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First eigenvalue of the p-Laplacian under integral curvature condition

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ABSTRACT

We give various estimates of the first eigenvalue of the p-Laplace operator on closed Riemannian manifold with integral curvature conditions.

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1. Introduction

On a compact Riemannian manifold (M^n, g), for $1 < p < \infty$, the p-Laplacian is defined by

$$\Delta_p(f) := \text{div}(|\nabla f|^{p-2}\nabla f). \tag{1.1}$$

It is a second order quasilinear elliptic operator and when $p = 2$ it is the usual Laplacian. The p-Laplacian has applications in many different contexts from game theory to mechanics and image processing. Corresponding to the p-Laplacian, we have the eigenvalue equation

$$\begin{cases} \Delta_p(f) = -\lambda|f|^{p-2}f & \text{on } M \\ \nabla_\nu f \equiv 0 \text{ (Neumann) or } f \equiv 0 \text{ (Dirichlet)} & \text{on } \partial M \end{cases} \tag{1.2}$$

where ν is the outward normal on ∂M. The first nontrivial Neumann eigenvalue for M is given by

$$\mu_{1,p}(M) = \inf\left\{ \frac{\int_M |\nabla f|^p}{\int_M |f|^p} \mid f \in W^{1,p}(M) \setminus \{0\}, \int_M |f|^{p-2} f = 0 \right\} \tag{1.3}$$

and the first Dirichlet eigenvalue of M is given by

$$\lambda_{1,p}(M) = \inf\left\{ \frac{\int_M |\nabla f|^p}{\int_M |f|^p} \mid f \in W_c^{1,p}(M) \setminus \{0\} \right\}. \tag{1.4}$$

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Though the regularity theory of the p-Laplacian is very different from the usual Laplacian, many of the estimates for the first eigenvalue of the Laplacian (when $p = 2$) can be generalized to general p. Matei [11] generalized Cheng's first Dirichlet eigenvalue comparison of balls [5] to the p-Laplacian. For closed manifolds with Ricci curvature bounded below by $(n - 1)K$, Matei for $K > 0$ [11], Valtora for $K = 0$ [17] and Naber–Valtora for general $K \in \mathbb{R}$ [12] give a sharp lower bound for the first nontrivial eigenvalue. Andrews–Clutterbuck [1,2] also gave a proof using modulus of continuity argument. L.F. Wang [18] considered the case when the Bakry–Emery curvature has a positive lower bound for weighted p-Laplacians. Recently Y.-Z. Wang and H.-Q. Li [19] extended the estimates to smooth metric measure space and Cavalletti–Mondino [4] to general metric measure space. For a general reference on the p-Laplace equation, see [10]. See also [20] and references in the paper for related lower bound estimates.

In this paper, we extend the first eigenvalue estimates for p-Laplacian given in [11] to the integral Ricci curvature setting.

For each $x \in M^n$ let $\rho(x)$ denote the smallest eigenvalue for the Ricci tensor $\mathrm{Ric} : T_xM \to T_xM$, and $\mathrm{Ric}_-^K(x) = ((n-1)K - \rho(x))_+ = \max\{0, (n-1)K - \rho(x)\}$, the amount of Ricci curvature lying below $(n - 1)K$. Let

$$\|\mathrm{Ric}_-^K\|_{q,R} = \sup_{x\in M} \left(\int_{B(x,R)} (\mathrm{Ric}_-^K)^q dvol \right)^{\frac{1}{q}}. \tag{1.5}$$

Then $\|\mathrm{Ric}_-^K\|_{q,R}$ measures the amount of Ricci curvature lying below a given bound, in this case, $(n-1)K$, in the L^q sense. Clearly $\|\mathrm{Ric}_-^K\|_{q,R} = 0$ iff $\mathrm{Ric}_M \geq (n - 1)K$. Denote the limit as $R \to \infty$ by $\|\mathrm{Ric}_-^K\|_q$, which is a global curvature invariant. The Laplace and volume comparison, the basic tools for manifolds with pointwise Ricci curvature lower bound, have been extended to integral Ricci curvature bound [15], see Theorem 2.1.

