Rational Curves On K3 Surfaces

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Overview of the talk

- The problem: existence of rational curves on a K3 surface
- The conjecture: infinitely many rational curves on an algebraic K3
- The technique: lifting from a special K3 to a general K3
- The new approach: of Bogomolov-Hassett-Tschinkel, using the jumping of Picard ranks
- Theorem (L-Liedtke): When $\rho(X)$ is odd, then $X$ contains infinitely many rational curves.
The Problem

Problem: Find rational curves in a complex K3 surfaces.

K3 surfaces: smooth, compact complex surface $S$ satisfying

- $\pi_1(S) = \{1\}$;
- $K_S \cong \mathcal{O}_S$;
  - this is $\wedge^2 T_X \cong \mathcal{O}_S$;
  - By Yau’s celebrated theorem on Calabi conjecture, Complex K3 surfaces are Ricci flat Kahler metrics, and vice versa.

Examples:

- degree four hypersurfaces: $S = (p_4 = 0) \subset \mathbb{CP}^3$;
- resolutions of a torus quotient by involution: $S = (T^2_C/\langle \tau \rangle)^{\text{desing}}$. 
Rational curves: image of a $u: \mathbb{CP}^1 \to S$

- non-constant;
- generic one-one onto image;
- holomorphic;
- distinct rational curves if they have different images.

Distinct rational curves: if two have distinct images.

\[ \mathbb{CP}^1 \xrightarrow{u} S \]

blue = image of red.
Main questions: given a compact, complex K3 surface $S$,

Q1 : does $S$ contain a rational curve?
Q2 : if does, does it contain more than one rational curves?
Q1 : if does, does it contain infinitely many rational curves?
Q1 : if does, does it contain a continuous family of rational curves?
The existence of (at least) one rational curve

Theorem (Bogomolov-Mumford) The existence of \( L \not\sim O_S \) produces rational curves.

Outline

- \( L \not\sim O_S \) produces \( s \neq 0 \in H^0(L^\otimes n) \), \( C = s^{-1}(0) \) a curve in \( S \);
- \( c_1(L)^2 = 0 \) or \( = -2 \) are ok; (elliptic fibration; \((-2)\)-curves);
- \( c_1(L)^2 > 0 \) implies \( S \) is algebraic (projective).
For $S$ projective,

- find a K3 $X_0$ so that it has two smooth rationals $C_1$ and $C_2$ so that they intersects transversally; let $L_0 = \mathcal{O}_{X_0}(C_1 + C_2);
- find a deformation $(X_t, L_t)$ of $(X_0, L_0)$, with $(X_1, L_1) = (S, L);
- show that $C_1 \cup C_2 \subset X_0$ deforms to rationals $C_t \subset X_t$, for general $t$; (this argument will be revisited)
- the limit of rational curves are union of rational curves, so $X_1 = S$ contains rational curves.
Theorem (Lefschetz) A simply connected compact complex manifold $S$ has a line bundle $L \not\cong \mathcal{O}_S$ if (the Hodge classes)

$$H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Z}) \neq \{0\}.$$

Answer to Q1: $S$ has a rational curve if $H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Z}) \neq \{0\}$. 
Remarks We have three different proofs of this existence theorem

- **Bogomolov-Mumford**: find the existence of a special K3, using deformation to prove the existence for general K3, plus limit of rational curves is a union rational curves;
- **Mori**: using reduction to char= $p$ technique, not relying on deformation of K3 surfaces;
- **Yau-Zaslow**: using moduli of sheaves on K3, plus beautiful geometry of Hyperekahler manifolds; not relying on deformation of K3 surfaces;
- the theorem holds for any field $k = \bar{k}$. 
The existence of more than one rational curves

**Corollary:** The space \( H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Q}) \neq \{0\} \) is spanned by classes generated by rational curves.

**Answer to Q2:** The number of rational curves is

\[ \# \geq \dim H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Q}). \]
No family of rational curves

Answer to Q4: There can be no continuous family of rational curves.

Otherwise,

a. we can find a birational morphism \( \varphi : X \to S \), \( X \) is a ruled surface (generic \( \mathbb{P}^1 \)-bundle);

b. \( S \) K3 implies there is a non-zero \((2, 0)\)-form on \( S \);

c. \( \varphi^*(\Omega) = 0 \) from (b), \( \neq 0 \) from (a), a contradiction.
The Conjecture (on Q3)

**Conjecture**: Any smooth complex K3 surface $S$ contains infinitely many rational curves.

This is motivated by Lang’s conjecture:

**Lang Conjecture**: Let $X$ be a general type complex manifold. Then the union of the images of holomorphic $u : \mathbb{C} \to X$ lies in a finite union of proper subvarieties of $X$. 
Key to the existence of rational curves:
A class $\alpha \neq 0 \in H^2(S, \mathbb{Z})$ is Hodge (i.e. $\in H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Q})$) is necessary and sufficient for the existence of a union of rational curves $C_i$ so that $\sum [C_i] = \alpha$.

