THE CONSTANT RANK HESSIAN PROBLEM
FOR DEGENERATE ELLIPTIC FULLY NON-LINEAR EQUATIONS

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Outline of the Talk
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1. Convexity properties of elliptic equations: early works
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2. Convexity properties of elliptic equations: structural conditions
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3. Degenerate elliptic fully non-linear equations
   ▶ The homogeneous Monge-Ampère equation
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   - Geodesics in the space of Kähler potentials
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   - Geodesics in the space of volume forms: the Donaldson equation
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3. Some partial answers
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Convexity properties of elliptic equations: early works
The first eigenfunction of the Laplacian: Brascamp-Lieb

Consider the solution of the eigenvalue problem

$$\Delta u + \lambda u = 0 \text{ in } \Omega \subset \mathbb{R}^n, \quad u = 0 \text{ on } \partial \Omega,$$

where $\Omega$ is convex. Then

$$-\log u \text{ is convex}.$$
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Elliptic semilinear equations: Yau, Caffarelli-Friedman
Consider the solution of the Dirichlet problem

$$\Delta u = f(u) \quad \text{in} \; \Omega \subset \mathbb{R}^n, \quad u = M \quad \text{on} \; \partial \Omega,$$

where $\Omega$ is convex. Then there exists a function $u \rightarrow g(u)$, depending on $f$ and $M$, so that the function $g(u)$ is convex.
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\[ g(u) = (M^{p+1} - u^{p+1})^{-\frac{1}{2}}. \]
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Convexity of level sets, constant rank Hessians
Works of Korevaar, Kennington, Lewis, and others
Structural Conditions
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The structural condition of B. Bian and Pengfei Guan

Consider fully non-linear equations of the form

\[ F(u_{ij}(x), u_j(x), u, x) = 0 \]

where \( F(r, p, u, x) \) is a given function in \( C^{2,1}(S \times \mathbb{R}^n \times \mathbb{R} \times \Omega) \), with \( \Omega \) a domain in \( \mathbb{R}^n \), and \( S \) the space of symmetric matrices. Assume that

- Ellipticity: \( \frac{\partial F}{\partial r_{ij}}(u_{ij}, u_j, u, x) > 0 \) on \( \Omega \)

- Structural condition: \( F(r^{-1}, p, u, x) \) is locally convex in \( (r, u, x) \) for each \( p \) fixed.

Then the rank of the Hessian \( (u_{ij}(x)) \) is constant in \( \Omega \).
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**Observations**

- Builds on the method of Yau, Caffarelli-Friedman, and earlier conditions of Alvarez-Lasry-P.L. Lions
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- Earlier work of Caffarelli-Guan-Ma for \( F(r, p, u, x) = \tilde{F}(r) - \phi(p, u, x) \)
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- Key innovation of Bian-Guan: exploiting the convexity of quotients $\frac{\sigma_{k+1}(u_{ij}(x))}{\sigma_k(u_{ij}(x))}$
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- Similar quotients exploited in works of Huisken-Sinestrari
Degenerate Fully Non-Linear Equations
The homogeneous Monge-Ampère equation

Let $(M, \omega)$ be a compact Kähler manifold of dimension $d$ with boundary $\partial M$, and let $\omega_0$ be a closed, non-negative $(1,1)$-form on $M$. Consider the Dirichlet problem

$$(\omega_0 + \frac{i}{2} \partial \bar{\partial} \Phi)^d = 0 \quad \text{on} \quad M, \quad \Phi = \Phi_b \quad \text{on} \quad \partial M,$$

with $\omega_0 + \frac{i}{2} \partial \bar{\partial} \Phi \geq 0$. 
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- Similar questions for the real homogeneous Monge-Ampère equation
- In full generality, the answer is negative: the optimal regularity is \(C^{1,1}\), by classic examples of Nirenberg (real case), and Gamelin-Sibony (complex case). Recent counterexample to smooth solutions of maximum rank by Lempert-Vivas.
Geodesics in the space of Kähler potentials