We denote $\|f\|^*_{q,\Omega}$ the normalized q-norm on the domain Ω. Namely

$$\|f\|^*_{q,\Omega} = \left(\frac{1}{\mathrm{vol}(\Omega)} \int_\Omega |f|^q \right)^{\frac{1}{q}}.$$

Under the assumption that the integral Ricci curvature is controlled ($\|\mathrm{Ric}_-^K\|^*_q$ is small), we give the following first eigenvalue estimates:

Theorem 1.1 (*Cheng-type Estimate*). *Let (M^n, g) be a complete Riemannian manifold. For any $x_0 \in M$, $K \in \mathbb{R}$, $r > 0$, $p > 1$, $q > \frac{n}{2}$, denote $\bar{q} = \max\{q, \frac{p}{2}\}$, there exists an $\varepsilon = \varepsilon(n, p, \bar{q}, K, r)$ such that if $\partial B(x_0, r) \neq \emptyset$ and $\|\mathrm{Ric}_-^K\|^*_{\bar{q},B(x_0,r)} < \varepsilon$, then*

$$\lambda_{1,p}(B(x_0, r)) \leq \bar{\lambda}_{1,p}(B_K(r)) + C(n, p, \bar{q}, K, r) \left(\|\mathrm{Ric}_-^K\|^*_{\bar{q},B(x_0,r)} \right)^{\frac{1}{2}},$$

where $\mathbb{M}^n_K$ is the complete simply connected space of constant curvature K, $B_K(r) \subset \mathbb{M}^n_K$ is the ball of radius r and $\bar{\lambda}_{1,p}$ is the first Dirichlet eigenvalue of the p-Laplacian in the model space $\mathbb{M}^n_K$.

This generalizes the Dirichlet p-Laplacian first eigenvalue comparison in [11]. When $p = 2$, this is proved in [13].

Theorem 1.2 (*Lichnerowicz-Type Estimate*). *Let (M^n, g) be a complete Riemannian manifold. For $q > \frac{n}{2}$, $p \geq 2$ and $K > 0$, there exists $\varepsilon = \varepsilon(n, p, q, K)$ such that if $\|\mathrm{Ric}_-^K\|^*_q \leq \varepsilon$, then*

$$\mu_{1,p}^{\frac{2}{p}} \geq \frac{\sqrt{n}(p-2) + n}{(p-1)(\sqrt{n}(p-2) + n - 1)} \left[(n-1)K - 2\|\mathrm{Ric}_-^K\|^*_q \right]. \tag{1.6}$$

In particular, when Ric $\geq (n-1)K$, *we have*

$$\mu_{1,p}^{\frac{2}{p}} \geq \frac{\sqrt{n}(p-2)+n}{\sqrt{n}(p-2)+n-1} \cdot \frac{(n-1)K}{p-1} \geq \frac{(n-1)K}{p-1}. \tag{1.7}$$

Under these assumptions, Aubry's diameter estimate implies that M is closed [3]. That paper also has the proof for $p = 2$.

The explicit estimate (1.7) improves the estimate in [11, Theorem 3.2], where it is shown that $(\mu_{1,p})^{\frac{2}{p}} \geq \frac{(n-1)K}{p-1}$. When $p = 2$, the estimate (1.7) recovers the Lichnerowicz estimate that $\mu_{1,2} \geq nK$. The explicit estimate (1.6) is optimal when $p = 2$, but not optimal when $p > 2$. For optimal estimate we have the following Lichnerowicz–Obata-type estimate.

Theorem 1.3 (*Lichnerowicz–Obata-Type Estimate*)**.** *Let M^n be a complete Riemannian manifold. Then for any $\alpha > 1$, $K > 0$, $q > \frac{n}{2}$ and any $p > 1$, there is an $\varepsilon = \varepsilon(n, p, q, \alpha, K) > 0$ such that if $\|\mathrm{Ric}_-^K\|_q^* \leq \varepsilon$, then*

$$\alpha\mu_{1,p}(M) \geq \mu_{1,p}(\mathbb{M}_K^n).$$

When $\|\mathrm{Ric}_-^K\|_q^* = 0$, we can take $\alpha = 1$ and this gives Theorem 3.1 in [11].

This result is obtained from the following Faber–Krahn type estimate. Recall the classical Faber–Krahn inequality asserts that in $\mathbb{R}^n$ balls (uniquely) minimize the first eigenvalue of the Dirichlet–Laplacian among sets with given volume.

Theorem 1.4 (*Faber–Krahn-Type Estimate*)**.** *Under the same set up as in Theorem 1.3, let $\Omega \subset M$ be a domain and $B_K \subset \mathbb{M}_K^n$ be a geodesic ball in the model space such that*

$$\frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(M)} = \frac{\mathrm{vol}(B_K)}{\mathrm{vol}(M_K^n)}.$$

Then

$$\alpha^p \lambda_{1,p}(\Omega) \geq \lambda_{1,p}(B_K).$$

Again when $\|\mathrm{Ric}_-^K\|_q^* = 0$, we can take $\alpha = 1$ and this gives Theorem 2.1 in [11].

