THE FIRST NONZERO EIGENVALUE OF THE p-LAPLACIAN ON DIFFERENTIAL FORMS

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ABSTRACT. We introduce a generalization of the p-Laplace operator to act on differential forms and generalize an estimate of Gallot-Meyer [3] for the first nonzero eigenvalue on closed Riemannian manifolds.

1. Introduction

Let (M, g) be an n-dimensional closed Riemannian manifold. Motivated from the variational characterization of the Laplacian eigenvalue problem, we define the L^p-Dirichlet integral on k-forms (introduced in [12]) by

$$\mathcal{F}[\alpha] := \int_M \|d\alpha\|^p + \|d^*\alpha\|^p, \quad \alpha \in \Omega^k(M), \tag{1}$$

where d^* is the L^2-adjoint of the exterior derivative d. Note that $\mathcal{F}[\alpha] = 0$ if and only if $\alpha \in \mathcal{H}^k(M)$, that is, the minimum is zero and is attained for harmonic k-forms, i.e. $\alpha \in \ker(d) \cap \ker(d^*)$. For a nonzero infimum we consider the space

$$A_k := \left\{ \alpha \in \mathcal{W}^{1,p}(\Omega^k(M)) \mid ⨍_M \|\alpha\|^p = 1, \int_M \|\alpha\|^{p-2}\langle \alpha, \omega\rangle = 0, \omega \in \mathcal{H}^k(M) \right\}, \tag{2}$$

where the space $\mathcal{W}^{1,p}(\Omega^k(M))$ is the $(1,p)$-Sobolev space of differential k-forms defined in [12]. See §3 for the precise definition. Computing the Euler-Lagrange equation leads us to the defining the following operator

Definition 1.1 (p-Hodge Laplacian)**.**

$$\Delta_p \alpha := d^*(\|d\alpha\|^{p-2} d\alpha) + d(\|d^*\alpha\|^{p-2} d^*\alpha), \quad \alpha \in \Omega^k(M). \tag{3}$$

When $p = 2$, this becomes the usual Hodge Laplacian. For $p \neq 2$ and $\alpha \in C^\infty(M)$ Δ_p becomes the usual p-Laplacian. The corresponding eigenvalue equation is given by

$$\Delta_p \alpha = \lambda \|\alpha\|^{p-2} \alpha, \quad \alpha \in \Omega^k(M) \tag{4}$$

and the variational principle tells us that

$$\lambda_1 = \inf\{\mathcal{F}[\alpha] \mid \alpha \in A_k\}.$$

See §3 for details. When $p = 2$, there is much work on the spectrum of the Hodge-Laplacian acting on differential forms. Among many others, we point out the work of Gallot-Meyer [3], [4] who show an estimate of the first eigenvalue using bounds from the Weitzenböck curvature on compact Riemannian manifolds. For manifolds with boundary, among many others, see works of Kwong [5], Savo [11], Raulot-Savo [10], and references therein.

Date: December 17, 2019.

Key words and phrases. p-Laplacian, Hodge-Laplacian, Weitzenböck curvature.

For $p \neq 2$, the p-Laplace eigenvalue problem on 0-forms (functions) has attracted much attention. See notes by Lindqvist [6] for a general reference on the p-Laplace equation. For estimates on the first eigenvalue relating to the curvature, among many other works, see Matei [7], Naber-Valtorta [9], Seto-Wei [13] for eigenvalue estimates with $\mathrm{Ric} \geq K$, $K \in \mathbb{R}$.

Remark 1.1. There is also a related notion of p-harmonic k-forms which looks at the minimizer in a cohomology class of k-forms with finite L^p-norm, i.e.

$$\inf_{\alpha \in H_d^k(M)} \int_M \|\alpha\|^p.$$

The critical point of the variation leads to the following definition of p-harmonic, for closed k-forms α, if

$$d_p^* := d^*(\|\alpha\|^{p-2}\alpha) = 0$$

then α is p-harmonic. See [1] and references therein.