Let $L \to X$ be a positive line bundle over a compact complex $n$-manifold $X$, i.e., $L$ admits a metric $h_0$ whose curvature $\omega_0 = -\frac{i}{2} \partial \bar{\partial} \log h_0$ is strictly positive.
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- The space $\mathcal{K} = \{ \varphi \in C^\infty(X); \omega_\varphi \equiv \omega_0 + \frac{i}{2} \partial \bar{\partial} \varphi > 0 \}$ is an $\infty$-dimensional manifold with metric

$$||\delta \varphi||^2 = \int_X |\delta \varphi|^2 \omega^n_\varphi, \quad \delta \varphi \in T_\varphi(\mathcal{K}).$$
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- A path $[0, T) \ni t \to \varphi(\cdot, t) \in \mathcal{K}$ is a geodesic in $\mathcal{K}$ if

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- Set $\Phi(z, w) = \varphi(z, \log |w|)$. Then the geodesic equation for $\varphi(\cdot, t)$ is equivalent to the following homogeneous complex Monge-Ampère equation for $\Phi(z, w)$,

$$(\pi^* \omega_0 + \frac{i}{2} D \bar{D} \Phi)^{n+1} = 0 \text{ on } M = X \times A$$

where $A = \{ 1 \leq |w| < e^T \}.$
Motivation

- The conjecture of Yau: the existence of a Kähler metric in $c_1(L)$ of constant scalar curvature should be equivalent to the stability of the polarization $L \to X$ in the sense of geometric invariant theory.
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- Donaldson’s $\infty$-dimensional GIT: the stability of $L \to X$ can be defined as the strict positivity of a generalized Futaki invariant for any non-trivial geodesic ray (geodesics defined for $[0, \infty)$).
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Existence of generalized geodesics

- X.X. Chen: $C^{1,\alpha}$ solutions $\Phi(z, w)$ of the homogeneous Monge-Ampère equation exist and are unique. In fact, they arise as limits as $\epsilon \to 0$ of solutions of the following regularized equation ("$\epsilon$-geodesics")

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(\pi^*\omega_0 + \frac{i}{2} D\bar{D}\Phi_\epsilon)^{n+1} = \epsilon \Omega^n
$$

where $\Omega$ is a positive-definite $(1, 1)$-form on $M = X \times \{1 \leq |w| < e^T\}$. Note that it is not known whether $\Phi(z, w)$ is strictly $\omega_0$-plurisubharmonic in the $z$-variables.
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◮ More precisely, using methods of Caffarelli-Kohn-Nirenberg-Spruck and Yau, and $C^2$ boundary estimates of B. Guan, he shows that

$$\|\Phi_\epsilon\|_{C^0(M)} \leq C, \quad \|\Delta \Phi_\epsilon\|_{C^0(M)} \leq C,$$

with the key step an independent proof of the $C^1$ estimates.
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- Alternatively, the solution $\Phi(z, w)$ can be constructed as a limit as $k \to \infty$ of one-parameter subgroups in the space $\mathcal{K}_k \subset \mathcal{K}$ of Fubini-Study metrics arising from the Kodaira imbeddings of $L^k \to X$ (P.-Sturm, using the Tian-Yau-Zelditch approximation theorem)
Rank and regularity of $\varepsilon$-geodesics

The recent counterexample of Lempert and Vivas shows that there exists manifolds $X$, and metrics $h_0, h_1$ for which there exists no $C^3$ geodesic path $\varphi(\cdot, t)$ of potentials in $\mathcal{K}$ joining $h_0$ and $h_1$. 
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- However, certain manifolds $X$ do admit smooth paths of geodesics connecting two arbitrary potentials. A basic example is toric varieties: there Kähler metrics are of the form

$$\omega = \frac{i}{2} \frac{\partial^2 \varphi}{\partial \xi_k \partial \xi_j} dz^j \wedge d\bar{z}^k$$

with $z_j = \xi_j + i \eta_j$, and the Kähler potential $\varphi$ can be characterized by the symplectic potential $u$, which is its Legendre transform

$$u(x) = \sup_{\xi} (\phi(\xi) - x \cdot \xi)$$

The geodesic equation for $\varphi$ is equivalent to the trivial equation (D. Guan, 1999)

