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On the Asymptotic Expansion of the Bergman Kernel

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Shoo Seto

Dissertation Committee:
Professor Zhiqin Lu, Chair
Professor Song-Ying Li
Professor Jeffrey Streets

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DEDICATION

To all my family and friends.

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CURRICULUM VITAE

Shoo Seto

EDUCATION

Doctor of Philosophy in Mathematics **2015**
University of California, Irvine *Irvine, California*

Master of Science in Mathematics **2011**
University of California, Irvine *Irvine, California*

Bachelor of Arts in Mathematics **2009**
University of California, Berkeley *Berkeley, California*

RESEARCH EXPERIENCE

Graduate Research Assistant **2009–2015**
University of California, Irvine *Irvine, California*

TEACHING EXPERIENCE

Teaching Assistant **2009–2015**
University of California, Irvine *Irvine, California*

ABSTRACT OF THE DISSERTATION

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Shoo Seto

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Professor Zhiqin Lu, Chair

Let $(L, h) \to (M, \omega)$ be a polarized Kähler manifold. We define the Bergman kernel for $H^0(M, L^k)$, holomorphic sections of the high tensor powers of the line bundle L. In this thesis, we will study the asymptotic expansion of the Bergman kernel. We will consider the on-diagonal, near-diagonal and far off-diagonal, using $\mathcal{L}^2$ estimates to show the existence of the asymptotic expansion and computation of the coefficients for the on and near-diagonal case, and a heat kernel approach to show the exponential decay of the off-diagonal of the Bergman kernel for noncompact manifolds assuming only a lower bound on Ricci curvature and C^2 regularity of the metric.

Chapter 1

Introduction

Background and motivation

Let $(L, h) \to (M, \omega)$ be a positive Hermitian line bundle over a complex manifold M. Let L^k be the kth tensor power of L. An active field of research in complex geometry is to analyze the geometry of the manifold M when $k \to \infty$. A classical result in this direction is the celebrated Kodaira embedding theorem which states that such manifolds admitting positive line bundles can be embedded into a projective space of sufficiently high dimension. The main tools in this analysis are the holomorphic sections $H^0(M, L^k)$, which can be used to construct many objects in geometry, one of them being the Bergman kernel.

The Bergman kernel, as defined classically on pseudoconvex domains $\Omega \subset \mathbb{C}^n$, is the holomorphic integral kernel of the projection operator from square integrable to holomorphic square integrable functions. Its analogue on complex manifolds can be obtained by replacing holomorphic functions and the $\mathcal{L}^2$ inner product of functions with an $\mathcal{L}^2$ inner product induced from the Hermitian metric. In a local neighborhood, we can view holomorphic sections as local holomorphic functions, an observation we use in Chapter 4.

An explicit formula for the Bergman kernel, except for certain cases (c.f. §3.1.2, Lemma 4.1.1), is not possible. However by the works of Zelditch [33] and independently by Catlin [5], a complete asymptotic expansion of the Bergman kernel on the diagonal was given by using a result by Boutet de Monvel-Sjöstrand on the asymptotics of the Szegö kernel. The coefficients of the asymptotic expansion carry geometric information as demonstrated by Lu in [19] in which he explicitly computed the first four coefficients. In particular, the second coefficient is half the scalar curvature of the manifold. This fact was used by Donaldson [11] demonstrating the stability of polarized manifolds with constant scalar curvature Kähler metrics. The method employed by Lu in [19] uses certain holomorphic sections called *peak sections* constructed by Tian in [31]. These are sections which in a small neighborhood can be represented by monomials and by Hörmander's $\overline{\partial}$-estimate, extend to a global section with sufficient decay property.

Other methods to the analysis of the Bergman kernel include a heat kernel method by Dai, Liu, and Ma [9], where they obtained the full off-diagonal asymptotic expansion and Agmon-type estimates. Their results hold in a more general setting of the Bergman kernel of the spin^c Dirac operator associated to a positive line bundle on a compact symplectic manifold. Another approach, done by Berman, Berndtsson, and Sjöstrand in [3], involves using microlocal analysis techniques inspired by the calculus of pseudodifferential operators and Fourier integral operators with complex phase. In this thesis, we consider the diagonal and the near-diagonal expansion, and the off-diagonal Agmon-type decay estimates.

In chapter 2, we begin by reviewing the fundamentals of Kähler geometry and introduce the "polarized" setting that we will use. We introduce the primary object of this thesis, the Bergman kernel in Chapter 3, and discuss the on-diagonal behavior of the Bergman kernel. We give a computation of the coefficients up to the first order using methods of Tian [31] and Lu [19]. In Chapter 4, we give an elementary proof of the expansion based on a joint work by the author with Hezari, Kelleher, and Xu [14]. In Chapter 5, we prove an exponential decay of the Bergman kernel on the off-diagonal using a “perturbation” of the operator approach as seen in the works of [29].

Chapter 2

Preliminaries

2.1 Kähler Geometry

In this chapter, we review the basics of Kähler geometry and set up notations and conventions which will be used. We focus on the elements of Kähler geometry which will be pertinent to our analysis of the Bergman kernel for positive line bundles. We refer the reader to [30] for further detail.

