BERGMAN KERNEL ASYMPTOTICS THROUGH PERTURBATION

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Abstract. This article is a survey of methodology and results in *Asymptotic expansion of the Bergman kernel via perturbed Bargmann-Fock Model* [7] which I spoke on at the Analysis and Geometry in Several Complex Variables Conference at Texas A & M at Qatar on January 2015. This is joint work with H. Hezari, C. Kelleher, and H. Xu.

1. Introduction

In the study of several complex variables, the Bergman kernel $K(x, y)$ is the holomorphic reproducing kernel of the orthogonal projection onto square integrable holomorphic functions on a domain $D \subset \mathbb{C}^n$, i.e. for $\mathcal{P} : \mathcal{L}^2(D) \to H^0(D)$,

$$\mathcal{P}f(x) = \int_D f(y)\overline{K(y, x)}dV(y), \qquad f \in H^0(D), \quad x \in D. \tag{1.1}$$

An equivalent definition of the Bergman kernel can be given by first considering an orthonormal basis $\{\varphi_\alpha\}$ of holomorphic square integrable functions. Then the sum

$$K(y, x) := \sum_\alpha \varphi_\alpha(y)\overline{\varphi_\alpha(x)}$$

satisfies the orthogonal reproducing property hence is an alternate characterization of the Bergman kernel. It can be generalized to the complex manifold setting given a positive line bundle L and considering the holomorphic sections of L^k, the k-th tensor power of L and an $\mathcal{L}^2$ inner product induced from the Hermitian metric of the line bundle. In general, it is impossible to compute the Bergman kernel explicitly, however the asymptotic expansion of the kernel has been an active field of research. Initially Tian gave leading asymptotics on the diagonal using the method of peak sections [13] for the line bundle case. A complete expansion was given independently by Zelditch [14] and Catlin [4] by using the Boutet de Monvel and Sjöstrand parametrix [3]. The near-diagonal expansion follows immediately from the diagonal expansion. The symplectic case was given by Shiffman and Zelditch [12]. In particular, the near-diagonal asymptotic expansions, as $k \to \infty$, are of the form, with $b_l(x, \overline{y})$ certain hermitian functions ($b_l(x, \overline{y}) = \overline{b_l(y, \overline{x})}$),

$$K(x, y) = e^{k\psi(x,y)}k^n \left(1 + \sum_{l=1}^{\infty} \frac{b_l(x, \overline{y})}{k^l}\right), \ \mathrm{dist}(x, y) < \frac{1}{\sqrt{k}},$$

where $\psi(x, \overline{y})$ is the polarization of $\varphi(x)$, i.e. $\psi(x, \overline{y})|_{y=x} = \varphi(x)$. Lu demonstrates that the functions $b_l(x, \overline{x})$ is a polynomial of covariant derivatives of curvature of the underlying manifold M and computed the first 4 terms [10], and the off-diagonal terms $b_l(x, \overline{y})$ were computed by Lu and Shiffman [11] by expanding the kernel at the diagonal. There are various other approaches to show the existence of the asymptotic expansion such as using the asymptotics of the heat kernel done by Dai, Liu and Ma [5]. Berman, Berndtsson, and Sjöstrand, (BBS) [1] give an alternate approach to proving the existence of the expansion using microlocal analysis techniques. We propose a method to prove the existence and method of computing the coefficients via a perturbation approach. Unlike the existing methods which involve very technical machinery, we take a very elementary approach

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using only straightforward complex analysis and Hörmander's $\mathcal{L}^2$-estimate of the $\overline{\partial}$ operator to relate the local calculations to the global setting. As a byproduct, we obtain a relatively simple method to compute the near-diagonal coefficients of the Bergman kernel asymptotic expansion, independent of previous results on the diagonal. To be precise, we give a proof of the following:

Theorem 1.1 (Shiffman, Zelditch [12])**.** *The scaled near-diagonal Bergman kernel admits the following asymptotic expansion in the Bochner coordinates,*

$$K\left(\tfrac{u}{\sqrt{k}}, \tfrac{v}{\sqrt{k}}\right) = k^n e^{u\cdot\bar{v}} \left(1 + \sum_{j=2}^{\infty} \frac{c_j(u,\bar{v})}{\sqrt{k^j}}\right), \qquad |u|, |v| \leq 1.$$

Where each coefficient $c_j(u,\bar{v})$ is a polynomial of the form $\sum_{p,q} c_j^{p,q} u^p \bar{v}^q$ satisfying

$$\begin{cases} c_j^{p,q} = 0 & \text{for } p+q > 2j, \\ c_j^{p,q} = 0 & \text{for } p+q \neq j \mod 2. \end{cases}$$

In particular, by setting $u = v = 0$, this verifies the on-diagonal expansions of Zelditch [14] and Catlin [4].

