# FIRST EIGENVALUE OF THE LAPLACIAN ON CLOSED RIEMANNIAN MANIFOLDS

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ABSTRACT. These are the notes for my talk given at the Kansuron Summer Seminar 2017 held at Kyushu, Japan on September 6-8, 2017.

## 1. Brief History

Let $(M, g)$ be a closed Riemannian manifold. We can define the **Laplace-Beltrami operator** or the **Laplacian** locally in some coordinate $(x^i)$ by the formula

$$\Delta_g := \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left( g^{ij} \sqrt{\det g} \frac{\partial}{\partial x^j} \right).$$

Since $\partial M = \emptyset$, we consider the **closed eigenvalue problem**: Find all real numbers $\mu$ for which there exists a nontrivial solution $f \in C^2(M)$.

$$\Delta f + \mu f = 0.$$

By the divergence theorem, there is an immediate necessary condition for such a $f$, namely,

$$0 = \int_M \Delta f = -\mu \int_M f.$$

So if $\mu \neq 0$, we must have that $\int_M f = 0$. We can also get that the eigenvalues are non-negative as the following shows,

$$\mu \int_M f^2 = - \int_M f \Delta f = \int_M |\nabla f|^2 \geq 0.$$

On closed manifolds, we can take $f$ to be some constant so that $\mu_0 = 0$. It can also be shown that the eigenvalues of the Laplacian are nondecreasing and discrete, so we order them as such,

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots$$

There is also a variational characterization of the eigenvalues, called the **Min-Max Principle**:

$$\mu_1 = \inf \left\{ \frac{\int_M |\nabla f|^2}{\int_M f^2} \mid f \in H_1^2(M), \int f = 0 \right\}$$

$$\mu_k = \inf \left\{ \frac{\int_M |\nabla f|^2}{\int_M f^2} \mid f \in H_1^2(M), \int f f_i = 0, i = 1, \dots k-1, f_i \text{ are eigenfunctions of } \mu_i \right\}$$

Here $H_2^1$ denotes the completion of $C^\infty(M)$ with respect to the (Sobolev) norm

$$\|f\|_{1,2} := \int_M |f|^2 + |\nabla f|^2.$$

We will focus mainly on the first eigenvalue $\mu_1$. We now present a sharp estimate on the first nonzero eigenvalue of the Laplacian.

**Theorem 1.1** (Lichnerowicz-Obata [2], [5])**.** Let $M$ be an $n$-dimensional complete Riemannian manifold with Ricci curvature $\mathrm{Ric} \geq (n-1)K$, $K > 0$. Then the first nonzero eigenvalue satisfies $\mu_1 \geq nK = \mu_1(S^n_K)$. Furthermore, if $\mu_1 = nK$, then $M$ is isometric to the constant curvature space $S^n_K$.

We present a generalization of this result in two aspects: Integral Ricci curvature and the $p$-Laplacian.

1.1. **$p$-Laplacian.** The eigenvalue problem can be thought of as the minimizer of the following Rayleigh quotient (Dirichlet energy) functional

$$\mathcal{F}_2(f) = \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

We can generalize this to the $L^p$ norm by

$$\mathcal{F}_p(f) = \frac{\int_M |\nabla f|^p}{\int_M |f|^p}.$$

Computing the variation for $\mathcal{F}_p$ leads to the following eigenvalue equation for the **$p$-Laplacian**,

$$\Delta_p(f) := \mathrm{div}(|\nabla f|^{p-2} \nabla f) = -\mu |f|^{p-2} f.$$

The $p$-Laplace operator $\Delta_p$ is a second order quasilinear elliptic operator and when $p = 2$ it is the usual Laplacian. By direct computation, the relation between the $p$-Laplacian and the Laplacian is given by

$$\Delta_p f = (p-2)|\nabla f|^{p-4} \operatorname{Hess} f(\nabla f, \nabla f) + |\nabla f|^{p-2} \Delta f. \tag{1.1}$$

While the regularity theory of the $p$-Laplacian is very different for $p \neq 2$ (c.f. [3]), many of the estimates, including the Lichnerowicz-Obata theorem, can be generalized for general $p$.