2.1.1 Kähler Manifolds

Let (M, J) be a complex manifold of complex dimension n, where $J \in \mathrm{End}(TM)$ is the complex structure. A *Hermitian metric* on M is a Riemannian metric g such that for $v, w \in TM$, we have $g(v, w) = g(Jv, Jw)$. Under a holomorphic local coordinate system $(z_i)_{i=1}^n$, g can be written as

$$g := g(\partial_i, \partial_{\bar{j}}) dz^i \otimes d\bar{z}^j := g_{i\bar{j}} dz^i \otimes d\bar{z}^j,$$

and $(g_{i\bar{j}})$ can be viewed as an $n \times n$ Hermitian matrix. Here and throughout, we will use the Einstein summation convention for repeated indices. Next define the *Kähler form* (or fundamental 2-form) ω by

$$\omega(v, w) := \frac{1}{2\pi} g(Jv, w), \quad v, w \in TM,$$

which in local coordinates is given by

$$\omega = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

We say a metric is *Kähler* if $d\omega = 0$, and call a manifold equipped with such a metric a *Kähler manifold*. The condition $d\omega = 0$ in local coordinates is given by

$$\partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}}.$$

An immediate consequence of the symmetries of the metric is the existence of a *holomorphic normal coordinate system.*

Lemma 2.1.1 (Holomorphic normal coordinates)**.** At each point x_0 on a Kähler manifold (M, g), there exists a holomorphic coordinate chart $(U, (z^i)_{i=1}^n)$ centered at x_0 such that

$$g_{i\bar{j}}(x_0) = \delta_{ij} \quad \text{and} \quad \partial_k g_{i\bar{j}}(x_0) = 0, \quad \text{for } i, j, k \in \{1, \dots, n\}.$$

Proof. Let $x_0 \in M$. We first choose coordinates $(w^i)_{i=1}^n$ centered at x_0 such that $g_{i\bar{j}}(0) = \delta_{ij}$, which can be found since $(g_{i\bar{j}}(0))$ is Hermitian symmetric, and expand the metric at 0 to obtain:

$$\omega = \sqrt{-1} \left(\delta_{ij} + \partial_l g_{i\bar{j}}(0) w^l + \partial_{\bar{l}} g_{i\bar{j}}(0) \overline{w}^l + O(|w|^2) \right) dw^i \wedge d\overline{w}^j.$$

We use the following holomorphic change of variables

$$\Gamma^l_{ij} := g^{l\bar{q}}\partial_j g_{i\bar{q}},$$
$$w^l := z^l - \frac{1}{2}\Gamma^l_{ij}(0)z^i z^j.$$

Then

$$dw^l = dz^l - \Gamma^l_{ij}(0)z^i dz^j,$$

so that

$$\begin{aligned}\omega &= \sqrt{-1}\left(\delta_{ij} + \partial_l g_{i\bar{j}}(0)z^l + \partial_{\bar{l}} g_{i\bar{j}}(0)\bar{z}^l - \Gamma^j_{il}(0)z^l - \overline{\Gamma^i_{jl}}(0)\bar{z}^l + O(|z|^2)\right) dz^i \wedge d\bar{z}^j \\ &= \sqrt{-1}\left(\delta_{ij} + O(|z|^2)\right) dz^i \wedge d\bar{z}^j.\end{aligned}$$

□

A crucial point is that for Riemannian metrics normal coordinates can be constructed via the exponential map, however the coordinates may not be *holomorphic*, hence we require the Kähler condition to construct such holomorphic coordinates. The strategy of the proof of Lemma 2.1.1 can be extended further yielding the result below

Lemma 2.1.2 (K-coordinate system)**.** With the same notation and hypotheses as Lemma 2.1.1, for any choice of $p_i \in \mathbb{Z}_+$ with $p = \sum_{i=1}^n p_i$, there exists a holomorphic coordinate chart $(U, (z_i))$ centered at $x_0 \in M$ such that

$$g_{i\bar{j}}(x_0) = \delta_{ij} \quad \text{and} \quad \frac{\partial^p g_{i\bar{j}}}{(\partial z_1)^{p_1}\cdots(\partial z_n)^{p_n}}(x_0) = 0, \quad \text{for } i,j \in \{1,\dots,n\}.$$

In other words, we can find a coordinate system where the derivative of the metric vanishes for purely holomorphic and purely antiholomorphic directions. These coordinates play an important role in the computation of the coefficients of the Bergman kernel asymptotic expansion.

The Kähler condition also implies additional basic results in Kähler geometry, known as the $\partial\overline{\partial}$-lemmas.

Lemma 2.1.3 (Local $\partial\overline{\partial}$-lemma)**.** Let (M,ω) be a Kähler manifold. For any $x_0 \in M$, there is a neighborhood U and a real function $\varphi(z,\overline{z})$, called the *local Kähler potential*, such that

$$\omega = \sqrt{-1}\partial\overline{\partial}\varphi.$$

It can be derived as a special case of the following global version,

Lemma 2.1.4 (Global $\partial\overline{\partial}$-lemma)**.** Let (M,ω) be a Kähler manifold and $[\omega]$ the cohomology class of (p,q) forms. Given $\omega, \omega' \in [\omega]$, there exists $\varphi \in H^{p-1,q-1}$ such that

$$\omega - \omega' = \partial\overline{\partial}\varphi.$$

Proof. Since $\omega, \omega' \in [\omega]$ there exists $\alpha \in \Lambda^{p+q-1}$ such that

$$d\alpha = \omega - \omega'$$

Decomposing into types $\alpha = \alpha^{p,q-1} + \alpha^{p-1,q}$ we have

$$\partial\alpha^{p,q-1} = 0, \quad \overline{\partial}\alpha^{p-1,q} = 0.$$

By the Hodge decomposition,

$$\alpha^{p-1,q} = h + \overline{\partial}f$$

where h is $\overline{\partial}$-harmonic. Furthermore, on Kähler manifolds,

$$\Delta_\partial h = \Delta_{\overline{\partial}} h = 0$$

so that $\partial h = 0$. Applying the same argument, $\alpha^{p,q-1} = \tilde{h} + \partial \tilde{f}$, with $\overline{\partial}\tilde{h} = 0$, we have

$$\begin{aligned} \omega - \omega' &= d\alpha \\ &= (\partial + \overline{\partial})(\tilde{h} + \partial \tilde{f} + h + \overline{\partial} f) \\ &= \partial\overline{\partial}(f - \tilde{f}) \end{aligned}$$