2. Outline of the Proof

2.1. **Localization and construction.** Let $(L,h) \to (M^n, \omega)$ be a positive hermitian holomorphic line bundle over a compact complex manifold. The Kähler form ω induced by h is given by

$$\omega := -\tfrac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(h).$$

The localization argument of the kernel is based on the one given in BBS [1]. Given a point $x \in M$ and a neighborhood U_x with local holomorphic coordinates $(z_1, \dots z_n)$ and local trivialization e_L of L, the hermitian metric, by positivity of the line bundle, can by locally expressed as $h(e_L, e_L) = e^{-\varphi}$, where φ is a local strictly plurisubharmonic function. Local holomorphic sections of L^k are represented by holomorphic functions on U_x, and the *local* norm is a weighted L^2-norm over U_x, i.e.

$$\|f\|_{\mathcal{L}^2(U,k\varphi)} = \int_U |\tilde{f}(z)|^2 e^{-k\varphi(z)} dV_g, \quad f = \tilde{f} e_L^k \in H^0(U, L^k) \tag{2.1}$$

where we naturally extended the metric and frame as h^k and e_L^k. By using the observation that the Bergman kernel is concentrated in the near diagonal, we first construct, for $N \in \mathbb{Z}_+$, a *local* Bergman kernel, $K_N^{loc}(x,y)$, which will satisfy a reproducing property locally, up to an error $O(k^{-\frac{N+1}{2}})$, i.e.

$$f(x) = \left\langle \chi_k(y) f(y), K_N^{loc}(y,x) \right\rangle_{\mathcal{L}^2(B,k\varphi)} + O\left(k^{-\frac{N+1}{2}}\right) \|f\|_{\mathcal{L}^2(B,k\varphi)}, \; f \in H^0(U, L^k). \tag{2.2}$$

Where BBS use a certain contour integral to show the local reproducing property of $e^{k\theta\cdot(x-y)}$, then perturbing the candidate kernel by a *negligible amplitude* (defined in [1]) to obtain the local kernel. Our approach however begins by using Böchner coordinates which allows us to write the Kähler potential φ as

$$\varphi(z) = |z|^2 + R(z) \tag{2.3}$$

where $R(z) = O(|z|^4)$. Then using the rescaling $z \mapsto \frac{v}{\sqrt{k}}$ and choosing a smooth cutoff function χ such that

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2} k^{-\frac{1}{4}-\varepsilon} \\ 0 & \text{if } |x| \geq k^{-\frac{1}{4}-\varepsilon}, \qquad 0 < \varepsilon < \frac{1}{4} \end{cases} \tag{2.4}$$

we can re-write the norm as

$$\|f\| = \int_{\mathbb{C}^n} \left| \chi\left(\frac{v}{\sqrt{k}}\right) f\left(\frac{v}{\sqrt{k}}\right) \right|^2 e^{-|v|^2 + kR(\frac{v}{\sqrt{k}})} dV_E, \tag{2.5}$$

where dV_E is the Euclidean volume form. Since the cut-off function ensures that $|v| \le k^{\frac{1}{4}-\varepsilon}$ so that $kR(\frac{v}{\sqrt{k}}) = O(k^{-\varepsilon})$, we can do our calculations in a *perturbed Bargmann-Fock Space*. The ε is included so that it is indeed a small perturbation as $k \to \infty$. Also note that the support of $d\chi$ is

$$\operatorname{supp}(d\chi_k(\tfrac{u}{\sqrt{k}})) \subset \{z \mid \tfrac{1}{2}k^{\frac{1}{4}-\varepsilon} \le |z| \le k^{\frac{1}{4}-\varepsilon}\}$$