Example: Say we can have a family $S_t$, $t \in \text{disk}$,

- $\alpha \in H^{1,1}(S_0, \mathbb{C}) \cap H^2(S_0, \mathbb{Q})$ so that $S_0$ has $C_0 \cong \mathbb{CP}^1 \subset S_0$ with $[C_0] = \alpha$;

- in case $\alpha \not\in H^{1,1}(S_t, \mathbb{C})$ for general $t$, then $\mathbb{CP}^1 \cong C_0 \rightarrow S_0$ can not be extended to holomorphic $u_t : \mathbb{CP}^1 \rightarrow S_t$. 

No extension of the curve $C_0$
We will consider polarized K3 surfaces \((S, H), c_1(H) > 0;\)

we can group them according to \(H^2 = 2d:\)

\[
\mathcal{M}_{2d} = \{(S, H) \mid H^2 = 2d\}.
\]

each \(\mathcal{M}_{2d}\) is smooth, of dimension 19;

each \(\mathcal{M}_{2d}\) is defined over \(\mathbb{Z}\). (defined by equation with coefficients in \(\mathbb{Z}\).)

to show that \((S, H)\) contains infinitely many rational curves, it suffices to show that

- for any \(N\), there is a rational curve \(R \subset S\) so that \([R] \cdot H \geq N\).

we define \(\rho(S) = \dim H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Q})\), called the rank of the Picard group of \(S\).
The existence of $\infty$ rational curves on complex algebraic K3 surfaces

**Theorem (Chen, 99):** Very general polarized complex K3 surfaces (i.e. $(S, H)$) contains infinitely many rational curves.

**Theorem:** A complex elliptic K3 surface contains infinitely many rational curves.

**Theorem:** A complex elliptic K3 surface with infinite automorphism group contains infinitely many rational curves.
Theorem (Bogomolov-Hassett-Tschinkel, 10) A degree 2 polarized complex K3 surface \((S, H)\) (i.e. \(H^2 = 2\)) having \(\rho(S) = 1\) contains infinitely many rational curves.

Extending their method, we prove

Theorem (L–Liedtke, 11) A polarized complex K3 surface \((S, H)\) having \(\rho(S) = 1\) contains infinitely many rational curves.

- \(\rho\) can run between 1 and 20;
- A K3 with \(\rho \geq 5\) is an elliptic K3, thus covered by an earlier theorem.
Technical approach

Extension Problem:

- a family of K3 surface $S_t$, $t \in T$ (a parameter space);
- $C_0 \subset S_0$ a union of rational curves; let $\alpha = [C_0] \in H^2(S, \mathbb{Z})$; (it is a Hodge class);
- suppose $\alpha \in H^{1,1}(S_t, \mathbb{C})$ for all $t \in T$;

We like to show that

- exists a family of curves $C_t \subset S_t$, such that
  - $C_t$ are union of rational curves;
  - $C_0 = C_0$. 

Can $C_0$ be extended to union of rationals?

Know $\alpha \in H^2(S_t, \mathbb{C})$
Use moduli of genus 0 stable maps

- A genus zero stable map is a holomorphic map $u : C \to X$ such that
  - $C$ is a nodal, genus 0 complex curve (i.e. a tree of $\mathbb{C}P^1$'s);
  - for any $\Sigma \cong \mathbb{C}P^1 \subset C$ with $u|_\Sigma = \text{const.}$, $\Sigma$ contains $\geq 3$ nodes of $C$. 

![Diagram of genus 0 stable maps](image-url)
Moduli of genus 0 stable maps

- A genus zero stable map is a holomorphic map $u : C \to X$ such that
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  - for any $\Sigma \cong \mathbb{C}P^1 \subset C$ with $u|\Sigma = \text{const.}$, $\Sigma$ contains $\geq 3$ nodes of $C$.

Let $S$ be a K3 surface, $\alpha \in H^2(S, \mathbb{Z})$,

- $\overline{M}_0(S, \alpha) = \{ [u : C \to S] \mid \text{genus 0 stable maps with } u_*([C]) = \alpha \}$
  - is an algebraic space;
  - has exp. dimension $-1$.
    - (expected, as when $\alpha$ is not Hodge, then $\overline{M}_0 = \emptyset$.)
Extension Lemma

- \( S_t, \ t \in T \), be a family of K3 surfaces;
- \( \alpha \in H^2(S_0, \mathbb{Z}) \) is Hodge for all \( t \);
- \([u_0] \in \overline{M}_0(S_0, \alpha)\).