1.2. **Integral Ricci curvature.** For each $x \in M^n$, denote the smallest eigenvalue for the Ricci tensor $\mathrm{Ric} : T_xM \to T_xM$, and define $\mathrm{Ric}^K_- := ((n-1)K - \rho(x))_+ := \max\{0, (n-1)K - \rho(x)\}$. Let

$$\| \mathrm{Ric}^K_- \|_{q,R} = \sup_{x \in M} \left( \int_{B(x,R)} (\mathrm{Ric}^K_-)^q dV \right)^{\frac{1}{q}}.$$

The quantity $\| \mathrm{Ric}^K_- \|_{q,R}$ measure the amount of Ricci curvature lying below a given bound, this this case, $(n-1)K$, in the $L^q$ sense. Write the limit as $R \to \infty$ by $\| \mathrm{Ric}^K_- \|_q$. This generalizes the pointwise Ricci lower bound since $\| \mathrm{Ric}^K_- \|_{q,R} = 0$ if and only if $\mathrm{Ric} \geq (n-1)K$. A convenient quantity we will work with is the normalized $L^q$ norm, namely

$$\|f\|^*_{q,\Omega} = \left( \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega |f|^q dV \right)^{\frac{1}{q}}.$$

Many results with the assumption of pointwise Ricci curvature lower bound, for instance Laplace and volume comparison, have been extended to the integral Ricci curvature setting [6].

## 2. Eigenvalue lower bound

**Definition 2.1** ($p$-Laplacian)**.** Define the $p$-Laplacian, $\Delta_p$ and the corresponding eigenvalue problem

$$\Delta_p(f) := \mathrm{div}(|\nabla f|^{p-2} \nabla f) = -\mu |f|^{p-2} f.$$

Let $\mu_{1,p}$ be the first non-zero eigenvalue.

In [7], we extend the eigenvalue results of Aubry [1], Matei [4], to the setting of *integral Ricci curvature*:

**Theorem 2.1** (Lichnerowicz-type estimate (Theorem 1.2 [7])). Let $(M^n, g)$ be a complete Riemannian manifold. For $q > \frac{n}{2}$, $p \geq 2$ and $K > 0$, there exists $\varepsilon = \varepsilon(n, p, q, K)$ such that if $\| \operatorname{Ric}_-^K \|_q^* \leq \varepsilon$, then

$$\mu_{1,p}^{\frac{2}{p}} \geq \frac{\sqrt{n}(p-2)+n}{(p-1)(\sqrt{n}(p-2)+n-1)} \left[(n-1)K - 2\| \operatorname{Ric}_-^K \|_q^*\right].$$

In particular, when $\operatorname{Ric} \geq (n-1)K$, we have

$$\mu_{1,p}^{\frac{2}{p}} \geq \frac{\sqrt{n}(p-2)+n}{\sqrt{n}(p-2)+n-1} \cdot \frac{(n-1)K}{p-1} \geq \frac{(n-1)K}{p-1}.$$

When $p = 2$, the estimate recovers the Lichnerowicz estimate. For an optimal estimate, we have

**Theorem 2.2** (Lichnerowicz-Obata-type estimate (Theorem 1.3 [7])). Let $M^n$ be a complete Riemannian manifold. Then for any $\alpha > 1$, $K > 0$, $q > \frac{n}{2}$, and any $p > 1$, there is an $\varepsilon = \varepsilon(n, p, q, \alpha, K) > 0$ such that if $\| \operatorname{Ric}_-^K \|_q^* \leq \varepsilon$, then

$$\alpha \mu_{1,p}(M) \geq \mu_{1,p}(S_K^n).$$

*Proof of Theorem 2.1 lower bound.* We will first need to extend the Bochner formula to the $p$-Laplacian.

**Lemma 2.1** ($p$-Bochner formula).

$$\begin{aligned}(2.1)\qquad &\frac{1}{p}\Delta(|\nabla f|^p) \\ &= (p-2)|\nabla f|^{p-2}|\nabla|\nabla f||^2 + \frac{1}{2}|\nabla f|^{p-2}\left[|\operatorname{Hess} f|^2 + \langle \nabla f, \nabla \Delta f\rangle + \operatorname{Ric}(\nabla f, \nabla f)\right].\end{aligned}$$

We also will need a Sobolev inequality in the integral Ricci curvature setting:

**Proposition 2.1.** Given $q > \frac{n}{2}$ and $K > 0$, there exists $\varepsilon = \varepsilon(n, q, K)$ such that if $M^n$ is a complete manifold with $\| \operatorname{Ric}_-^K \|_q^* \leq \varepsilon$, then there is a constant $C_s(n, q, K)$ such that

$$\left(\fint_M f^{\frac{2q}{q-1}}\right)^{\frac{q-1}{q}} \leq C_s(n,q,K)\fint_M |\nabla f|^2 + 2\fint_M f^2.$$

for all functions $f \in W^{1,2}$.