$$\ddot{u} = 0$$

so that smooth solutions always exist.
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So the problems of regularity and maximum rank for $\Phi(z, w)$, as well as uniform bounds in $\epsilon$ for the Hessian of $\Phi(z, w)$ are of particular interest in this context.
The Donaldson equation
Let \((X, g_{ij})\) be a compact Riemannian manifold. Then the Donaldson equation is the following Dirichlet problem

\[ u_{tt}(n + \Delta u) - g^{ij} u_{ti} u_{tj} = 0, \quad u(0, x) = u_0(x), \quad u(x, 1) = u_1(x) \]

for a function \(u(x, t)\) on \(M = X \times [0, 1]\) satisfying \(n + \Delta u \geq 0\).
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for a function \(u(x, t)\) on \(M = X \times [0, 1]\) satisfying \(n + \Delta u \geq 0\).

The elliptic regularization of this equation is, for \(\epsilon > 0\),

\[ u_{tt}(n + \Delta u) - g^{ij} u_{t_i} u_{t_j} = \epsilon, \quad u(0, x) = u_0(x), \quad u(x, 1) = u_1(x) \]
The Donaldson equation
Let \((X, g_{ij})\) be a compact Riemannian manifold. Then the Donaldson equation is the following Dirichlet problem

\[ u_{tt}(n + \Delta u) - g^{ij} u_{ti} u_{tj} = 0, \quad u(0, x) = u_0(x), \quad u(x, 1) = u_1(x) \]

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Motivation

- Geodesics in the space of volume forms: consider the space \(\mathcal{H}\) defined by

\[ \mathcal{H} = \{ u \in C^\infty(X); \quad n + \Delta u > 0 \}. \]

with the metric

\[ \|\delta u\|^2 = \int_X |\delta u|^2 (n + \Delta u) \sqrt{g} \]

Then \(t \to u(\cdot, t)\) is a geodesic path in \(\mathcal{H}\) if and only if it satisfies the Donaldson equation.
Free boundary problems: as shown by Donaldson, the Donaldson equation is essentially equivalent to several free boundary problems, e.g., given positive functions $\rho_0, \rho_1$ with $\int_X \rho_i d\mu = \int_X d\mu$, find a domain $\Omega \subset X \times \mathbb{R}$ defined by $\Omega = \{(x, z) : H_0(x) < z < H_1(z)\}$, and a function $\theta$ on $\Omega$ which is equal to 0 and 1 on $H_0$ and $H_1$ respectively, with fluxes $\rho_i$, and satisfying the equation

$$(\epsilon \partial_z^2 + \Delta_X)\theta = 0.$$
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**Nahm’s equation:** the classical Nahm’s equation is the ODE system

$$\frac{dT_i}{dt} = [T_j, T_k]$$

for cyclic permutations of the three indices 1, 2, 3 and $T_i$ matrices in $U(n)$. The Donaldson equation can be interpreted as a Nahm’s equation with $U(n)$ replaced by the group of Hamiltonian diffeomorphisms of a surface with a fixed area form.
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The existence of \( C^{1,\alpha} \) solutions of the Donaldson equation has been established by Chen and He. But the optimal regularity and the rank of the Hessian of the solution is not known at this time.
The Counter-Example of Lempert and Vivas
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Let \((X, \omega_0)\) be a compact Kähler manifold of dimension \(n\), and let \(\omega_1 = \omega_0 + \frac{i}{2} \partial \bar{\partial} \psi\) be another Kähler form in the same class. Consider the Dirichlet problem

\[
(\pi^* \omega_0 + \frac{i}{2} D \bar{D} \Phi)^{n+1} = 0 \quad \text{on} \quad M = X \times A, \quad A = \{1 < |w| < e\}
\]

\[
\Phi = 0 \quad \text{on} \quad X \times 0, \quad \Phi = \psi \quad \text{on} \quad X \times 1,
\]