Letting $\varphi = f - \tilde{f}$, the result follows. □

Definition 2.1.1 (Holomorphic Vector Bundle)**.** Let M be a complex manifold. A *holomorphic vector bundle* of rank k over M is a complex manifold E together with a holomorphic projection map $\pi : E \to M$ satisfying:

1. For any $p \in M$, the fiber $E_p := \pi^{-1}(p)$ is a complex vector space of complex dimension k.

2. There exists an open covering $\{U_i\}$ of M and homeomorphic maps $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^k$ commuting with the projection to U_i such that for each $p \in U_i$, the restriction of $\varphi_i|_{\pi^{-1}(p)} \to \{p\} \times \mathbb{C}^k$ is an isomorphism of the complex vector spaces.

3. On the overlap $p \in U_i \cap U_j$, the induced transition maps

$$\varphi_{ij}(x_0) := \varphi_i \circ \varphi_j^{-1}(x_0, \cdot) : \mathbb{C}^k \to \mathbb{C}^k$$

are holomorphic $\mathbb{C}$-linear maps.

The primary object of study on vector bundles are sections, which can be thought of as generalization of functions.

Definition 2.1.2 (Sections)**.** Let E be a complex vector bundle over M. A section of E is a map $s : M \to E$ such that $(\pi \circ s)(x_0) = x_0$ for all $x_0 \in M$.

The key idea is that it is a map taking values in E which maps p to objects in the fiber E_p. The space of smooth sections is denoted at $\Gamma(M, E)$. In particular, if E is holomorphic, then we denote $H^0(M, E)$ as the space of holomorphic sections. Restricting the vector bundle E to an open set $U \subset M$ preserves the vector bundle structure and we can define the space of sections on the restricted bundle, denoted as $\Gamma(U, E) := \Gamma(U, E|_U)$.

Definition 2.1.3 (Local frame)**.** The set of sections $\{e_i\}$, with each $e_i \in \Gamma(U, E)$, is called a *local frame* if for each $x_0 \in M$ the collection $\{e_i(x_0)\}$ forms a basis for the fiber E_{x_0} as a vector space.

2.1.2 Calculus on vector bundles

We now introduce the notion of a Hermitian metric on a vector bundle E.

Definition 2.1.4. Let E be a holomorphic vector bundle. A *Hermitian metric* h of E is an assignment of a Hermitian inner product $h(\cdot, \cdot)$ on each E_{x_0}, which varies smoothly with respect to $x_0 \in M$. To be explicit, let $\{e_i\}$ be a local frame. Then

$$h_{i\bar{j}}(x_0) := h(e_i, \overline{e}_j)|_{x_0}$$

where the matrix $(h_{i\bar{j}}(x_0))$ is a Hermitian $n \times n$ matrix for each x_0.

Connections

Here we introduce the notion of a connection, which is a way to differentiate sections by "connecting" them between different fibers.

Definition 2.1.5 (Connection)**.** Let $E \to M$ be a complex vector bundle. A connection D on E is a $\mathbb{C}$-linear operator

$$D : \Gamma(M, E) \to \Gamma(M, \Lambda^1(E))$$

satisfying $D(fs) = df \otimes s + fDs$ for $f \in C^\infty(M)$ and $s \in \Gamma(M, E)$. The connection above induces a connection on the bundle $\Lambda^p(E)$ given by

$$D : \Gamma(M, \Lambda^p(E)) \to \Gamma(M, \Lambda^{p+1}(E))$$

by

$$D(v \wedge s) = dv \wedge s + (-1)^p v \wedge Ds,$$

where $v \in \Gamma(M, \Lambda^p(E))$.

Let (E, h) be a holomorphic vector bundle over a Kähler manifold M. We will focus mainly on a particular type of connection.

Definition 2.1.6. Let D be a connection satisfying the following conditions:

$$Dh = 0 \quad \text{(metric compatibility)},$$

$$DJ = 0 \quad \text{(complex structure compatibility)}.$$

Such a connection is called a *Hermitian connection*.

2.1.3 Line bundles

The following are two important examples of line bundles.

Example 2.1.1 (Canonical line bundle)**.** Let M be a complex manifold of dimension n. The nth exterior power of the holomorphic cotangent bundle forms a line bundle called the *canonical bundle* of M

$$K_M := \Lambda^n T^{*(1,0)}(M) = \det(T^{*(1,0)}(M)).$$

Let $\{U_\alpha, (z_i)_\alpha\}$ be a holomorphic atlas of M. A local holomorphic frame on U_α is given by $\{(dz^1 \wedge \ldots \wedge dz^n)_\alpha\}$ and the transition function is given by

$$(dz^1 \wedge \ldots \wedge dz^n)_\alpha = \det\left(\frac{\partial z^i_\alpha}{\partial z^j_\beta}\right)(dz^1 \wedge \cdots \wedge dz^n)_\beta \quad \text{ on } U_\alpha \cap U_\beta \neq \emptyset.$$