In this paper we prove the following lower bound estimate for the first eigenvalue

Theorem 1.1. Let M^n be a closed Riemannian manifold with the eigenvalues of the curvature operator bounded below by $H \in \mathbb{R}$ and $p \geq 2$. Then

$$\lambda_1 \geq \left(\frac{k(n-k)}{2^{\frac{2}{p}-1}\left(C + \frac{(p-2)}{2}\right)} H \right)^{\frac{p}{2}},$$

where

$$C = \max\left\{ \frac{k}{k+1}, \frac{n-k}{n-k+1} \right\}.$$

Remark 1.2. When $p = 2$, the above recovers the estimate due to Gallot-Meyer [3] (see also [4]), for $1 \leq k \leq \frac{n}{2}$,

$$\lambda_1 \geq k(n-k+1)H.$$

The organization of this paper is as follows. In §2 we review some known estimates for differential k-forms. In §3 we show that the infimum can be characterized as an eigenvalue problem. In §4 we give the main estimate. In §5 we give a brief discussion on boundary conditions for differential forms and possible future directions.

Acknowledgments: The author would like to thank Prof. Zhiqin Lu for very helpful discussions, suggestions and constant support, and to Prof. Guofang Wei and Prof. Qi Zhang for discussions and encouragement on the problem. We also thank Prof. Nguyen Thac Dung for letting of know of an improvement on the constant, see Lemma 2.2.

2. Some estimates on $\Omega^k(M)$

We first recall the Weitzenböck curvature

Definition 2.1. Let $p \in M$ and let $\{E_i\}_{i=1}^n$ be an orthonormal frame at p. Then for $\alpha \in \Omega^k(M)$, define the Weitzenböck curvature W_k by

$$W_k(\alpha)(X_1, \ldots, X_k) := \sum (R(E_j, X_i)\alpha)(X_1, \ldots, E_j, \ldots, X_k).$$

Note that on 1-forms, this is simply the Ricci tensor.

If the eigenvalues of the curvature operator are bounded by $H \in \mathbb{R}$, we can show that

$$(W_k(\alpha), \alpha) \geq k(n+1-k)H\|\alpha\|^2. \tag{5}$$

The Weitzenböck curvature shows up in the main tool we will use in obtaining our estimate is the Bochner-Weitzenböck formula for k-forms

$$\frac{1}{2}\Delta\|\alpha\|^2 = (\Delta\alpha, \alpha) - \|\nabla\alpha\|^2 - (W_k(\alpha), \alpha), \tag{6}$$

where $\Delta := \Delta_2 = dd^* + d^*d$. Note that for exact 1-form $\alpha = df$, since $\nabla df = \operatorname{Hess} f$, the usual Cauchy-Schwarz inequality will give us an estimate on the middle term. For k-forms, we will need the following proved by Gallot-Meyer

Lemma 2.1 ([3])**.** Let $\alpha \in \Omega^k(M)$, $1 \leq k \leq n-1$. Then

$$\|\nabla\alpha\|^2 \geq \frac{1}{k+1}\|d\alpha\|^2 + \frac{1}{n-k+1}\|d^*\alpha\|^2. \tag{7}$$

We give a proof for completeness. The proof we give is in the context of conformal Killing forms and can be found in various sources, for instance, [8].

Proof. Consider the two linear maps

$$\begin{aligned} &\iota : TM \otimes \Omega^k(M) \to \Omega^{k-1}(M) \\ &\iota(v, \alpha) = \iota_v\alpha \end{aligned}$$

and

$$\begin{aligned} &\wedge : \Omega^1(M) \otimes \Omega^k(M) \to \Omega^{k+1}(M) \\ &\wedge(\beta, \alpha) = \beta \wedge \alpha. \end{aligned}$$

Let ι^* and $\wedge^*$ be their metric adjoint. Then

$$\wedge \circ \iota^*(\alpha) = 0 \text{ and } \iota \circ \wedge^*(\alpha) = 0,$$

so that we get the decomposition

$$TM \otimes \Omega^k(M) \simeq \operatorname{im}(\iota^*) \oplus \operatorname{im}(\wedge^*) \oplus Y$$

where Y is the orthogonal complement. By direct computation, we have for $\alpha \in \Omega^k(M)$,

$$\iota \circ \iota^*(\alpha) = (n-k+1)\alpha \text{ and } \wedge \circ \wedge^*(\alpha) = (k+1)\alpha.$$