with \(\Phi(x, w)\) strictly \(\omega_0\)-plurisubharmonic in \(z\).
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- **Theorem of Lempert and Vivas:** Let \((X, \omega_0)\) be a compact connected Kähler manifold. If \(X\) admits a holomorphic isometry \(\Psi : X \rightarrow X\) with an isolated fixed point and \(\Psi^2 = \text{Id}_X\), then there exists a Kähler form \(\omega_1\) which cannot be connected to \(\omega_0\) be a smooth geodesic.
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- An example of such a manifold is a complex torus \(X = \mathbb{C}^m/\Gamma, \omega_0\) a translation invariant Kähler form, and and \(\Psi\) is the reflection \(\Psi : z \rightarrow -z\).
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- The homogeneous complex Monge-Ampère equation defines a foliation by \(C^1\) complex curves, the tangent vector of which is the degenerate direction of the Hessian.
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(p^* \omega_0 + \frac{i}{2} D \bar{D} \Phi)^{n+1} = 0 \quad \text{on} \quad M = X \times A, \quad A = \{1 < |w| < e\}
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- The homogeneous complex Monge-Ampère equation defines a foliation by \(C^1\) complex curves, the tangent vector of which is the degenerate direction of the Hessian.

- Lempert-Vivas show that the complex curve through the fixed point \(z_0\) is \(A \times \{z_0\}\), and that the tangent spaces to the curves nearby in the foliation must satisify an algebraic inequality.
The Real Monge-Ampère Equation

Let $M = X^n \times T$, with $X^n = (\mathbb{R}/\mathbb{Z})^n$ and $T = [0, 1]$. We consider the Dirichlet problem

$$\det(D^2_{xt}u + I_{n+1}) = \epsilon, \quad D^2_{xt}u + I_{n+1} \geq 0,$$

$$u(x, 0) = u^0(x), \quad u(x, 1) = u^1(x)$$

where $\epsilon > 0$, and the boundary data satisfy the strict partial convexity condition

$$D^2_xu^0 + I_n \geq \lambda, \quad D^2_xu^0 + I_n \geq \lambda$$

for some strictly positive constant $\lambda$. 
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for some strictly positive constant $\lambda$.

- **Theorem 1 (P. Guan and P.):** Let $u(x, t)$ be the solution of the above Dirichlet problem. Assume that either $n = 1$ or $n = 2$. Then for all $(x, t) \in M$ and all $\epsilon > 0$, we have

\[
D^2_x u(x, t) + I_n \geq \lambda.
\]
The Real Monge-Ampère Equation

Let \( M = X^n \times T \), with \( X^n = (\mathbb{R}/\mathbb{Z})^n \) and \( T = [0, 1] \). We consider the Dirichlet problem

\[
\det(D_{xt}^2 u + I_{n+1}) = \epsilon, \quad D_{xt}^2 u + I_{n+1} \geq 0, \\
u(x, 0) = u^0(x), \quad u(x, 1) = u^1(x)
\]

where \( \epsilon > 0 \), and the boundary data satisfy the strict partial convexity condition

\[
D_x^2 u^0 + I_n \geq \lambda, \quad D_x^2 u^0 + I_n \geq \lambda
\]

for some strictly positive constant \( \lambda \).

\begin{itemize}
  \item \textbf{Theorem 1 (P. Guan and P.)} Let \( u(x, t) \) be the solution of the above Dirichlet problem. Assume that either \( n = 1 \) or \( n = 2 \). Then for all \( (x, t) \in M \) and all \( \epsilon > 0 \), we have

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  \]

  \item \textbf{Theorem 2 (P. Guan and P.)} Assume that \( n = 1 \). Then there exists a constant \( C \) depending only on \( N, \lambda \), the Dirichlet data \( u^0 \) and \( u^1 \), and independent of \( \epsilon > 0 \), so that

  \[
  \sum_{a+b \leq N} \| D_x^a D_t^b u \|_{C^0(X^1 \times T)} \leq C_N.
  \]
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Let $M = X^n \times T$, with $X^n = (\mathbb{R}/\mathbb{Z})^n$ and $T = [0, 1]$. We consider the Dirichlet problem

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- The proof of Theorem 2 relies on a partial Legendre transform, inspired by works of D. Guan, P. Guan, and Rios-Sawyer-Wheeden.
The Donaldson Equation
Let $M = X^n \times T$, with $X^n = (\mathbb{R}/\mathbb{Z})^n$ and $T = [0, 1]$. Consider the Dirichlet problem

$$u_{tt}(n + \Delta u) - g^{jk} u_{tj} u_{tk} = \epsilon,$$
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The Donaldson Equation

