mathcal{T}_g$ to its thick part $\mathcal{T}_g(\varepsilon)$. 
$L^2$-cohomology of noncompact Riemannian manifold $(M, ds^2)$.

$H^i_{(2)}(M)$ defined by the complex of $L^2$-differential forms.

(If $M$ is compact, then the $L^2$-cohomology is the de Rham cohomology.)

Similarly, we can define $L^p$-cohomlogy groups

They only depend on the quasi-isometry class
For general noncompact Riemannian manifolds, it is not easy to determine $L^p$-cohomology.

De Rham theorem gives a topological interpretation of the $L^2$-cohomology groups.

In general, we also want a topological interpretation.

In this case, locally symmetric spaces of finite volumes are probably the simplest ones.

Let $\Gamma \backslash X$ be a noncompact arithmetic locally symmetric varieties (i.e., arithmetic locally Hermitian symmetric space).

It has a Baily-Borel compactification, $\overline{\Gamma \backslash X}^{BB}$, a normal projective variety.

**example.** When $\Gamma \backslash X$ is a hyperbolic surface, then it is obtained by adding finitely many points, one for each end.
Another example is the Siegel modular variety $A_n$, adding $A_{n-1}, \cdots, A_1, A_0$.

**Fact.** It is a projective variety defined over a number field.

It admits the intersection cohomology group with respect to the middle perversity $IH^i(\Gamma \backslash X^{BB})$.

The key point of the intersection cohomology is that it satisfies the Poincare duality property.
Zucker conjecture, solution by Looijenga, Saper-Stern

**Theorem** (Looijenga, Saper-Stern). The $L^2$-cohomology of $\Gamma \backslash X$ is canonically isomorphic to the intersection cohomology of $\Gamma \backslash X^{BB}$:

$$H^i_{(p)}(\Gamma \backslash X) \cong IH^i(\Gamma \backslash X^{BB}).$$
Zucker’s result on $L^p$-cohomology of locally symmetric spaces

Reductive Borel-Serre compactification $\Gamma \backslash X^{RBS}$ of arithmetic locally symmetric spaces $\Gamma \backslash X$.

The singularities of $\Gamma \backslash X^{BB}$ are big.

$\Gamma \backslash X^{RBS}$ is mapped onto $\Gamma \backslash X^{BB}$ and resolve the singularities topologically to certain extent.

(The link of the singularities of $\Gamma \backslash X^{RBS}$ is less complicated.)

Even if $\Gamma \backslash X$ is a Hermitian locally symmetric space, $\Gamma \backslash X^{RBS}$ is not a complex space. This was introduced by Zucker in the study of $L^2$-cohomology of $\Gamma \backslash X$. 
The reason why is called reductive Borel-Serre compactification is that it is a reduction of the Borel-Serre compactification of $\Gamma \backslash X$ mentioned above.

The fibers of the map $\overline{\Gamma \backslash X}^{BS} \to \overline{\Gamma \backslash X}^{RBS}$ are nilmanifolds, and the boundary of $\overline{\Gamma \backslash X}^{RBS}$ consists of reductive locally symmetric spaces.

**Example.** Borel-Serre compactification for hyperbolic surfaces obtained by adding a circle to each end, and for the reductive Borel-Serre compactification, add one point.

**Theorem.** (Zucker) For $p \gg 0$, The $L^p$-cohomology $H^i_p(\Gamma \backslash X)$ is isomorphic to $H^i(\overline{\Gamma \backslash X}^{RBS})$. 
Generalizations to $\mathcal{M}_g$.

Let $\overline{\mathcal{M}}_g^{DM}$ be the Deligne-Mumford compactification, an orbifold.

As mentioned before, $L^p$-cohomology groups only depend on the quasi-isometry class of the metrics.

On Teichmuller space and moduli space, there are two different kinds of metrics.

One is the Weil-Petersson metric, and another is Teichmuller metrics and other complete metrics quasi-isometric to it.
We recall **Weil-Petersson metric**

It is a metric defined on $T_g$, invariant with respect to $\text{Mod}_g$.

For any point $p = (\Sigma_g, \varphi) \in T_g$, its cotangent space $T^*_p T_g$ is the space of holomorphic quadratic forms $Q(\Sigma_g)$.