To prove these results, since we do not have pointwise Ricci curvature lower bound, one key is to control the error terms.

We now give a quick overview of the paper. In Section 2 we prove the Cheng-type upper bound using the first eigenfunction of Δ_p for the model case as a test function in the L^p-Rayleigh quotient and using the Laplacian comparison and volume doubling for integral curvature (Theorem 2.1) to control the error. In Section 3, we prove the Lichnerowicz-type lower bound by using the p-Bochner formula and the Sobolev inequality. In Section 4, to prove a Faber–Krahn-type lower bound, a necessary tool we need is an integral curvature version of the Gromov–Levy isoperimetric inequality, which we first show. The proof of the eigenvalue estimate then follows from an argument using the co-area formula.

2. Proof of Theorem 1.1

First we recall the Laplace and volume comparison for integral Ricci curvature proved by the second author joint with Petersen [14,15].

Let M^n be a complete Riemannian manifold of dimension n. Given $x_0 \in M$, let $r(x) = d(x_0, x)$ be the distance function and $\psi(x) = \left(\Delta r - \bar{\Delta}^K r\right)_+$, where $\bar{\Delta}^K$ is the Laplacian on the model space $\mathbb{M}_K^n$. The classical Laplace comparison states that if $\mathrm{Ric}_M \geq (n-1)K$, then $\Delta r \leq \bar{\Delta}_K r$, i.e., if $\mathrm{Ric}_-^K \equiv 0$, then $\psi \equiv 0$. In [15] this is generalized to integral Ricci lower bound.

Theorem 2.1 (*Laplace and Volume Comparison [14,15]*). *Let M^n be a complete Riemannian manifold of dimension n. If $q > \frac{n}{2}$, then*

$$\|\psi\|^*_{2q.B(x,r)} \leq C(n,q)\left(\|\mathrm{Ric}^K_-\|^*_{q,B(x,r)}\right)^{\frac{1}{2}}. \tag{2.1}$$

There exists $\varepsilon = \varepsilon(n,q,K,r) > 0$ such that, if $\|\mathrm{Ric}^K_-\|^*_{q,B(x,r)} \leq \varepsilon$, then

$$\frac{\mathrm{vol}(B(x,r))}{\mathrm{vol}(B(x,r_0))} \leq 2\frac{\mathrm{vol}B_K(r)}{\mathrm{vol}B_K(r_0)}, \quad \forall r_0 \leq r. \tag{2.2}$$

For p-Laplacian of radial function, we have the following comparison.

Proposition 2.1 (*p-Laplace Comparison*). *If f is a radial function such that $f' \leq 0$, then*

$$\Delta_p f \geq \bar{\Delta}^K_p f + f'|f'|^{p-2}\psi. \tag{2.3}$$

Proof. From the definition of the p-Laplacian (1.1),

$$\begin{aligned}\Delta_p f = \mathrm{div}(|\nabla f|^{p-2}\nabla f) &= \langle \nabla|\nabla f|^{p-2}, \nabla f\rangle + |\nabla f|^{p-2}\Delta f \\ &= (p-2)|\nabla f|^{p-4}\,\mathrm{Hess}\, f(\nabla f, \nabla f) + |\nabla f|^{p-2}\Delta f.\end{aligned} \tag{2.4}$$

Hence when $f = f(r)$ is a radial function

$$\begin{aligned}\Delta_p f &= (p-2)|f'|^{p-2}f'' + |f'|^{p-2}(f'' + \Delta r f') \\ &= (p-1)|f'|^{p-2}f'' + \Delta r f'|f'|^{p-2} \\ &= (p-1)|f'|^{p-2}f'' + \bar{\Delta}^K r f'|f'|^{p-2} + \left(\Delta r - \bar{\Delta}^K r\right) f'|f'|^{p-2} \\ &= \bar{\Delta}^K_p r + \left(\Delta r - \bar{\Delta}^K r\right) f'|f'|^{p-2}.\end{aligned} \tag{2.5}$$

When $f' \leq 0$, $\left(\Delta r - \bar{\Delta}^K r\right) f'|f'|^{p-2} \geq \psi f'|f'|^{p-2}$, which gives the estimate. □

Let $\bar{f} > 0$ be the first eigenfunction for the Dirichlet problem for Δ_p in $B_K(r) \subset \mathbb{M}^n_K$. By [7] $\bar{f}$ is radial. Below we show that $\bar{f}$ is a decreasing function of the radius. For $p \geq 2$, this was shown in [11]. Our proof is much shorter.