If g is a Hermitian metric on M, then it induces a metrics a natural Hermitian metric on K_M is given by

$$h = (\det g)^{-1}$$

Its dual, denoted K_M^{-1}, is called the *anti-canonical bundle.*

Example 2.1.2 (Line bundles over $\mathbb{CP}^n$)**.** On $\mathbb{CP}^n$ (defined in 2.1.6), we can construct a line bundle, called the *tautological line bundle* or $\mathcal{O}(-1)$, by assigning to each point the line that the point represents and viewing it as a line subbundle of the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$. To be precise, let $[p] := [p_0 : \cdots : p_n]$. Then at $[p]$, attach the line in $\mathbb{C}^{n+1}$ defined by the vector $\langle p_0, \ldots, p_n\rangle$. Let $U_i = \{[p] \mid p_i \neq 0\}$. Since

$$\langle p_0, \cdots, p_n\rangle = p_i \left\langle \frac{p_0}{p_i}, \ldots, \frac{p_n}{p_i}\right\rangle$$

we can use p_i as a local trivialization. Then the transition function g_{ij} must satisfy

$$g_{ij} = \frac{p_i}{p_j}.$$

Let $s \in H^0(\mathbb{CP}^n, \mathcal{O}(-1))$ be a global holomorphic section. Since $\mathcal{O}(-1)$ is a subbundle of the trivial bundle, under a global non-vanishing frame, we can view s as a holomorphic map $s : \mathbb{CP}^n \to \mathbb{C}^{n+1}$. The components of s are then holomorphic functions on a compact complex manifold, hence they must be constant. According to the transition map, on $U_\alpha \cap U_\beta = \not\emptyset$ it must satisfy $s_\alpha = g_{\alpha\beta} s_\beta$, hence

$$\frac{s_\alpha}{p_\alpha} = \frac{s_\beta}{p_\beta},$$

which for constants is only satisfied when $s = 0$.

The dual of $\mathcal{O}(-1)$, sometimes called the *hyperplane bundle*, is denoted $\mathcal{O}(1)$ and its tensor powers $\mathcal{O}(m) := \mathcal{O}(1)^m$. The transition functions for $\mathcal{O}(m)$ are obtained by inversions from the tautological bundle, i.e.

$$g_{ij}^m = \left(\frac{p_j}{p_i}\right)^m.$$

The global sections of $\mathcal{O}(m)$ can be thought of as homogeneous polynomials in $p_0, \ldots, p_n$ of degree m.

By looking at the transition functions, it can also be shown that $K_{\mathbb{CP}^n} \cong \mathcal{O}(-n-1)$.

2.1.4 Curvature

In this subsection, we define and establish the curvature and sign conventions we will use. Let ∇ be the Levi-Civita connection on a Kähler manifold (M, ω).

Definition 2.1.7 (Curvature Tensor)**.** The curvature tensor R is defined as

$$R(v_1, v_2, v_3, v_4) = g(\nabla_{v_1}\nabla_{v_2}v_3 - \nabla_{v_2}\nabla_{v_1}v_3 - \nabla_{[v_1,v_2]}v_3, v_4)$$

for $v_i \in TM$

On Kähler manifolds, the local coordinate formula for the curvature tensor is greatly simplified due to the compatibility with the complex structure. Let $v, w \in T^{1,0}M$. Then

$$\nabla_v w \in T^{1,0}M,$$
$$\nabla_v \overline{w} = 0 = \nabla_{\overline{v}} w.$$

For local coordinates $(z_i)_{i=1}^n$ with coordinate holomorphic vector fields $\{\partial_i\}_{i=1}^n$, the curvature tensor is determined completely by terms of the form

$$\begin{aligned} R_{i\bar{j}k\bar{l}} := R(\partial_i, \partial_{\bar{j}}, \partial_k, \partial_{\bar{l}}) &= -g(\nabla_{\bar{j}} \nabla_i \partial_k, \partial_{\bar{l}}) \\ &= -g_{m\bar{l}} \partial_{\bar{j}} \Gamma^m_{ik} \\ &= -\partial_i \partial_{\bar{j}} g_{k\bar{l}} + g^{p\bar{q}} (\partial_{\bar{j}} g_{p\bar{l}})(\partial_k g_{i\bar{q}}). \end{aligned}$$

Definition 2.1.8 (Ricci and scalar curvature)**.** The *Ricci curvature* Ric is the trace of the Riemann curvature tensor and the *scalar curvature* ρ is the trace of the Ricci curvature, i.e.

$$\begin{aligned} &\mathrm{Ric}_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}}, \quad &&\text{(Ricci)} \\ &\rho = g^{i\bar{j}} R_{i\bar{j}}. &&\text{(scalar)} \end{aligned}$$

A useful identity for the Ricci curvature which holds for Kähler manifolds is the following

Lemma 2.1.5.

$$\mathrm{Ric}_{i\bar{j}} = -\partial_i \partial_{\bar{j}} (\log \det g). \tag{2.1}$$

Proof. Follows from the two identities,

$$\partial_{\bar{j}} \det g = g^{k\bar{l}}(\partial_{\bar{j}} g_{k\bar{l}}) \det g$$
$$\partial_i g^{k\bar{l}} = -g^{p\bar{l}} g^{k\bar{q}}(\partial_i g_{p\bar{q}})$$

□

2.1.5 First Chern Class

We will use the identity (2.1) to define an important cohomology class associated to a manifold called the *(first) Chern class*. First define the *Ricci form* to be

$$\mathrm{Ric}(\omega) = \frac{\sqrt{-1}}{2\pi} \mathrm{Ric}_{i\bar{j}}\, dz^i \wedge d\bar{z}^j = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \det g$$

Let h be another Kähler metric on M. Then $\frac{\det h}{\det g}$ be a globally defined function and the difference of the Ricci forms is given by

$$\mathrm{Ric}(h) - \mathrm{Ric}(g) = -\sqrt{-1}\partial\bar{\partial} \log \frac{\det h}{\det g}$$

Hence $[\mathrm{Ric}(g)] \in H^2(M, \mathbb{R})$ defines a cohomology class independent of the metric and we define the *first Chern class of M*