Viewing $\nabla\alpha \in \Gamma(TM \otimes \Omega^k(M))$, From the decomposition,

$$\nabla\alpha = \iota^*\beta + \wedge^*\gamma + \delta,$$

applying ι, we have

$$\iota\nabla\alpha = (n-k+1)\beta.$$

So the projection operator onto $\operatorname{im}(\iota^*)$ is given by

$$\pi_{\iota^*}\nabla\alpha = \frac{1}{n-k+1}\iota^*\iota\nabla\alpha$$

and similarly

$$\pi_{\wedge^*}\nabla\alpha = \frac{1}{k+1}\wedge^*\wedge\nabla\alpha.$$

Let $T\alpha := \pi_T(\nabla\alpha)$ the projection onto the orthogonal complement space. Since

$$d\alpha = \wedge(\nabla\alpha) \text{ and } d^*\alpha = -\iota(\nabla\alpha),$$

we have the decomposition

$$T\alpha(X) = \nabla_X\alpha - \frac{1}{k+1}\iota_X d\alpha + \frac{1}{n-k+1}X^* \wedge d^*\alpha$$

and taking the norm gives us

$$\|\nabla\alpha\|^2 = \|T\alpha\|^2 + \frac{1}{k+1}\|d\alpha\|^2 + \frac{1}{n-k+1}\|d^*\alpha\|^2,$$

which implies (7). □

Remark 2.1. The projection operator T defined above is called the twistor operator and a form $\alpha \in \Omega^k(M)$ is called a conformal Killing form if $T\alpha = 0$.

The following lemma was pointed out by N.T. Dung and gives us a way to control the interior product by using an orthogonal decomposition of forms as the image under an interior product.

Lemma 2.2 (Lemma 3.5 [2])**.** Let $V \in TM$, $\alpha \in \Omega^{k+1}$, $\beta \in \Omega^k$. Then

$$|\langle \iota_V\alpha, \beta\rangle| \le \|V\|\|\alpha\|\|\beta\|.$$

3. Variational characterization of the eigenvalue

In this section we will compute the Euler-Lagrange equation of (1) and show that the extremal problem can be reformulated as an eigenvalue problem. Analogous to the 0-form (function) case, we will look at weak solutions lying the $(1,p)$-Sobolev space of differential k-forms first defined by Scott in [12] as

$$\mathcal{W}^{1,p}(\Omega^k(M)) := \left\{\alpha \in W(\Omega^k(M)) \mid \alpha, d\alpha, d^*\alpha \in L^p(\Omega^*(M))\right\}$$

where $W(\Omega^k(M))$ is the classical Sobolev space of k-forms, i.e., α is locally integrable and admits a generalized gradient.

Definition 3.1. We say that λ is an eigenvalue, if there exists a k-form $\alpha \in \mathcal{W}^{1,p}(\Omega^k(M))$ such that

$$\int_M \|d\alpha\|^{p-2}\langle d\alpha, d\beta\rangle + \int_M \|d^*\alpha\|^{p-2}\langle d^*\alpha, d^*\beta\rangle = \lambda\int_M \|\alpha\|^{p-2}\langle\alpha,\beta\rangle,$$

for any $\beta \in C^\infty(\Omega^k(M))$.

We will show the first nonzero eigenvalue λ_1 can be characterized as the infimum of the L^p-Dirichlet energy over the space A_k given in (2).

Proposition 3.1. For closed manifolds M and $p \ge 2$,

$$\lambda_1 = \inf\left\{\int_M \|d\alpha\|^p + \|d^*\alpha\|^p \mid \alpha \in A_k\right\}.$$

Proof. Let ω be a fixed harmonic form and let $\beta(t) \in A$ for small $t > 0$ such that $\beta(0) = \alpha$. Computing the first variation of (1), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}[\beta(t)]\Big|_{t=0} &= p\int_M \|d\alpha\|^{p-2}\langle d\alpha, d\beta'(0)\rangle + \|d^*\alpha\|^{p-2}\langle d^*\alpha, d^*\beta'(0)\rangle \\ &= p\int_M \langle \Delta_p\alpha, \beta'(0)\rangle.\end{aligned}$$