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$$u_{tt}(n + \Delta u) - g^{jk} u_{tj} u_{tk} = \epsilon, \quad D_{xt}^2 u + l_{n+1} \geq 0,$$

$$u(x, 0) = u^0(x), \quad u(x, 1) = u^1(x)$$

where $\epsilon > 0$, and the boundary data satisfy the strict partial convexity condition

$$D_x^2 u^0 + l_n \geq \lambda, \quad D_x^2 u^0 + l_n \geq \lambda$$

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**Theorem 3** (P. Guan and P.) Assume that $n = 1$ or $n = 2$. Then under the above condition, for any $\epsilon > 0$, the solution $u(x, t)$ satisfies the condition

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The Donaldson Equation

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- **Theorem 4 (P. Guan and P.)** Assume that $n = 3$. Let $\Omega$ be a domain in $\mathbb{R}^3$, and $\delta > 0$. Then if $u(x,t)$ is a solution of the Donaldson equation on $\Omega \times (0, \delta)$ with $D^2_x u + l_n \geq 0$, then the rank of $D^2_x u + l_n$ is constant in $\Omega \times (0, \delta)$. 
The Donaldson Equation

Let $M = X^n \times T$, with $X^n = (\mathbb{R}/\mathbb{Z})^n$ and $T = [0, 1]$. Consider the Dirichlet problem

$$
\begin{align*}
\frac{\partial^2 u}{\partial t^2} (n + \Delta u) - g^{jk} u_{tj} u_{tk} &= \epsilon, \\
D_{xt}^2 u + l_{n+1} &\geq 0, \\
u(x, 0) &= u^0(x), \\
u(x, 1) &= u^1(x)
\end{align*}
$$

where $\epsilon > 0$, and the boundary data satisfy the strict partial convexity condition

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- In particular, when the boundary data satisfies the strict partial convexity condition and $n = 3$, we have $D_x^2 u(x, t) + l_n > 0$ for all $(x, t) \in X \times T$. 
Sketch of Proofs
Both the Monge-Ampère equation and the Donaldson equation are of the form

$$F(D_{xt}^2 u + I_{n+1}) = \epsilon$$

We set $F^{\alpha \beta} = \frac{\partial F}{\partial r_{\alpha \beta}}$, $F^{\alpha \beta, \gamma \delta} = \frac{\partial^2 F}{\partial r_{\alpha \beta} \partial r_{\gamma \delta}}$, etc.
Both the Monge-Ampère equation and the Donaldson equation are of the form

\[ F(D_{xt}^2 u + I_{n+1}) = \epsilon \]

We set \( F_{\alpha\beta} = \frac{\partial F}{\partial r_{\alpha\beta}} \), \( F_{\alpha\beta,\gamma\delta} = \frac{\partial^2 F}{\partial r_{\alpha\beta} \partial r_{\gamma\delta}} \), etc.

**The strong maximum principle**

For \( \epsilon > 0 \), the equation is elliptic and the following version of the strong maximum principle holds: if \( \varphi \) is a non-negative function satisfying \( \varphi(x_0, t_0) = 0 \) and

\[ F_{\alpha\beta} \varphi_{\alpha\beta} \leq C (\varphi + |\nabla \varphi|) \]

then \( \varphi \) vanishes in a neighborhood of \((x_0, t_0)\).
Both the Monge-Ampère equation and the Donaldson equation are of the form

\[ F(D^2_{xt}u + I_{n+1}) = \epsilon \]

We set \( F^{\alpha\beta} = \frac{\partial F}{\partial r^{\alpha\beta}}, F^{\alpha\beta,\gamma\delta} = \frac{\partial^2 F}{\partial r^{\alpha\beta} \partial r^{\gamma\delta}}, \) etc.

### The strong maximum principle

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then \( \varphi \) vanishes in a neighborhood of \((x_0, t_0)\).