Lemma 2.1. *For $t \in (0,r)$ and $p > 1$, $\bar{f}'(t) \leq 0$.*

Proof. Write the volume element of $\mathbb{M}^n_K$ in geodesic polar coordinate $dvol = \mathcal{A}(t)dtd\theta_{S^{n-1}}$. As the first eigenfunction $\bar{f}$ is radial, by (2.5) it satisfies the ODE

$$(\mathcal{A}|f'|^{p-2}f')' = -\lambda_1\mathcal{A}|f|^{p-2}f. \tag{2.6}$$

As $\mathcal{A}(0) = 0$ and $p > 1$, integrating both sides from 0 to t we get

$$\mathcal{A}|f'|^{p-2}f'(t) = -\lambda_1\int_0^t |f|^{p-2}f\mathcal{A} \leq 0. \quad \square$$

Now we are ready to prove Theorem 1.1.

Proof. Let $\bar{f}$ be a first eigenfunction for the Dirichlet problem for Δ_p in $B_K(r) \subset \mathbb{M}^n_K$ with $\bar{f}(0) = 1$. Hence $0 \leq \bar{f} \leq 1$. Let $r = r(x) = d(x_0, x)$ be the distance function on M centered at the point x_0. Then $\bar{f}(r) \in W^{1,p}_0(B(x_0,r))$. Denote $Q = \frac{\int_B |\nabla\bar{f}|^p}{\int_B |\bar{f}|^p}$, where $B := B(x_0,r)$. By (1.4) we have,

$$\lambda_{1,p}(B(x_0,r)) \leq Q. \tag{2.7}$$

Using integration by part, $\bar{f}' \leq 0$, $0 \leq \bar{f} \leq 1$ and the p-Laplacian comparison (2.3), we have

$$\begin{aligned} Q &= -\frac{\int_B \Delta_p \bar{f} \cdot \bar{f}}{\int_B |\bar{f}|^p} \\ &\leq -\frac{\int_B \bar{\Delta}_p^K \bar{f} \cdot \bar{f} + \psi \bar{f}' |\bar{f}'|^{p-2} \bar{f}}{\int_B |\bar{f}|^p} \\ &= \bar{\lambda}_{1,p}(B_K(r)) - \frac{\int_B \psi \bar{f}' |\bar{f}'|^{p-2} \bar{f}}{\int_B |\bar{f}|^p} \\ &\leq \bar{\lambda}_{1,p}(B_K(r)) + \frac{\int_B \psi |\bar{f}'|^{p-1}}{\int_B |\bar{f}|^p}. \end{aligned}$$

By Hölder inequality

$$\int_B \psi |\bar{f}'|^{p-1} \leq \left(\int_B \psi^p \right)^{\frac{1}{p}} \left(\int_B |\bar{f}'|^p \right)^{1-\frac{1}{p}}.$$

Let $r_0 = r_0(n, K, r) \in (0, r)$ such that $\bar{f}(r_0) = \frac{1}{2}$. Then $\bar{f} \geq \frac{1}{2}$ on $[0, r_0]$, and

$$\int_B |\bar{f}|^p \geq \left(\int_B |\bar{f}|^p \right)^{1-\frac{1}{p}} \cdot \left(\int_{B(x_0,r_0)} |\bar{f}|^p \right)^{\frac{1}{p}} \geq \left(\int_B |\bar{f}|^p \right)^{1-\frac{1}{p}} \cdot \left(\mathrm{vol} B(x_0, r_0) 2^{-p} \right)^{\frac{1}{p}}.$$

Hence the error term

$$\begin{aligned} \frac{\int_B \psi |\bar{f}'|^{p-1}}{\int_B |\bar{f}|^p} &\leq 2 \left(\frac{\int_B \psi^p}{\mathrm{vol} B(x_0, r_0)} \right)^{\frac{1}{p}} \cdot \left(\frac{\int_B |\bar{f}'|^p}{\int_B |\bar{f}|^p} \right)^{1-\frac{1}{p}} \\ &= 2\, Q^{1-\frac{1}{p}} \left(\fint_B \psi^p \right)^{\frac{1}{p}} \left(\frac{\mathrm{vol} B(x_0, r)}{\mathrm{vol} B(x_0, r_0)} \right)^{\frac{1}{p}} \\ &\leq 2\, Q^{1-\frac{1}{p}} \|\psi\|^*_{2\bar{q}, B(x_0,r)} \left(\frac{\mathrm{vol} B(x_0, r)}{\mathrm{vol} B(x_0, r_0)} \right)^{\frac{1}{p}}. \end{aligned}$$