Next we compute the variation of the constraints so that

$$\frac{d}{dt}\int_M \|\beta\|^p\Big|_{t=0} = p\int_M |\alpha|^{p-2}\langle \alpha, \beta'(0)\rangle$$

and

$$\frac{d}{dt}\int_M \|\beta\|^{p-2}\langle \beta, \omega\rangle\Big|_{t=0} = (p-2)\int_M \|\alpha\|^{p-4}\langle \alpha, \beta'(0)\rangle\langle \alpha, \omega\rangle + \|\alpha\|^{p-2}\langle \beta'(0), \omega\rangle.$$

By Lagrange multiplier method, there must be some λ and μ such that for $\beta \in \Omega^k(M)$,

$$\int_M \langle \Delta_p\alpha, \beta\rangle = \lambda\int_M \|\alpha\|^{p-2}\langle \alpha, \beta\rangle + \mu\int_M \|\alpha\|^{p-4}\langle \alpha, \beta\rangle\langle \alpha, \omega\rangle + \|\alpha\|^{p-2}\langle \beta, \omega\rangle.$$

Setting $\beta = \omega$, we have

$$0 = \mu\int_M \|\alpha\|^{p-4}\langle \alpha, \omega\rangle^2 + \|\alpha\|^{p-2}\|\omega\|^2$$

so that $\mu = 0$. Therefore,

$$\Delta_p\alpha = \lambda\|\alpha\|^{p-2}\alpha.$$

□

4. Proof of theorem 1.1

We will consider the following integral

$$\int_M \langle \Delta_p\alpha, \Delta\alpha\rangle = \int_M \langle \Delta_p\alpha, dd^*\alpha\rangle + \int_M \langle \Delta_p, d^*d\alpha\rangle.$$

Let $\alpha \in \Omega^k(M)$ be an eigenform satisfying (4). Then

$$\begin{aligned}\int_M \langle \Delta_p\alpha, d^*d\alpha\rangle &= \lambda\int_M \|\alpha\|^{p-2}\langle \alpha, d^*d\alpha\rangle \\ &= \lambda\int_M \langle d(\|\alpha\|^{p-2}\alpha), d\alpha\rangle \qquad (8)\\ &= \lambda\int_M \langle d(\|\alpha\|^{p-2})\wedge\alpha, d\alpha\rangle + \lambda\int_M \|\alpha\|^{p-2}\|d\alpha\|^2\end{aligned}$$

and

$$\begin{aligned}\int_M \langle \Delta_p\alpha, dd^*\alpha\rangle &= \lambda\int_M \|\alpha\|^{p-2}\langle \alpha, dd^*\alpha\rangle \\ &= \lambda\int_M \langle d^*(\|\alpha\|^{p-2}\alpha), d^*\alpha\rangle \qquad (9)\\ &= \lambda\int_M \|\alpha\|^{p-2}\|d^*\alpha\|^2 - \lambda\int_M \langle \iota_{\nabla\|\alpha\|^{p-2}}\alpha, d^*\alpha\rangle.\end{aligned}$$

On the other hand, by using the Bochner-Weitzenböck formula (6) we have

$$\begin{aligned}\int_M \langle \Delta_p \alpha, \Delta\alpha\rangle &= \lambda \int_M \|\alpha\|^{p-2}\langle \alpha, \Delta\alpha\rangle \\ &= \lambda \int_M \left((p-2)\|\alpha\|^{p-2}|\nabla\|\alpha\||^2 + \|\alpha\|^{p-2}\|\nabla\alpha\|^2 + \|\alpha\|^{p-2}(W_k(\alpha),\alpha)\right).\end{aligned} \tag{10}$$