### The set-up for the constant rank problem

Let \( \mu_0 \) be the minimum over \( X^n \times T \) of the lowest eigenvalue of \( D^2_x u + I_n \). We shall show that the set where the matrix \( D^2_x u + I_n - \mu_0 I_n \) has a zero eigenvalue is open. For each \( K \), let \( \lambda_i, 1 \leq i \leq n \), be the eigenvalues of the matrix \( D^2_x u + I_n - \mu_0 I_n \), and set

\[ \varphi = \sum_{i_1 < \cdots < i_{n-K+1}} \lambda_{i_1} \cdots \lambda_{i_{n-K+1}} = \sigma_{n-K+1}(\lambda_i) \]
Good vs Bad Eigenvalues
Good vs Bad Eigenvalues

\[ \{1, \cdots, n\} = G \cup B \]

where the good set \( G \) consists of eigenvalues \( \lambda \) which do not vanish at \((x_0, t_0)\), and the bad set \( B \) of eigenvalues which do vanish at \((x_0, t_0)\). (\( \#G = n - K \) and \( \#B = K \)). Set

\[ v_{ij} = u_{ij} + \delta_{ij} - \mu_0 \delta_{ij}. \]

Then

\[ c_1 \sum_{m \in B} v_{mm} \leq \varphi \leq c_2 \sum_{m \in B} v_{mm}. \]
Good vs Bad Eigenvalues

\{1, \cdots, n\} = G \cup B

where the good set \( G \) consists of eigenvalues \( \lambda \) which do not vanish at \( (x_0, t_0) \), and the bad set \( B \) of eigenvalues which do vanish at \( (x_0, t_0) \). (\( \# G = n - K \) and \( \# B = K \)).

Set

\[ v_{ij} = u_{ij} + \delta_{ij} - \mu_0 \delta_{ij}. \]

Then

\[ c_1 \sum_{m \in B} v_{mm} \leq \varphi \leq c_2 \sum_{m \in B} v_{mm}. \]

The general formula for the linearized operator

\[ F^{\alpha\beta} \varphi_{\alpha\beta} = -\left( \prod_{g \in G} v_{gg} \right) \left( \sum_{m \in B} F^{\alpha\beta, \gamma\delta} u_{\alpha\beta m} u_{\gamma\delta m} + 2 \sum_{m \in B} \sum_{g \in G} \frac{F^{\alpha\beta}}{v_{gg}} u_{mg\alpha} u_{mg\beta} \right) + O(\varphi + |\nabla \varphi|) \]
The case of the real Monge-Ampère equation

In this case,

\[ F(r) = \det(r_{\alpha\beta}) - \epsilon \]
The case of the real Monge-Ampère equation

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\[ F(r) = \det(r_{\alpha\beta}) - \epsilon \]

The cases that we can treat correspond to \( \#G = 1 \). The main terms in the preceding identity turn out to be respectively given by, modulo \( O(\varphi + |\nabla \varphi|) \),

\[ F^{\alpha\beta} u_{mg\alpha} u_{mg\beta} = (u_{gg} + 1)(F^{tt,gg} u_{tgm}^2 + 2 \sum_{i \in B} F^{it,gg} u_{mgi} u_{mgT} - \sum_{\alpha\beta} F^{\alpha\beta,gg} u_{mgg} u_{\alpha\beta m}) \]
The case of the real Monge-Ampère equation
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$$F^{\alpha\beta,\gamma\delta} u_{\alpha\beta m} u_{\gamma\delta m} = -2(F^{tt,gg} u_{tgm}^2 + 2 \sum_{i \in B} F^{it,gg} u_{mgi} u_{mgt} - \sum_{\alpha\beta} F^{\alpha\beta,gg} u_{mgg} u_{\alpha\beta m})$$
The case of the real Monge-Ampère equation

In this case,

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The cases that we can treat correspond to \( \# G = 1 \). The main terms in the preceding identity turn out to be respectively given by, modulo \( O(\varphi + |\nabla \varphi|) \),

\[
F^{\alpha\beta} u_{mg\alpha} u_{mg\beta} = (u_{gg} + 1)(F^{tt, gg} u_{tgm}^2 + 2 \sum_{i \in B} F^{it, gg} u_{mgi} u_{mgt} - \sum_{\alpha\beta} F^{\alpha\beta, gg} u_{mgg} u_{\alpha\beta m})
\]