Choose $\varepsilon \leq \varepsilon_0$ in Theorem 2.1, using (2.1) and (2.2), and combining above, we have

$$Q \leq \bar{\lambda}_{1,p}(B_K(r)) + C(n, p, \bar{q}, K, r) \left(\|\mathrm{Ric}_-^K\|^*_{\bar{q}, B(x_0,r)} \right)^{\frac{1}{2}} Q^{1-\frac{1}{p}}. \tag{2.8}$$

Applying Young's inequality to the last term, we have

$$Q \leq \bar{\lambda}_{1,p}(B_K(r)) + \frac{1}{p} C(n, p, \bar{q}, K, r) \left(\|\mathrm{Ric}_-^K\|^*_{\bar{q}, B(x_0,r)} \right)^{\frac{p}{2}} + \frac{p-1}{p} Q.$$

Moving Q to the left hand side, we obtain

$$Q \leq p \bar{\lambda}_{1,p}(B_K(r)) + C(n, p, \bar{q}, K, r) \left(\|\mathrm{Ric}_-^K\|^*_{\bar{q}, B(x_0,r)} \right)^{\frac{p}{2}}.$$

Applying this to (2.8) so that the $Q^{1-\frac{1}{p}}$ can be bounded in terms of the fixed quantities, we obtain

$$Q \leq \bar{\lambda}_{1,p}(B_K(r)) + C(n, p, \bar{q}, K, r) \left(\|\mathrm{Ric}_-^K\|^*_{\bar{q}, B(x_0,r)} \right)^{\frac{1}{2}}. \quad \square$$

3. Proof of Theorem 1.2

To prove Theorem 1.2, we need the following Bochner formula for p power.

Lemma 3.1 (*p-Bochner*)**.**

$$\begin{aligned}&\frac{1}{p}\ \Delta(|\nabla f|^p)\\&= (p-2)|\nabla f|^{p-2}|\nabla|\nabla f||^2 + \frac{1}{2}|\nabla f|^{p-2}\left\{|\mathrm{Hess}\, f|^2 + \langle \nabla f, \nabla \Delta f\rangle + \mathrm{Ric}(\nabla f, \nabla f)\right\}.\end{aligned} \tag{3.1}$$

One can find this implicitly in the literature, see e.g. [6,12,16]. The proof is very simple, for completeness, we present it here.

Proof. One computes

$$\frac{1}{p}\Delta(|\nabla f|^p) = \frac{1}{p}\Delta(|\nabla f|^2)^{\frac{p}{2}} = (p-2)|\nabla f|^{p-2}|\nabla|\nabla f||^2 + \frac{1}{2}|\nabla f|^{p-2}\Delta(|\nabla f|^2). \tag{3.2}$$

Recall the Bochner formula

$$\frac{1}{2}\Delta(|\nabla f|^2) = |\mathrm{Hess}\, f|^2 + \langle \nabla f, \nabla \Delta f\rangle + \mathrm{Ric}(\nabla f, \nabla f).$$

Plugging this into (3.2) gives (3.1). $\square$

We also need the following Sobolev inequality, which follows from Gallot's isoperimetric constant estimate for integral curvature [8] and Aubry's diameter estimate [3].

Proposition 3.1. *Given $q > \frac{n}{2}$ and $K > 0$, there exists $\varepsilon = \varepsilon(n, q, K)$ such that if M^n is a complete manifold with $\|\mathrm{Ric}_-^K\|_q^* \le \varepsilon$, then there is a constant $C_s(n, q, K)$ such that*

$$\left(\fint_M f^{\frac{2q}{q-1}}\right)^{\frac{q-1}{q}} \le C_s(n, q, K)\fint_M |\nabla f|^2 + 2\fint_M f^2 \tag{3.3}$$

for all functions $f \in W^{1,2}$.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. When $p = 2$ the result is proved in [3]. In the rest we assume $p > 2$.