which proves to be important when we want to obtain an exponential decay of our kernel. With this framework, we formally solve for coefficients such that for any polynomial F,

$$F\left(\frac{u}{\sqrt{k}}\right) = \int_{\mathbb{C}^n} F\left(\frac{v}{\sqrt{k}}\right) e^{u\cdot\bar{v} - |v|^2} \left(\sum_{j=0}^{N} \sum_{j,p,q} \frac{c_j^{p,q}}{\sqrt{k^j}} u^p \bar{v}^q \right) \left(\sum_{m \le d} \sum_{r+s=0}^{2m} \frac{a_m^{r,s} v^r \bar{v}^s}{\sqrt{k^m}} \right) dV_E, \tag{2.6}$$

where

$$e^{-kR\left(\frac{v}{\sqrt{k}}\right)} \Omega\left(\tfrac{v}{\sqrt{k}}\right) = \sum_{m} \sum_{r+s=0}^{2m} \frac{a_m^{r,s} v^r \bar{v}^s}{\sqrt{k^m}}, \tag{2.7}$$

the power series expansion of the perturbed volume form term. Note that $e^{u\cdot v}$ is the reproducing kernel for the Bargmann-Fock space, $(\mathbb{C}^n, e^{-|v|^2})$, hence the reproducing property holds trivially for $N = 0$. For $N > 0$, the coefficients c_j must cancel with a_m, which allows us to solve for the coefficients. After some lengthy combinatorial arguments, we are able to show the existence of the coefficients with the stated properties in Theorem 1.1. It still remains to show that this candidate kernel,

$$K^{loc}\left(\tfrac{u}{\sqrt{k}}, \tfrac{v}{\sqrt{k}}\right) = k^n e^{u\bar{v}} \left(1 + \sum_{j=2}^{\infty} \frac{c_j(u,\bar{v})}{\sqrt{k}^j} \right), \tag{2.8}$$

where

$$c_j(u,\bar{v}) = \sum_{p,q} c_j^{p,q} u^p \bar{v}^q, \tag{2.9}$$

satisfies the local reproducing property when truncated to the N-th term as in (2.2). To achieve the desired property, we must show a series of error estimates between the finitely expanded terms, which are defined on all of $\mathbb{C}^n$ and the terms localized by the cut-off functions. Combining the estimates, we are able to show that the our candidate kernel indeed satisfies the desired local reproducing property.

Proposition 2.1 (Local reproducing property)**.** *Let $f \in H^0(B)$, and c_j quantities as constructed above in 2.6. Then for $u \in B$, the following equality holds*

$$f\left(\frac{u}{\sqrt{k}}\right) = \left\langle \chi_k\left(\frac{v}{\sqrt{k}}\right) f\left(\frac{v}{\sqrt{k}}\right), e^{\bar{u}\cdot v} \left(\sum_{j}^{N} \frac{c_j(v,\bar{u})}{\sqrt{k^j}} \right) \right\rangle_{\mathcal{L}^2\left(B(\sqrt{k}), k\varphi_k(v)\right)} + O\left(\frac{1}{\sqrt{k}^{N+1-2n}} \right) \|f\|_{\mathcal{L}^2(B,k\varphi)}.$$

2.2. **Local to global.** To carry our locally constructed kernel to the global setting, we use Hörmander's $\mathcal{L}^2$ estimate for the $\bar{\partial}$ operator, which is the standard procedure used to show the existence of global sections with desired estimates. It was done in BBS and we base our proof on that. A difference between the BBS construction and the one presented is that we have the analyticity of our local kernel without any assumption on our metric potential φ, since we have explicitly constructed our kernel and only take finitely many terms. To be precise,

Theorem 2.2 (Local to global)**.** *The following equality relates the truncated local Bergman kernel K_N^{loc} to the global Berman kernel K.*

$$K\left(\frac{u}{\sqrt{k}}, \frac{v}{\sqrt{k}}\right) = K_N^{loc}\left(\frac{u}{\sqrt{k}}, \frac{v}{\sqrt{k}}\right) + O\left(k^{2n-\frac{N+1}{2}}\right).$$