\[
F^{\alpha\beta, \gamma\delta} u_{\alpha\beta m} u_{\gamma\delta m} = -2(F^{tt, gg} u_{tgm}^2 + 2 \sum_{i \in B} F^{it, gg} u_{mgi} u_{mgt} - \sum_{\alpha\beta} F^{\alpha\beta, gg} u_{mgg} u_{\alpha\beta m})
\]

and hence

\[
F^{\alpha\beta} \varphi_{\alpha\beta} = 2\mu_0 F^{\alpha\beta, \gamma\delta} u_{\alpha\beta m} u_{\gamma\delta m} \leq 0
\]

by the concavity of \( F \).
The case of the Donaldson equation

- In this case take \( \varphi = \sigma_{n-K+1} + \frac{\sigma_{n-K+2}}{\sigma_{n-K+1}} \)
- We find

\[
F^{\alpha \beta} \varphi_{\alpha \beta} \leq -C \sum_{m \in B} Q_m + O(\varphi + |\nabla \varphi|)
\]

with the term \( Q_m \) defined by

\[
Q_m \equiv u_{ttm} \Delta u_m - \sum_{k \in G} u_{k tm}^2 + u_{tt} \sum_{j, k \in G} \frac{u_{mjk}^2}{1 + u_{jj}}
+ (n + \Delta u) \sum_{j \in G} \frac{u_{mjt}^2}{1 + u_{jj}} - 2 \sum_{j, k \in G} \frac{u_{tk u_{tjm} u_{mjk}}}{1 + u_{jj}}
\]

- The key inequality is that, for \( \#G = 1 \) and \( \#G = 2 \),

\[
Q_m \geq 0
\]
In fact, it follows from the Donaldson equation that, when \(\#G = 2\), we have

\[
Q_m = \frac{1}{2} \sum_{j \neq k} \left\{ \left( \frac{u_{tt}}{(1 + u_{jj})(1 + u_{kk})(n + \Delta u)} \right)^{\frac{1}{2}} ((1 + u_{kk})u_{jj1} - (1 + u_{jj})u_{kk1}) \right. \\
+ \left. \left( \frac{u_{tk}u_{tk1}}{1 + u_{kk}} - \frac{u_{tj}u_{tj1}}{1 + u_{jj}} \right) \left( \frac{1 + u_{jj}(1 + u_{kk})}{u_{tt}(n + \Delta u)} \right)^{\frac{1}{2}} \right\}^2 \\
+ \frac{1}{2} \sum_{j \neq k} \frac{1}{(1 + u_{jj})(1 + u_{kk})} \left[ u_{1jk} \sqrt{u_{tt}(n + \Delta u)} \right. \\
- \left. \frac{u_{tk}u_{tj1}(1 + u_{kk}) + u_{tj}u_{tk1}(1 + u_{jj})}{\sqrt{u_{tt}(n + \Delta u)}} \right]^2 \\
+ \sum_{j} \frac{1}{u_{tt}(n + \Delta u)} u_{1jt}^2 \sum_{k \neq j} \frac{1 + u_{kk}}{1 + u_{jj}} \left( \sum_{\ell \neq j, k} u_{t\ell}^2 + \epsilon \right)
\]
Maximum rank and partial Legendre transform

- If $u$ has maximum rank, then the partial Legendre transform on $M = (\mathbb{R}/\mathbb{Z})^n \times T$ is well-defined,

\[ x \to y = \frac{\partial u}{\partial x} + x, \quad t \to s = t, \]

\[ u(x, t) \to f(y, s) = -\frac{1}{2}|x - y|^2 - u(x, t). \]
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$$\det(D^2_{xt}u + I_{n+1}) = \epsilon \iff \frac{\partial^2 f}{\partial s^2} + \epsilon \det(D^2_y f + I_n) = 0$$
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- In \( 1 + 1 \) dimensions, this leads to \( C^\infty \) bounds for \( u \), independent of \( \epsilon > 0 \),

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\partial_y^m x = -\frac{1}{(1 + u_{xx})^m} \partial_x^{m+1} u + \frac{P(u, \cdots, \partial_x^m u)}{(1 + u_{xx})^{2m-1}}
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with \( P \) a polynomial in its variables.
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- In $n + 1$ dimensions, $n \geq 2$, it is unclear what uniform bounds hold for the transformed equation, and consequently for $u(x, t)$.