By Aubry's diameter estimate [3], M is closed. Integrating (3.1) on M we have

$$0 = \fint_M |\nabla f|^{p-2}\left\{(p-2)|\nabla|\nabla f||^2 + |\mathrm{Hess}\, f|^2 + \langle \nabla f, \nabla \Delta f\rangle + \mathrm{Ric}(\nabla f, \nabla f)\right\}. \tag{3.4}$$

For the Hessian term, using the Cauchy–Schwarz inequalities

$$|\mathrm{Hess}(\nabla f, \nabla f)|^2 \le |\nabla f|^4 |\mathrm{Hess}\, f|^2,$$

$$|\Delta f|^2 \le n|\mathrm{Hess}\, f|^2$$

and the formula for p-Laplacian (2.4), we have

$$\begin{aligned}\fint_M |\nabla f|^{p-2}|\operatorname{Hess} f|^2 &\geq \fint_M |\nabla f|^{p-2}\frac{(\Delta f)^2}{n}\\ &= \frac{1}{n}\fint_M \Delta f \Delta_p f - \frac{p-2}{n}\fint_M \Delta f|\nabla f|^{p-4}\operatorname{Hess} f(\nabla f,\nabla f)\\ &\geq \frac{1}{n}\fint_M \Delta f \Delta_p f - \frac{p-2}{n}\fint_M |\Delta f||\nabla f|^{p-2}|\operatorname{Hess} f|\\ &\geq \frac{1}{n}\fint_M \Delta f \Delta_p f - \frac{p-2}{\sqrt{n}}\fint_M |\nabla f|^{p-2}|\operatorname{Hess} f|^2.\end{aligned}$$

Hence

$$\fint_M |\nabla f|^{p-2}|\operatorname{Hess} f|^2 \geq \frac{1}{n+\sqrt{n}(p-2)}\fint \Delta f \Delta_p f. \tag{3.5}$$

For the third term,

$$\fint_M |\nabla f|^{p-2}\langle \nabla f, \nabla(\Delta f)\rangle = -\fint_M \Delta_p f \Delta f.$$

For the curvature term,

$$\begin{aligned}\fint_M |\nabla f|^{p-2}\operatorname{Ric}(\nabla f,\nabla f) &\geq (n-1)K\fint_M |\nabla f|^p - \fint_M |\operatorname{Ric}_-^K|\,|\nabla f|^p\\ &\geq (n-1)K\fint_M |\nabla f|^p - \|\operatorname{Ric}_-^K\|_q^*\left(\fint_M |\nabla f|^{\frac{pq}{q-1}}\right)^{\frac{q-1}{q}}.\end{aligned}$$

Applying the Sobolev inequality (3.3) to the function $|\nabla f|^{\frac{p}{2}}$ gives

$$\begin{aligned}\left(\fint_M \left(|\nabla f|^{\frac{p}{2}}\right)^{\frac{2q}{q-1}}\right)^{\frac{q-1}{q}} &\leq C_s\fint_M |\nabla|\nabla f|^{\frac{p}{2}}|^2 + 2\fint_M |\nabla f|^p\\ &= C_s\frac{p^2}{4}\fint_M |\nabla f|^{p-2}|\nabla|\nabla f||^2 + 2\fint_M |\nabla f|^p.\end{aligned}$$

Plugging these into (3.4), we have

$$\begin{aligned}0 \geq &-\frac{n-1+\sqrt{n}(p-2)}{n+\sqrt{n}(p-2)}\fint_M \Delta_p f\Delta f + ((n-1)K - 2\|\operatorname{Ric}_-^K\|_q^*)\fint_M |\nabla f|^p\\ &+\left((p-2) - C_s\|\operatorname{Ric}_-^K\|_q^*\frac{p^2}{4}\right)\fint_M |\nabla f|^{p-2}|\nabla|\nabla f||^2.\end{aligned}$$

Choosing $\|\operatorname{Ric}_-^K\|_q^*$ small so that $\left((p-2) - C_s\|\operatorname{Ric}_-^K\|_q^*\frac{p^2}{4}\right) \geq 0$. Then we can throw the last term away and get

$$0 \geq -\frac{n-1+\sqrt{n}(p-2)}{n+\sqrt{n}(p-2)}\fint_M \Delta_p f\Delta f + ((n-1)K - 2\|\operatorname{Ric}_-^K\|_q^*)\fint_M |\nabla f|^p. \tag{3.6}$$