Combining (8), (9), and (10), we obtain

$$\begin{aligned}&\int_M \langle d(\|\alpha\|^{p-2})\wedge\alpha, d\alpha\rangle - \int_M \langle \iota_{\nabla\|\alpha\|^{p-2}}\alpha, d^*\alpha\rangle + \int_M \|\alpha\|^{p-2}\|d\alpha\|^2 + \int_M \|\alpha\|^{p-2}\|d^*\alpha\|^2 \\ &= \int_M \left((p-2)\|\alpha\|^{p-2}|\nabla\|\alpha\||^2 + \|\alpha\|^{p-2}\|\nabla\alpha\|^2 + \|\alpha\|^{p-2}(W_k(\alpha),\alpha)\right).\end{aligned} \tag{11}$$

Using Lemma 2.2, the first term of (11) can be estimated as

$$\begin{aligned}\int_M \langle d(\|\alpha\|^{p-2})\wedge\alpha, d\alpha\rangle &= \int_M \langle \alpha, \iota_{\nabla\|\alpha\|^{p-2}}(d\alpha)\rangle \\ &\le \int_M \|\nabla\|\alpha\|^{p-2}\|\|d\alpha\|\|\alpha\| \\ &= (p-2)\int_M \|\alpha\|^{\frac{p-2}{2}}\|\nabla\|\alpha\|\|\|\alpha\|^{\frac{p-2}{2}}\|d\alpha\| \\ &\le \frac{(p-2)}{2}\int_M \|\alpha\|^{p-2}\|\nabla\|\alpha\|\|^2 + \frac{(p-2)}{2}\int_M \|\alpha\|^{p-2}\|d\alpha\|^2\end{aligned}$$

and similarly for the second term,

$$\begin{aligned}-\int_M \langle \iota_{\nabla\|\alpha\|^{p-2}}\alpha, d^*\alpha\rangle &\le \int_M \|\nabla\|\alpha\|^{p-2}\|\alpha\|\|d^*\alpha\| \\ &= (p-2)\int_M \|\alpha\|^{\frac{p-2}{2}}\|\nabla\|\alpha\|\|\|\alpha\|^{\frac{p-2}{2}}\|d^*\alpha\| \\ &\le \frac{(p-2)}{2}\int_M \|\alpha\|^{p-2}\|\nabla\|\alpha\|\|^2 + \frac{(p-2)}{2}\int_M \|\alpha\|^{p-2}\|d^*\alpha\|^2.\end{aligned}$$

Applying these estimates to (11), we get

$$\begin{aligned}&\frac{(p-2)+2}{2}\int_M \|\alpha\|^{p-2}\|d\alpha\|^2 + \frac{(p-2)+2}{2}\int_M \|\alpha\|^{p-2}\|d^*\alpha\|^2 \\ &\quad\ge \int_M \|\alpha\|^{p-2}\|\nabla\alpha\|^2 + \int_M \|\alpha\|^{p-2}(W_k(\alpha),\alpha) \\ &\quad\ge \frac{1}{k+1}\int_M \|\alpha\|^{p-2}\|d\alpha\|^2 + \frac{1}{n-k+1}\int_M \|\alpha\|^{p-2}\|d^*\alpha\|^2 + \int_M \|\alpha\|^{p-2}(W_k(\alpha),\alpha).\end{aligned}$$

Let

$$C := \max\left\{\frac{k}{k+1}, \frac{n-k}{n-k+1}\right\}.$$

Using

$$\int_M \|\alpha\|^{p-2}\|d\alpha\|^2 \le \left(\int_M \|\alpha\|^p\right)^{1-\frac{2}{p}}\left(\int_M \|d\alpha\|^p\right)^{\frac{2}{p}}$$

and

$$\int_M \|\alpha\|^{p-2}\|d^*\alpha\|^2 \le \left(\int_M \|\alpha\|^p\right)^{1-\frac{2}{p}} \left(\int_M \|d^*\alpha\|^p\right)^{\frac{2}{p}},$$

we have

$$\left(C+\frac{(p-2)}{2}\right)\left(\int_M \|\alpha\|^p\right)^{1-\frac{2}{p}}\left[\left(\int_M \|d\alpha\|^p\right)^{\frac{2}{p}}+\left(\int_M \|d^*\alpha\|^p\right)^{\frac{2}{p}}\right]$$
$$\ge \int_M \|\alpha\|^{p-2}(W_k(\alpha),\alpha).$$