Definition 2.1.9 (First Chern class of M)**.**

$$c_1(M) = [\mathrm{Ric}(g)]$$

More generally, we can define the first Chern class of a Hermitian line bundle (L, h) as the cohomology class of form of the line bundle. Let s be a local non-vanishing holomorphic section of L. As in the case for Ricci curvature, the curvature form is locally defined by

$$F(h) = -\sqrt{-1}\partial\overline{\partial}\log h(s)$$

where $h(s) = \langle s, s\rangle_h$. Given any other Hermitian metric, it can be written as $e^{-f}h$ for a globally defined function f and so

$$F(e^{-f}h) - F(h) = \sqrt{-1}\partial\overline{\partial}f$$

thus we can define the first Chern class of the line bundle L to be the cohomology class

$$c_1(L) = \frac{1}{2\pi}[F(h)]$$

By the $\partial\overline{\partial}$-lemma, we have that every real $(1,1)$-form in $c_1(L)$ is the curvature of some Hermitian metric on L. With this viewpoint, we see that the first Chern class of M is the first Chern class of the anti-canonical bundle of M,

$$c_1(M) = c_1(K_M^{-1}).$$

2.1.6 Example: Complex Projective Space

The model case of a Kähler manifold which we will consider is the *complex projective space*, $\mathbb{CP}^n$. It is constructed from $\mathbb{C}^{n+1} - \{0\}/\sim$, where the equivalence relation is given by $p \sim q$ if and only if for $p = (p_0, \cdots, p_n)$, $q = (q_0, \cdots, q_n)$, there is a nonzero complex number $\lambda \in \mathbb{C}^*$ such that $p = \lambda q$.

A local coordinate system is given by the following. Let

$$U_i = \{[p_0 : \cdots : p_n] \in \mathbb{CP}^n \mid p_i \neq 0\}$$

for $i = 0, \cdots, n$. Then the map z_i given by

$$z_i([p]) = \left(\frac{p_0}{p_i}, \cdots, \frac{\widehat{p_i}}{p_i}, \cdots, \frac{p_n}{p_i}\right)$$

where $\frac{\widehat{p_i}}{p_i}$ is removed, gives a local holomorphic coordinate chart. The holomorphic structure is the one induced from $\mathbb{C}^{n+1}$ and the transition maps are easily seen to be holomorphic.

Fubini-Study Metric

A Kähler metric that is commonly equipped to $\mathbb{CP}^n$ is the Fubini-Study metric.

Definition 2.1.10 (Fubini-Study Metric)**.** Let $[p_0 : \cdots : p_n]$ be homogeneous coordinates on $\mathbb{CP}^n$. The *Fubini-Study metric* is defined as the 2-form

$$\omega_{FS} := \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\sum_{i=0}^{n}|p_i|^2).$$

On a neighborhood, say $U_0 = \{p_0 \neq 0\}$, it can be written in local coordinates $z_i = \frac{p_i}{p_0}$

$$\omega_{FS} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(1 + \sum_{i=1}^{n}|z_i|^2).$$

The metric is a homogeneous metric, that is, for any $A \in U(n+1) \subset \mathrm{Aut}(\mathbb{CP}^n)$, the holomorphic automorphism group, acts transitively and leaves the form ω_{FS} invariant. As such, to check that the form ω_{FS} is indeed a metric, we need to only check that it is positive definite at one point, say $p = [1 : 0 : \cdots : 0]$. Directly computing, we have

$$\begin{aligned}\omega_{FS}(p) &= \frac{\sqrt{-1}}{2\pi}\left(\frac{\delta_{ij}(1+|z|^2) - z_j\overline{z}_i}{(1+|z|^2)^2}\right)dz^i \wedge d\overline{z}^j|_p \\ &= \frac{\sqrt{-1}}{2\pi}dz^i \wedge d\overline{z}^i > 0\end{aligned} \tag{2.2}$$

Curvature of Fubini-Study metric

As an example, we will compute the curvature terms of the Fubini-Study metric. Using normal coordinates at the point $p = [1 : 0 : \cdots : 0]$, the curvature tensor is given by

$$R_{i\bar{j}k\bar{l}} = -\partial_k \partial_{\bar{l}} g_{i\bar{j}}.$$

Using $g_{i\bar{j}}$ given in (2.2), computing while dropping the terms which will evaluate to 0,

$$\begin{aligned} -\partial_k \partial_{\bar{l}} \left(\frac{\delta_{ij}}{1+|z|^2} - \frac{z_j \bar{z}_i}{(1+|z|^2)^2} \right) \bigg|_p &= \partial_k \left(\frac{\delta_{ij} z_l}{(1+|z|^2)^2} + \frac{z_j \delta_{il}}{(1+|z|^2)^2} \right) \bigg|_p \\ &= \delta_{ij}\delta_{kl} + \delta_{jk}\delta_{il} \end{aligned}$$

Hence the curvature tensor is given by

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}} g_{k\bar{l}} + g_{k\bar{j}} g_{i\bar{l}}$$

and tracing gives the Ricci curvature

$$\mathrm{Ric}_{i\bar{j}} = (n+1) g_{i\bar{j}}$$

and the scalar curvature

$$\rho = n(n+1)$$

The Fubini-Study metric on $\mathbb{CP}^n$ is in fact a Kähler-Einstein metric with positive first Chern class.

2.2 Kodaira Embedding

Definition 2.2.1 (Ample line bundle)**.** A line bundle L over M is *very ample*, if for suitable sections $s_0, \dots, s_N$ of L, the *Kodaira map*

$$\begin{aligned} \varphi : M &\to \mathbb{CP}^N \\ p &\mapsto [s_0(p) : \dots : s_N(p)] \end{aligned} \tag{2.3}$$

defines an embedding of M into $\mathbb{CP}^N$. A line bundle L is *ample* if L^k is very ample for sufficiently large $k \in \mathbb{Z}_+$.