Extension Lemma (Ran, Bogomolov-Tschinkel, –): Suppose \([u_0] \in \overline{M}_0(S_0, \alpha)\) is isolated, then \( u_0 \) extends to \( u_t \in \overline{M}_0(S_t, \alpha) \) for general \( t \in T \).

\[ \text{Ext. 1. } u \text{ does not deform in } S_0 \]
\[ \text{Ext. 2. } \alpha \text{ remains Hodge} \]

\( \Rightarrow \) exists family of rational curves.
Extension Lemma (Ran, Bogomolov-Tschinkel, –): Suppose $[u_0] \in \overline{M}_0(S_0, \alpha)$ is isolated, then $u_0$ extends to $u_t \in \overline{M}_0(S_t, \alpha)$ for general $t \in T$.

Outline of the proof

- let $U$ be an open neighborhood of all deformations of $S_0$; $U$ is a smooth complex manifold of $\mathbb{C}$ dimension 20;
- let $\mathcal{X} \to U$ be the tautological family of K3 surfaces;
- form $\overline{M}_0(\mathcal{X}, \alpha)$, the moduli of genus zero stable morphisms $u : C \to \mathcal{X}_z$, for some $z \in U$.
- exp. dim $\overline{M}_0(S_0, \alpha) = -1$ plus dim $U = 20$ implies
- exp. dim $\overline{M}_0(\mathcal{X}, \alpha) = 19$.

Let $[u_0] \in W \subset \overline{M}_0(\mathcal{X}, \alpha)$ be an irreducible component; dim $W \geq 19$. 
Look at $\pi : W \to U$, $\pi(W) \subset U$ has dimension at least 19;

$U(\alpha) \subset U$ is the locus where $\alpha \in H^2(S_z, \mathbb{Z})$ remain Hodge; $\dim U(\alpha) = 19$;

$\pi(W) \subset U(\alpha)$;

by assumption, $T \subset U(\alpha)$;

This implies that $T \subset \pi(W)$; thus $u_0$ lifts to a family $u_t : C_t \to S_t$, $t \in T$. 
Extension criterion

**Definition:** We say a map \([u] \in \overline{M}_0(S, \alpha)\) rigid if \([u]\) is an isolated point in \(\overline{M}_0(S, \alpha)\).

**Extension principle:** In case (a genus zero stable map) \(u : C \to S\) is rigid, then \(u\) extends to nearby K3 surfaces as long as the class \(u_*[C] \in H^2(S, \mathbb{Z})\) remains Hodge.
Lifting to a general K3 surface

Given an \((S, H)\) and an integer \(N\), to construct a rational \(R \subset S\) of \([R] \cdot H \geq N\), we will

- find a family of K3 surfaces \((X_t, H_t)\), \(t \in T\) a parameter space, such that
  - \((S, H)\) is a general member of \((X_t, H_t)\);
  - for a \(0 \in T\), we have \(C_0 \subset X_0\) a union of rational curves, and an irreducible \(R_0 \subset C_0\) so that \([R_0] \cdot H_0 \geq N\);
  - \([C_0] = nH_0\);
  - we can represent \(C_0\) as the image of a rigid genus zero stable map \([u_0]\).

By Extension criterion:

- we can extend \(u_0\) to \(u_t\) for general \(t\);
- since \(t\) is general, there is an irreducible component \(R_t\) of \(\text{image}(u_t)\) so that \([R_t] \cdot H_t \geq [R_0] \cdot H_0 \geq N\);
- since \((S, H)\) is a general member, we can assume \((S, H) = (X_t, H_t)\).
\text{deg} \geq N \quad \text{deg} \geq N

\text{general}
Theorem (Chen) Very general K3 \( (S, H) \in \mathcal{M}_{2d} \) contains infinitely many rational curves.

Proof

- Form a family, so that \( (X_0, H_0) \) is a singular K3 surface, thus \( C_0 \subset X_0 \) can be constructed explicitly.
- Applying the lifting property. So for any \( N \), all K3’s in a dense open subset in \( \mathcal{M}_{2d} \) contains rational curves of degree \( \geq N \).
Theorem (Bogomolov-Hassett-Tschikel). Fix an \( r \). Assume that every K3 \((X, H) \in \mathcal{M}_{2d}\) (of \( \rho(X) = r \)) defined over a number field \( K \) contains infinitely many rational curves in \( X_{\overline{Q}} \), then every \((X, H) \in \mathcal{M}_{2d}\) contains infinitely many rational curves.

- Let \( X \) be any K3, not defined over a number field, then \( X \) is the generic fiber of \( \mathcal{X} \to S \), \( S \) a variety over \( K \);

Example \( X = (\pi x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0) \subset \mathbb{CP}^3 \).  