For $f, g \in Q(\Sigma)$, define an inner product

$$\langle f, g \rangle = \int_{\Sigma_g} f \overline{g} (ds^2)^{-1},$$

where $ds^2$ is the area form of the hyperbolic metric on $\Sigma_g$.

This is the Weil-Petersson metric $\omega_{WP}$. 
Fact: (1) W-P metric $\omega_{WP}$ is invariant under $\text{Mod}_g$.

(2) It is a Kahler metric and has negative curvature.

(3) It is incomplete but is geodesically convex (every two points are connected by a unique geodesic).

(4) The volume of $\mathcal{M}_g$ in $\omega_{WP}$ is finite.

(5) The completion of $\mathcal{M}_g$ with respect to $\omega_{WP}$ is the Deligne-Mumford compactification.
**Theorem** [J-Zucker]

(1) For any Riemannian metric quasi-isometric to the Teichmüller metric, and any $p < +\infty$,

$$H^i_{(p)}(\mathcal{M}_g) \cong IH^i(\mathcal{M}^{DM}_g) = H^i(\mathcal{M}^{DM}_g).$$

(2) For any metric quasi-isometric to Weil-Petersson metric, when $p \geq 4/3$,

$$H^i_{(p)}(\mathcal{M}_g) \cong H^i(\mathcal{M}^{DM}_g),$$

when $p < 4/3$,

$$H^i_{(p)}(\mathcal{M}_g) \cong H^i(\mathcal{M}_g).$$
(Note that $L^p$-cohomology groups only depend on the quasi-isometry class of the metric.)

This depends on the asymptotics of the W-P metric near the boundary divisor.

This result shows a rank-1 phenomenon of $\mathcal{M}_g$ since when the rank of a Hermitian locally symmetric space $\Gamma \backslash X > 1$, $\overline{\Gamma \backslash X}^{RBS}$ is different from $\overline{\Gamma \backslash X}^{BB}$. 
**Spectral theory**

Since this is a conference on geometric analysis, I will conclude a result on the spectral theory of the W-P metric of $\mathcal{M}_g$.

Before studying spectral theory of $(\mathcal{M}_g, \omega_{WP})$, we recall some basic facts about self-adjoint extension of the Laplace operator.

For any smooth Riemannian manifold $M$, its Laplacian $\Delta$ is a symmetric operator with the domain $\mathcal{C}_0^\infty(M)$.

**Fact.** If $M$ is a complete Riemannian manifold, then $\Delta$ admits a unique self-adjoint extension to $L^2(M)$. ($\Delta$ is also called essentially self-adjoint).

(First proved by Gaffney, later Yau, Chernoff)

The completeness of $M$ is important.
**Fact.** If $M$ is a compact Riemannian manifold, then $\Delta$ has only a discrete spectrum, and its counting function satisfies the Weyl asymptotic law.

Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be its eigenvalues.

$$N(\lambda) = |\{\lambda_i \leq \lambda\}|.$$

Then as $\lambda \to +\infty$,

$$N(\lambda) \sim c \, \text{vol}(M) \lambda^{\frac{n}{2}},$$
$c$ depends on the dimension.

In the above result, the compactness of $M$ is crucial.
But for \((\mathcal{M}_g, \omega_{WP})\), both the completeness and compactnese conditions fail.

**Theorem** (J-Mazzeo-Muller-Vasy). (1) The Laplacian \(\Delta_{WP}\) of the W-P metric of \(\mathcal{M}_g\) is essentially self-adjoint.

(2) \(\Delta_{WP}\) has only a discrete spectrum and its counting function satisfies the Weyl asymptotic law.

In some sense, \((\mathcal{M}_g, \omega_{WP})\) behaves like a compact Riemannian manifold.

We do have a compact orbifold, the Deligne-Mumford compactification, but the metric \(\omega_{WP}\) is singular near the divisors.
The spectral theory of Teichmüller metric is probably not easy. For example, there are different definitions of the Laplacian operator.

\( \mathcal{M}_g \) admits many natural complete metrics which are quasi-isometric to the Teichmüller metric.

For example, the Bergman metric, Kahler-Einstein metric, the Ricci metric, McMullen metric.

It is easy to see that the spectrum of the Laplacian of these metrics is not discrete.

It should be continuous.

A natural problem is to understand the spectrum and generalized eigenfunctions of the continuous spectrum.

For example, geometric scattering theory.