Let f be the first eigenfunction for Δ_p, that is, $\Delta_p f = -\mu|f|^{p-2}f$. Then

$$\begin{aligned}\fint_M \Delta_p f \Delta f &= -\mu \fint_M |f|^{p-2} f \Delta f \\ &= \mu \fint_M \langle \nabla(|f|^{p-2}f), \nabla f\rangle \\ &= \mu(p-1) \fint_M |f|^{p-2}|\nabla f|^2 \\ &\le (p-1)\mu \left(\fint_M |f|^p\right)^{1-\frac{2}{p}} \left(\fint_M |\nabla f|^p\right)^{\frac{2}{p}} \\ &= (p-1)(\mu)^{\frac{2}{p}} \fint_M |\nabla f|^p,\end{aligned}$$

where we use the fact that f is the first eigenfunction, so we have

$$\fint_M |f|^p = \frac{1}{\mu}\fint_M |\nabla f|^p.$$

This gives

$$(\mu)^{\frac{2}{p}} \left[(p-1)\frac{n-1+\sqrt{n}(p-2)}{n+\sqrt{n}(p-2)}\right] \ge (n-1)K - 2\|\mathrm{Ric}_-^K\|_q^*,$$

which is (1.6). □

4. Proof of Theorem 1.4

To prove the Faber–Krahn-type estimate, we will need a version of the Gromov–Levy isoperimetric inequality for integral curvature. The inequality follows from the following volume comparison for tubular neighborhood of hypersurface of Petersen–Sprouse.

Proposition 4.1 (*[13], Lemma 4.1*). *Suppose that $H \subset M$ is a hypersurface with constant mean curvature $\eta \ge 0$, and that H divides M into two domains $\Omega_\pm$, where Ω_+ is the domain in which mean curvature is positive. Furthermore, let $d_\pm > 0$ such that $d_+ + d_- \le \mathrm{diam}(M) \le D$ and $\Omega_\pm \subset B(H, d_\pm)$. Let $\bar{H} = S(x_0, r_0) \subset \mathbb{M}_K^n$, a sphere of constant positive mean curvature η, and let $\bar{\Omega}_+ = B(x_0, D) - B(x_0, r_0)$, $\bar{\Omega}_- = B(x_0, r_0)$. Finally assume that $d_+ \le D - r_0$ and $d_- \le r_0$. Then for any $\alpha > 1$, there is an $\varepsilon(n, p, \alpha, K) > 0$ such that if $\|\mathrm{Ric}_-^K\|_q^* \le \varepsilon$, then*

$$\mathrm{vol}(\Omega_\pm) \le \alpha \frac{\mathrm{area}(H)}{\mathrm{area}(\bar{H})} \mathrm{vol}(\bar{\Omega}_\pm).$$

Using this, the Gromov–Levy isoperimetric inequality for the integral curvature case can be shown by following the original proof given in [9] page 522 and keeping track of the error term coming from the integral curvature.

Proposition 4.2. *Let $\Omega \subset M$ be a domain. Then for any $\alpha > 1$, there is an $\varepsilon = \varepsilon(n, p, \alpha, K) > 0$ such that if $\|\mathrm{Ric}_-^K\|_q^* \le \varepsilon$, then*

$$\mathrm{area}(\partial B_K(r_0)) \le \alpha\, \mathrm{area}(\partial\Omega) \frac{\mathrm{vol}(B_K(r_0))}{\mathrm{vol}(\Omega)},$$

where $B_K(r_0) \subset \mathbb{M}_K^n$ is the ball of radius r_0 in constant curvature K space. When $\|\mathrm{Ric}_-^K\|_q^* = 0$, we can take $\alpha = 1$.

Now we prove Theorem 1.4, the Faber–Krahn inequality.

Proof. Without loss of generality, we can suppose that our test functions are Morse functions to ensure that the level sets of f are closed regular hypersurfaces for almost all values. Let $\Omega_t := \{x \in \Omega \mid f > t\}$ and consider the decreasing rearrangement of f defined by

$$\bar{f}(s) = \inf\{t \geq 0 \mid |\Omega_t| < s\}.$$

It is a non-increasing function on $[0, |\Omega|]$. We define the spherical rearrangement $\bar{\Omega}$ of Ω as the ball in $\mathbb{M}^n_K$ centered at some fixed point such that $\beta|\bar{\Omega}| = |\Omega|$, where $\beta := \frac{\mathrm{vol}(M)}{\mathrm{vol}(\mathbb{M}^n_K)}$. By abuse of notation, we define the spherical decreasing rearrangement $\bar{f} : \bar{\Omega} \to \mathbb{R}$ to be

$$\bar{f}(x) = \bar{f}(C_n|x|^n)$$

for $x \in \bar{\Omega}$, where $|x|$ is the distance from the center of $\bar{\Omega}$ and C_n is the volume S^n_K. Note that

$$\mathrm{vol}(\{f > t\}) = \mathrm{vol}(\{\bar{f} > t\}). \tag{4.1}$$

Now by construction, we have

$$\int_\Omega f^p = \int_0^{|\Omega|} (\bar{f}(s))^p ds = \beta \int_{\bar{\Omega}} (\bar{f})^p.$$