Theorem 2.2.1 (Kodaira Embedding Theorem)**.** Let L be a line bundle over a compact complex manifold M. Then L is ample if and only if $c_1(L) > 0$, i.e. L is positive.

Proof for ample implies positive. By the embedding, L^k can be identified with the restriction of the $\mathcal{O}(1)$ bundle of some complex projective space. In particular, L^k is positive. Let h be the positively curved Hermitian metric of L^k. Then $L^{\frac{1}{k}}$ is a positively curved Hermitian metric on L. □

In proving positive implies ample, we first make the following lemma

Lemma 2.2.1. Let L be a positive holomorphic line bundle. If for every $x, y \in M$ with $x \neq y$ and every $v \in \mathbb{C}^n$, there exists elements $S, T \in H^0(M, L^N)$ such that $S(y) \neq 0$, $S(x) = 0$, $T(x) = 0$, and $dT(x) = v$, then L is ample.

Proof. Let $s_0, \dots, s_N$ be a basis for $H^0(M, L^k)$. Since at each point $y \in M$, we can find a non-vanishing section $s(y) \neq 0$, we see that the at least one of $s_i(y) \neq 0$, hence the Kodaira map φ (2.6) defined by the basis is well defined. Suppose $\varphi(x) = \varphi(y)$. Then there exists nonzero complex number $\lambda \in \mathbb{C}^*$ such that

$$(s_0(y), \cdots, s_N(y)) = \lambda(s_0(x), \ldots, s_N(x))$$

but this would contradict the existence of a section such that $S(x) = 0$ and $S(y) \neq 0$, hence φ is injective. It remains to prove that $d\varphi$ is injective, hence we want to show it has maximal rank. For any $x \in M$, let $T_1, \ldots, T_n$ be sections of L^k such that $T_j(x) = 0$ for $1 \leq j \leq n$. Let e_L^N be a local frame at x . Then each T_j has a local representative $T_j = t_j e_L^N$, where $t_j \in H^0(U)$. Also let $T_0 = e_L^N$. By assumption, we can assign for each j, $dt_j = v_j$, where $\{v_j\}$ is the standard basis vector on $\mathbb{C}^n$. Then define the immersion

$$x \mapsto [T_0 : T_1 : \cdots : T_n : 0 : \cdots : 0].$$

Since the $T_i = a_{ij} S_j$, we have the rank of φ must be n as well. $\square$

Theorem 2.2.2 (L^2 estimate construction)**.** Let $p_1, \cdots, p_s \in M$ and $K_1, \cdots, K_s \in \mathbb{Z}_+$. Then for k sufficiently large, there is a holomorphic section $S \in H^0(M, L^k)$ such that at each p_i, S has the prescribed derivatives up to order K_i.

Proof. Let U_i be a coordinate neighborhood of p_i, with coordinates $(z_1, \ldots, z_n)$. By prescribed derivatives, we mean to assign the values for a holomorphic function f_i at p_i,

$$\frac{\partial^{K_i} f_i}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(p_i)$$

Let η_i be a smooth cut-off function whose supports are within U_i and is 1 in a neighborhood of p_i. We then define a global smooth section of L^m by

$$w = \sum_i \eta_i f_i.$$

Since we need a global holomorphic section, we consider the $\overline{\partial}$ equation

$$\overline{\partial} f = \overline{\partial} w$$

If the solution f exists, then $w - f$ will be holomorphic. To ensure that we have a section with the required conditions for the derivatives, that is, the vanishing order be at least $K_i + 1$ at each p_i.

Consider the Laplace equation

$$(\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^* \overline{\partial}) u = \Delta_{\overline{\partial}} g = \overline{\partial} w \tag{2.4}$$

Taking $\overline{\partial}$ on both sides, we have $\overline{\partial}\,\overline{\partial}^* \overline{\partial} u = 0$. Then

$$0 = (\overline{\partial} u, \overline{\partial}\,\overline{\partial}^* \overline{\partial} u) = \|\overline{\partial}^* \overline{\partial} u\|^2$$

So that $\overline{\partial}^* \overline{\partial} u = 0$. Letting $\overline{\partial}^* u = f$, we have

$$\overline{\partial} f = \overline{\partial} w.$$

To solve (2.4), we first use the Weitzenbock formula to get a lower bound estimate for the first eigenvalue. Let $e^{-\eta} h^k$ be a Hermitian metric on L^k. For any L^k-valued $(0,1)$-form, $a_{\overline{\alpha}} d\overline{z}^{\alpha}$,

$$\Delta_{\overline{\partial}}(a_{\overline{\alpha}} d\overline{z}^{\alpha}) = -\nabla_{\beta} \nabla_{\overline{\beta}}(a_{\overline{\alpha}} d\overline{z}^{\alpha}) + (\mathrm{Ric})_{\alpha\overline{\gamma}} a_{\overline{\alpha}} d\overline{z}^{\gamma} + m \,\mathrm{Ric}(h)_{\alpha\overline{\gamma}} a_{\alpha\overline{\gamma}} d\overline{z}^{\gamma} + (\partial_{\alpha} \partial_{\overline{\gamma}} \eta) a_{\overline{\alpha}} d\overline{z}^{\gamma}$$