Sketch of proof. We first apply the local reproducing property to the global Bergman kernel K to obtain:

$$K = \langle K, K_N^{loc}\rangle + O(k^{\frac{N+1}{2}-n})\|K\|$$

Using the well known upper bound

$$\|K\| \leq Ck^n,$$

we obtain

$$K = \langle K, K_N^{loc}\rangle + O(k^{\frac{N+1}{2}-2n})$$

Next to estimate

$$\chi K_N^{loc} - \langle \chi K_N^{loc}, K\rangle$$

which is the $\mathcal{L}^2$ minimal solution to

$$\overline{\partial}(u) = \overline{\partial}(\chi K_N^{loc}).$$

Since $\overline{\partial}\chi$ is only defined in an annulus and K_N^{loc} is explicitly known, we obtain an upper bound estimate $O(k^{-\delta\sqrt{k}})$ for some $\delta > 0$. Combining the two estimates, we obtain our result. □

2.3. C^μ **convergence.** The convergence for higher order regularity can be achieved from the C^0 convergence. However, since the C^μ norm depends on the coordinates, we must give some care in how we choose the coordinates. The C^0 convergence was proved using Böchner coordinates centered at a point $p \in M$. To obtain the C^μ convergence, we first choose a smooth family of Böchner coordinates, an example of such a family is given in [9]. The issue is that the asymptotic expansion is given by centering a "nice" holomorphic coordinate system at a point and that the C^μ norm depends on both the variables of the expansion and the point of the expansion. For the variables of the expansion, since they are holomorphic and anti-holomorphic in each variable, an argument using the Cauchy estimates on balls of radius $\frac{1}{\sqrt{k}}$ will give us the result. For the variable at the point of expansion, we use the Böchner-Martinelli integral representation to bound the difference between the local candidate and global kernel in a neighborhood of size $\frac{1}{\sqrt{k}}$. Details are given in [7].

2.4. **Further Consideration.** Define the *Bergman function* $\mathcal{B}(x)$ to the the norm of $K(x,x)$ as a section fo the bundle $L^k \otimes \overline{L}^k$. In [9] they consider the case when the Hermitian metric on the bundle and hence the metric on the manifold itself is real analytic. Then the asymptotic expansion of the Bergman kernel function is a convergent series.

Theorem 2.3 (Theorem 1.3 [9])**.** *Suppose that the Hermitian metric h of L is real analytic at a point $x \in M$. Then the series of the asymptotic expansion of the Bergman function at x*

$$\mathcal{B}(x) = \sum_{j=0}^{\infty} \frac{b_j(x,x)}{k^j}$$

is convergent in C^μ $(\mu \geq 0)$ for k large. Moreover, we have

$$\left\|\mathcal{B}(x) - k^n \sum_{j=0}^{\infty} \frac{b_j(x)}{k^j}\right\|_{C^\mu} \leq k^n e^{-\varepsilon(\log k)^3}$$

for some absolute constant $\varepsilon > 0$.

As our method in the proof of the existence of the expansion is constructive, if we can show that the coefficients on the diagonal satisfy an inequality of the form $b_j \leq C^j$ for some uniform C, we will be able to show that the expansion of the Bergman function converges and by using the $\mathcal{L}^2$-estimate for the $\overline{\partial}$-operator, we will have our exponentially decaying error estimate.

In our formulas, we need to obtain an estimate for the coefficients $c_j^{0,0}$. We use the following recurrence relation

$$\sum_p c_\tau^{p,0} u^p = -\sum_{j=0}^{\tau-1} \sum_{s+q\leq r} \sum_m c_j^{m,q} a_{\tau-j}^{r,s} u^{m+r-q-s} \frac{r!}{(r-q-s)!}.$$

Since we only need the $c_\tau^{0,0}$ portion, we compare the coefficients in front of the constant term as a polynomial in u. We have the condition $m+r-q-s=0$, however we know $m\geq 0$ and $r-q-s\geq 0$ hence $m=0$ and $r=q+s$. The formula then reduces to

$$c_\tau^{0,0} = -\sum_{j=0}^{\tau-1} \sum_{s+q=r} c_j^{0,q} a_{\tau-j}^{r,s}\ r!.$$

Heuristically, the coefficients $c_j^{p,q}$ share the same properties as the $a_m^{r,s}$ which are known and with the added assumption that the metric is real analytic, we can provide some upper bound to relate the series to a geometric series, which will imply the convergence.

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