Now we proceed to prove Theorem 2.1. Integrating (2.1), we have

$$0 = \fint_M |\nabla f|^{p-2}\left[(p-2)|\nabla|\nabla f||^2 + |\operatorname{Hess} f|^2 + \langle \nabla f, \nabla \Delta f\rangle + \operatorname{Ric}(\nabla f, \nabla f)\right]$$

The key is to show appropriate estimates for each term. By the Cauchy-Schwarz inequalty,

$$|\operatorname{Hess}(\nabla f, \nabla f)|^2 \leq |\nabla f|^4 |\operatorname{Hess} f|^2$$
$$|\Delta f|^2 \leq n|\operatorname{Hess} f|^2.$$

Combining with (1.1), we can compute that

$$\fint_M |\nabla f|^{p-2}|\operatorname{Hess} f|^2 \geq \frac{1}{n}\fint_M \Delta f \Delta_p f - \frac{p-2}{\sqrt{n}}\fint_M |\nabla f|^{p-2}|\operatorname{Hess} f|^2.$$

so that

$$\fint_M |\nabla f|^{p-2}|\operatorname{Hess} f|^2 \geq \frac{1}{n+\sqrt{n}(p-2)}\fint_M \Delta f \Delta_p f.$$

Integrating by parts,

$$\fint_M |\nabla f|^{p-2}\langle \nabla f, \nabla(\Delta f)\rangle = -\fint_M \Delta_p f \Delta f.$$

The Ricci curvature term can be bounded by

$$\fint_M |\nabla f|^{p-2} \operatorname{Ric}(\nabla f, \nabla f) \geq (n-1)K \fint_M |\nabla f|^p - \| \operatorname{Ric}_-^K \|_q^* \left( |\nabla f|^{\frac{pq}{q-1}} \right)^{\frac{q-1}{q}}.$$

Applying the Sobolev inequality,

$$\left( \fint_M \left( |\nabla f|^{\frac{p}{2}} \right)^{\frac{2q}{q-1}} \right)^{\frac{q-1}{q}} \leq C_s \frac{p^2}{4} \fint_M |\nabla f|^{p-2} |\nabla |\nabla f||^2 + 2 \fint_M |\nabla f|^p.$$

Combining these inequalities, we get

$$\begin{aligned} 0 \geq &-\frac{n-1+\sqrt{n}(p-2)}{n+\sqrt{n}(p-2)} \fint_M \Delta_p f \Delta f + ((n-1)K - 2\| \operatorname{Ric}_-^K \|_q^*) \fint_M |\nabla f|^p \\ &+ \left( (p-2) - C_s \| \operatorname{Ric}_-^K \|_q^* \frac{p^2}{4} \right) \fint_M |\nabla f|^{p-2} |\nabla |\nabla f||^2. \end{aligned}$$

Choose $\| \operatorname{Ric}_-^K \|_q^*$ small so that $\left( (p-2) - C_s \| \operatorname{Ric}_-^K \|_q^* \frac{p^2}{4} \right) \geq 0$, we can throw the last term away (here we possibly lose the sharpness of the estimate), we get

$$0 \geq -\frac{n-1+\sqrt{n}(p-2)}{n+\sqrt{n}(p-2)} \fint_M \Delta_p f \Delta f + ((n-1)K - 2\| \operatorname{Ric}_-^K \|_q^*) \fint_M |\nabla f|^p.$$

Let $f$ be the first eigenfunction for $\Delta_p$, i.e., $\Delta_p f = -\mu |f|^{p-2} f$. Then integrating by parts, we have

$$\begin{aligned} \fint_M \Delta_p f \Delta f &= \mu(p-1) \fint_M |f|^{p-2} |\nabla f|^2 \\ &\leq \mu(p-1) \left( \fint_M |f|^p \right)^{1-\frac{2}{p}} \left( \fint_M |\nabla f|^p \right)^{\frac{2}{p}} \\ &= (\mu)^{\frac{2}{p}} (p-1) \fint_M |\nabla f|^p, \end{aligned}$$

where we used the fact that $f$ is the first eigenfunction so that

$$\fint_M |f|^p = \frac{1}{\mu} \fint_M |\nabla f|^p.$$

Combining these, we obtain

$$(\mu)^{\frac{2}{p}} \left[ (p-1) \frac{n-1+\sqrt{n}(p-2)}{n+\sqrt{n}(p-2)} \right] \geq (n-1)K - 2\| \operatorname{Ric}_-^K \|_q^*.$$

□

For the optimal lower bound, the eigenvalue of the $p$-Laplacian for the sphere is not explicitly known. Hence, we will first need to show a Faber-Krahn type estimate for the $p$-Laplacian. The estimate gives a lower bound of the first eigenvalue of a domain with Dirichlet boundary conditions comparing to the first eigenvalue of a ball in constant curvature space. Applying the estimate to each nodal domain and using the fact that the nodal domain of the first Dirichlet eigenvalue on a sphere is a hemisphere, we can show that the lower bound is achieved by a sphere. See [7] for details.

## References

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