Let ψ be an eigensection corresponding to the lowest eigenvalue, λ_1. Then

$$\begin{aligned}\lambda_1 \int_M \|\psi\|^2 e^{-\eta} dV_g &= \int_M \langle \Delta_{\overline{\partial}}(\psi), \psi \rangle e^{-\eta} dV_g \\ &\leq \int_M \|\nabla \psi\|^2 e^{-\eta} dV_g + (kC_1 - C_2) \int_M \|\psi\|^2 e^{-\eta} dV_g\end{aligned}$$

for some $C_1, C_2 > 0$. Hence $\lambda_1 \geq kC_1 - C_2$. By the first eigenvalue estimate $\Delta_{\overline{\partial}} u \geq \lambda_1 u$, hence

$$\|\overline{\partial} w\|^2_{L^2,\eta} = \|\Delta_{\overline{\partial}} u\|^2_{L^2,\eta} \geq \lambda_1^2 \|u\|^2_{L^2,\eta} \geq (kC_1 - C_2) \|u\|^2_{L^2,\eta}$$

Thus we have

$$\begin{aligned}\int_M \|f\|^2 e^{-\eta} dV_g = \int_M \|\overline{\partial}^* u\|^2 e^{-\eta} dV_g &= \int_M \langle u, \overline{\partial}\overline{\partial}^* u \rangle e^{-\eta} dV_g \\ &\leq \|u\|_{L^2,\eta} \|\overline{\partial} w\|_{L^2,\eta} \leq \frac{\|\overline{\partial} w\|^2_{L^2,\eta}}{kC_1 - C_2}\end{aligned}$$

By shrinking the neighborhoods if necessary, we assume that the U_i are all disjoint. Then in a small neighborhood of p_i,

$$\overline{\partial}(w) = \sum_i (\overline{\partial}\eta_i) f_i + \eta_i (\overline{\partial} f_i) = 0.$$

Hence $\|\overline{\partial} w\|^2_{L^2,\eta} < \infty$. Now we choose an appropriate cut-off function η for the vanishing order of f. Consider η to be the function such that at each point near p_i,

$$\eta(x) = \lambda \log(r_i)$$

where r_i is the distance $d(x, p_i)$. The value of λ will control the vanishing order of the section.

For the integral on the neighborhood U_i, we have

$$\int_{U_i} |f_i|^2 e^{-\eta} = \int_{U_i} \frac{|f_i|^2}{r^\lambda} \sim \int_0^1 |f_i|^2 r^{2n-1-\lambda} < \infty$$

Since $\int_0^1 \frac{1}{r^p} < \infty$ when $p < 1$. Let $|f_i|^2 \sim r^{2q}$. Then we require $\lambda - 2q - 2n < 1$, so choosing λ sufficiently large, we can push the vanishing order of f_i to be large. □

2.3 Polarized Kähler Manifolds

Let $L \to M$ be a Hermitian line bundle over M with Hermitian metric h. We say that the line bundle L is positive if the curvature $\mathrm{Ric}(h)$ is positive definite. We define a Kähler form ω_g by the curvature form of L, i.e. for a fixed point $p \in M$ with local coordinates $(z^i)_{i=1}^n$ at p,

$$\omega_g = -\frac{\sqrt{-1}}{2\pi} \frac{\partial^2}{\partial z^i \partial \overline{z}^j} \log \tilde{h} dz^i \wedge d\overline{z}^j = \frac{\sqrt{-1}}{2\pi} g_{i\overline{j}} dz^i \wedge d\overline{z}^j,$$

where $\tilde{h}$ is the local representation of the Hermitian metric h. We refer to this setting as *polarized Kähler manifold* with *polarization* L. For each $k \in \mathbb{Z}_+$, the Hermitian metric h induces a Hermitian metric h^k on $L^k := \underbrace{L \otimes \cdots \otimes L}_{k \text{ times}}$. The Hermitian metric h^k further induces an L^2 inner product on $H^0(M, L^k)$, the space of holomorphic global sections of L^m as follows: Choose an orthonormal basis $\{S_\alpha^k\}$ of $H^0(M, L^k)$. Then define the inner product

$$\langle S_\alpha^k, S_\beta^k \rangle_{L^2} := \int_M (S_\alpha^k, S_\beta^k)_{h^k} \frac{\omega^n}{n!} \tag{2.5}$$

We use the following notations

$$\mathcal{L}^2(M, L^k) = \{f \in \Gamma(M, L^k) \mid \|f\|_{\mathcal{L}^2} := \int_M |f|^2_{h^k} dV_g < \infty\}$$
$$H^0(M, L^k) = \{f \in \mathcal{L}^2 \mid f \text{ holomorphic section of } L^k\}$$

By the Kodaira embedding theorem, for sufficiently high tensor power k, the basis $\{S^k_\alpha\}$ induces a holomorphic embedding of M into $\mathbb{CP}^N$, $N + 1 = \dim H^0(M, L^k)$ given by the mapping

$$\begin{aligned} \varphi_k :& M \to \mathbb{CP}^N \\ & x \mapsto [S^k_0(x) : \ldots : S^k_N(x)] \end{aligned} \tag{2.6}$$

Let g_{FS} be the standard Fubini-Study metric on $\mathbb{CP}^N$, i.e., for homogeneous coordinate system $[Z_0 : \cdots : Z_N]$ of $\mathbb{CP}^N$,

$$\omega_{FS} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left(\sum_{i=0}^N |Z_i|^2 \right)$$

Definition 2.3.1 (Bergman metric)**.** The $\frac{1}{k}$-multiple of g_{FS} on $\mathbb{CP}^N$ restricts to a polarized Kähler metric $\frac{1}{k}\varphi_k^* g_{FS}$ on M, where φ_k is the Kodaira map defined above. The metric is called the *Bergman metric* with respect to L.

The Bergman metric and polarized metrics are related by the following

Theorem 2.3.1 (Tian)**.** With notation as above,

$$\left\| \frac{1}{k} \varphi_k^* g_{FS} - g \right\|_{C^\infty} = O\left(\frac{1}{k^2}\right)$$

Tian [31] originally proved the C^2 convergence with remainder $O\left(\frac{1}{\sqrt{k}}\right)$ and was improved to C^∞ and $O\left(\frac{1}{k}\right)$ by Ruan [26]. Zelditch [33] and Catlin [5] independently generalized the above theorem by giving the asymptotic expansion of the Bergman kernel using the Szegö kernel on the unit circle bundle of L^* over M.