Next we want to compare the L^p norm of ∇f and $\nabla \bar{f}$. Now $\partial\Omega_t = \{x \in \Omega \mid f = t\}$ and since $\bar{f}$ is a radial function, we have

$$|\nabla \bar{f}| = \left|\frac{\partial \bar{f}}{\partial r}\right|$$

which is a constant on $\partial\bar{\Omega}_t$. By Hölder's inequality, we have

$$\begin{aligned}
\mathrm{vol}(\{f = t\}) &= \int_{\{f=t\}} 1 \\
&= \int_{\{f=t\}} \frac{|\nabla f|^{\frac{p-1}{p}}}{|\nabla f|^{\frac{p-1}{p}}} \\
&\leq \left(\int_{\{f=t\}} \frac{1}{|\nabla f|}\right)^{\frac{p-1}{p}} \left(\int_{\{f=t\}} |\nabla f|^{p-1}\right)^{\frac{1}{p}}.
\end{aligned}$$

By Proposition 4.2

$$\alpha\, \mathrm{vol}(\{f = t\}) \geq \mathrm{vol}(\{\bar{f} = t\})$$

for some $\alpha > 1$. We have

$$\begin{aligned}
\mathrm{vol}(\{\bar{f} = t\}) &= \int_{\{\bar{f}=t\}} 1 \\
&= \left(\int_{\{\bar{f}=t\}} \frac{1}{|\nabla \bar{f}|}\right)^{\frac{p-1}{p}} \left(\int_{\{\bar{f}=t\}} |\nabla \bar{f}|^{p-1}\right)^{\frac{1}{p}}.
\end{aligned}$$

By the co-area formula, we have

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{vol}(\{f>t\}) &= \frac{\partial}{\partial t}\int_{\{f>t\}} 1 \\ &= \frac{\partial}{\partial t}\int_t^\infty \left(\int_{\{f=c\}} \frac{1}{|\nabla f|}\right) dc \\ &= -\int_{\{f=t\}} \frac{1}{|\nabla f|}.\end{aligned}$$

and similarly for $\bar{f}$ with (4.1) so that

$$\int_{\{f=t\}} \frac{1}{|\nabla f|} = \int_{\{\bar{f}=t\}} \frac{1}{|\nabla \bar{f}|}.$$

Combining and applying the co-area formula once more to integrate over Ω, we obtain

$$\alpha^p \int_\Omega |\nabla f|^p \geq \int_{\bar{\Omega}} |\nabla \bar{f}|^p$$

and by the Rayleigh quotient, we have

$$\frac{\int_\Omega |\nabla f|^p}{\int_\Omega |f|^p} \geq \frac{1}{\alpha^p}\frac{\int_{\bar{\Omega}} |\nabla \bar{f}|^p}{\int_{\bar{\Omega}} |\bar{f}|^p} \geq \frac{1}{\alpha^p}\lambda_{1,p}(\bar{\Omega}). \quad \square$$

To get Theorem 1.3 from Theorem 1.4, one follows the argument given in [11]. One first shows the relation between the first non-trivial Neumann eigenvalue and the first Dirichlet eigenvalue of its nodal domain. Namely, let f be a first nontrivial Neumann eigenfunction of Δ_p on M with $p > 1$, let $A_+ = f^{-1}(\mathbb{R}_+)$ and $A_- = f^{-1}(\mathbb{R}_-)$ be the nodal domains of f. Then

$$\mu_{1,p}(M) = \lambda_{1,p}(A_+) = \lambda_{1,p}(A_-).$$

Using the fact that the nodal domains of Δ_p for the first nontrival Neumann eigenfunction on spheres M^n_K are hemispheres $S^n_{K,\pm}$, in particular we have

$$\mu_{1,p}(M^n_K) = \lambda_{1,p}(S^n_{K,\pm}).$$

Applying the Faber–Krahn-type estimate (Theorem 1.4) to the nodal domain, we get Theorem 1.3.

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