For $p \ge 2$, and using the lower bound of the Weitzenböck curvature (5), we have

$$2^{\frac{2}{p}-1}\left(C+\frac{(p-2)}{2}\right)\left(\int_M \|\alpha\|^p\right)^{1-\frac{2}{p}}\left(\int_M \|d\alpha\|^p+\|d^*\alpha\|^p\right)^{\frac{2}{p}} \ge k(n-k)H\int_M \|\alpha\|^p.$$

Using the fact that $\int_M \|d\alpha\|^p + \|d^*\alpha\|^p = \lambda \int_M \|\alpha\|^p$ for eigenform α, we get

$$\lambda^{\frac{2}{p}} \ge \frac{k(n-k)}{2^{\frac{2}{p}-1}\left(C+\frac{(p-2)}{2}\right)}.$$

1 ## 5. Boundary conditions

In this section we briefly discuss the situation of a compact manifold M with nonempty smooth boundary ∂M. Let n denote the unit outer normal vector and let $J : \partial M \to M$ be the inclusion. Then $J^*\alpha$ is the restriction of a form to the boundary. Then d and its adjoint d^* are related with an additional boundary term given by

$$\int_M \langle d\alpha,\beta\rangle = \int_M \langle \alpha, d^*\beta\rangle + \int_{\partial M}\langle J^*(\alpha),\iota_n\beta\rangle, \quad \alpha\in\Omega^k(M), \beta\in\Omega^{k+1}(M).$$

and the corresponding Green's formula for the p-Laplacian is

$$(\Delta_p\alpha,\beta) = \int_M \|d\alpha\|^{p-2}\langle d\alpha, d\beta\rangle + \int_M \|d^*\alpha\|^{p-2}\langle d^*\alpha, d^*\beta\rangle$$
$$- \int_{\partial M}\langle \iota_n(\|d\alpha\|^{p-2}d\alpha), J^*(\beta)\rangle + \int_{\partial M}\langle \|d^*\alpha\|^{p-2}J^*(d^*\alpha),\iota_n\beta\rangle.$$

The two most common boundary conditions for the classical Laplacian eigenvalue problem are the Dirichlet and Neumann boundary condition. For the Hodge-Laplacian, the analogous boundary conditions are the absolute boundary condition

$$\begin{cases}\iota_n\alpha = 0\\ \iota_n d\alpha = 0, \quad \text{on } \partial M\end{cases}$$

and the relative boundary condition

$$\begin{cases}J^*(\alpha) = 0\\ J^*(d^*\alpha) = 0, \quad \text{on } \partial M.\end{cases}$$

The essential feature of the boundary condition is that if α satisfies either of the boundary conditions, then $\Delta_p\alpha = 0$ implies $d\alpha = 0$ and $d^*\alpha = 0$. The boundary terms that will be introduced to (11) are

$$\begin{aligned}\int_M \langle d(\|\alpha\|^{p-2}) \wedge \alpha, d\alpha\rangle - \int_M \langle \iota_{\nabla\|\alpha\|^{p-2}}\alpha, d^*\alpha\rangle + \int_M \|\alpha\|^{p-2}\|d\alpha\|^2 + \int_M \|\alpha\|^{p-2}\|d^*\alpha\|^2 \\ - \int_{\partial M} \|\alpha\|^{p-2}\langle J^*(\alpha), \iota_n(d\alpha)\rangle + \int_{\partial M} \|\alpha\|^{p-2}\langle J^*(d^*\alpha), \iota_n(\alpha)\rangle \\ = \int_M \left((p-2)\|\alpha\|^{p-2}|\nabla\|\alpha\||^2 + \|\alpha\|^{p-2}\|\nabla\alpha\|^2 + \|\alpha\|^{p-2}(W_k(\alpha), \alpha)\right).\end{aligned}$$

Since the boundary terms will vanish under either of the boundary conditions, we get the same estimate for the boundary value problem as well. It would be interesting to see what the Reilly formula, for instance a generalization of Theorem 3 in [10] would be in this context, however due to the asymmetry of the weight function in the p-Laplacian, it is not immediate what the appropriate Bochner-Weitzenböck type formula would be for Δ_p.

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