- Switch \( \pi \) by \( t \), we define \( \mathcal{X} = (tx_1^4 + x_2^4 + x_3^4 + x_4^4 = 0) \subset \mathbb{P}^3 \times \mathbb{A}^1 \), \( S = \mathbb{A}^1 \).
Theorem (Bogomolov-Hassett-Tschikel). Fix an \( r \). Assume that every K3 \((X, H) \in \mathcal{M}_{2d}\) (of \( \rho(X) = r \)) defined over a number field \( K \) contains infinitely many rational curves in \( X_\mathbb{Q} \), then every complex \((X, H) \in \mathcal{M}_{2d}\) contains infinitely many rational curves.

- Let \( X \) be any K3. Then \( X \) is the generic fiber of \( \mathcal{X} \to S \), \( S \) a variety over \( K \);
- By (Deligne), we can find \( \xi \in S \), over \( L/K \), s.t. \( \text{Pic}X = \text{Pic}\mathcal{X}_{\xi} \);
- By assumption, \( \exists C_0 \subset \mathcal{X}_\xi \), high degree; \([C_0 \to \mathcal{X}_\xi] \in \overline{M}_0(\mathcal{X}_\xi) \) rigid;
- applying lifting criterion, \([C_0 \to \mathcal{X}_\xi]\) lifts to generic fiber of \( \mathcal{X} \to B \); thus lifts to \( C \to X \), of high degree.
When \((S, H)\) is defined over a number field, say \(\mathbb{Q}\), after deleting a finite prime numbers, \((S, H)\) is the generic member of a family of K3’ over \(T = \text{spec}\mathbb{Z}\)–finite places:

\[
\begin{align*}
X_{(2)} & \quad X_{(3)} & \quad X_{(5)} & \quad \cdots & \quad X_{(\mathfrak{p})} & \quad x_{\mathbb{Q}} = S
\end{align*}
\]
Theorem (Bogomolov-Zarhin, Nygaard-Ogus). Let $X$ be a K3 over a number field $K$. Then

1. for all but finitely many places $p$ of $K$, the reductions $X_p$ are smooth, not supersingular;
2. for all such places with $\text{char } p \geq 5$, $\rho((X_p)_{\overline{F}_p})$ are even.

Proof

- Use Tate Conjecture for K3;
- Use Weil Conjecture for K3.
Theorem (Bogomolov-Hassett-Tschinkel). Let $(X, H)$ be a polarized K3 in $M_2$. Suppose $\rho(X) = 1$, then $X$ contains infinitely many rational curves.

Sketch

- we only need to prove the case where $X$ is defined over $K$, say $K = \mathbb{Q}$;
- for any places $p$, $\rho((X_p)_{\overline{F}_p}) \geq 2$ imply $\exists D_p \subset (X_p)_{\overline{F}_p}$ rational curves, not in $\mathbb{Z}H$;
- if $D_p \cdot H \leq N$ uniformly, implies $\rho(X_{\overline{Q}}) \geq 2$, a contradiction;
Sketch

- places \( p \), \( \exists D_p \subset (X_p\bar{\mathbb{F}}_p) \) rational curves, not in \( \mathbb{Z}H \), \( D_p \cdot H \) arbitrary large;
- using \( H^2 = 2 \), find \( D'_p \subset (X_p\bar{\mathbb{F}}_p) \) rational curves, \( D_p + D'_p \in |n_pH| \);
- \( D_p + D'_p \) is the image of a rigid genus zero stable map.

Applying extension criterion, we can find a rational curve of degree \( \geq N \) in the generic fiber, which is \( X \).
Theorem (– Liedtke) Let $(X, H)$ be a polarized complex K3 surface such that $\rho(X)$ is odd. Then $X$ contains infinitely many rational curves.

Outline of proof

- We only need to prove the Theorem for $(X, H)$ defined over a number field $K$;
- say $K = \mathbb{Q}$, we get a family $X_p$ for every prime $p \in \mathbb{Z}$, $X$ is the generic member of this family;
- $\forall p$, exists $D_p \subset X_p$, $D_p \notin \mathbb{Z}H$,
  - we have $\sup D_p \cdot H \to \infty$;
- pick $C_p \subset X_p$ union of rationals, $D_p + C_p \in |n_p H|$.

Difficulty: $D_p + C_p$ may not be representable as the image of a rigid genus zero stable map.
Solution: Suppose we can find a nodal rational curves $R \subset X$, of class $kH$ for some $k$, then for some large $m$ we can represent

$$C_p + D_p + mR,$$

which is a class in $(n + mk)H$, by a rigid genus zero stable map.
End of the proof: In general, $X$ may not contain any nodal rational curve in $|kH|$. However, we know a small deformation of $X$ in $\mathcal{M}_{2d}$ contains nodal rational curves in $|kH|$ (a Theorem of Chen). Using this, plus some further algebraic geometry argument, we can complete the proof.
Happy Birthday, Peter.