Chapter 3

The Bergman kernel

In this chapter, we will introduce the Bergman kernel and and its expansion on the diagonal. The first section will define the Bergman kernel, the second section will introduce the asymptotic expansion and in the third we will give an analysis of it via Tian's peak sections [31].

3.1 The Bergman Kernel

In this section, we introduce the primary object of our study, the *Bergman kernel.* We will be concerned with the Bergman kernel on manifolds, which is analogous to the classical Bergman kernel defined on pseudoconvex domains $\Omega \subset \mathbb{C}^n$. Some literature on the classical theory can be found in [1], [15].

3.1.1 Bergman kernel on manifolds

Let (M, ω) be a compact polarized Kähler manifold with a positive line bundle (L, h) such that $\mathrm{Ric}(h) = \omega$. Let h_α be the local representation of the Hermitian metric with respect to a local frame e_L on some neighborhood U_α, that is, $h_\alpha = h(e_L, e_L)$.

By the Hodge theorem, this space is finite-dimensional and a normal family argument shows that the space is a closed subspace of $\mathcal{L}^2(M, L^m)$. We then consider the orthogonal projection $\mathcal{P}_m : \mathcal{L}^2(M, L^m) \to H^0(M, L^m)$.

Definition 3.1.1 (Bergman kernel)**.** The holomorphic integral kernel B_k of the projection is called the *Bergman kernel*, i.e., for $f \in \mathcal{L}^2(M, L^k)$,

$$\mathcal{P}_k(f)(x) = \langle f(y), B_k(y, x)\rangle_{\mathcal{L}^2}$$

If we choose an orthonormal basis $\{S^k_\alpha\}$ of $H^0(M, L^k)$, then the Bergman kernel is given by

$$B_k(x, y) = \sum_\alpha S^k_\alpha(x) \otimes \overline{S^k_\alpha(y)}$$

Definition 3.1.2. Let $B_k(x, x) \in \Gamma(L^k \otimes \overline{L}^k)$ be the Bergman kernel restricted to the diagonal. The *Bergman function*, sometimes called *density function*, $\mathcal{B}(x)$ is given by the point-wise norm, i.e.

$$\mathcal{B}(x) := \|B_m(x, x)\|_{h^m} = |\tilde{B}_m(x, x)|e^{-k\varphi(x)},$$

where $\tilde{B}(x, x)$ is the coefficient function of the Bergman kernel with respect to the frame $e^k_L \otimes \overline{e}^k_L$.

We have an extremal characterization of the Bergman kernel which we will be useful.

Lemma 3.1.1. Let $\mathcal{B}(x)$ be the Bergman function. Then

$$\mathcal{B}(x) = \sup_{\|s\|_{\mathcal{L}^2}=1} \|s(x)\|^2_{h^k}$$

Proof. We have

$$\mathcal{B}(x) = \sum_{i=0}^{N} \|S_i(x)\|^2_{h^k} \geq \|S(x)\|^2_{h^k}$$

for any $\|S(x)\|^2_{\mathcal{L}^2} = 1$, since we can choose an orthonormal basis $\{S_\alpha\}$ with $S_0 = S$. For the converse inequality, let $x \in M$ and consider the subset $Z_x \subset H^0(M, L^k)$ of sections that vanish at x. Since $B(x) > 0$, we know there exists a nonvanishing section $S_0(x) \neq 0$. Then for any $S \in H^0(M, L^k)$, consider the decomposition

$$S(x) = S(x) - \lambda S_0(x) + \lambda S_0(x)$$

where λ is chosen so that $S(x) - \lambda S_0(x) = 0$. We see that Z_x has codimension one. Let S_0 be in the orthogonal complement of Z_x and $\|S_0\|_{\mathcal{L}^2} = 1$ and extend to an orthonormal basis on $H^0(M, L^k)$. Then each section orthogonal to S_0 vanishes at x so

$$\mathcal{B}(x) = \|S_0(x)\|^2_{h^k}$$

which gives us the result. □

We first give a rough upper bound of the Bergman kernel.

Lemma 3.1.2 (Uniform upper bound on Bergman function)**.** Let $\mathcal{B}(x)$ be the Bergman function for the Bergman kernel of $H^0(M, L^k)$. Then there exists C dependent on M, and independent of k and x such that

$$\mathcal{B}(x) \leq Ck^n.$$

Proof. We use the extremal characterization of the Bergman function (4.9). On the compact Kähler manifold M, we fix a finite coordinate cover $\{U_\alpha\}$ and also fix a coordinate $(z^i)_{i=1}^n$ in U_α. For each U_α, we have a local Kähler potential $\varphi(z)$. We can assume that $U = B(0,2)$, and $\sup_{z\in B(0,2)} |D^2\varphi(z)| \leq C$, i.e. the second derivatives are uniformly bounded. Since φ is plurisubharmonic, we can assume the volume form $dV_g = (\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi)^n(z)$ is equivalent to $dV_E(z)$ in $B(0,2)$.

$$
\begin{aligned}
1 &\geq \int_{B(z_0,\frac{1}{\sqrt{k}})} |\tilde{s}(z)|^2 e^{-k\varphi(z)} dV_g \\
&\geq \frac{1}{C_1} \int_{B(z_0,\frac{1}{\sqrt{k}})} |\tilde{s}(z)|^2 e^{-k\varphi(z)} dV_E \\
&\geq \frac{1}{C_1} e^{-\sup_{B(0,2)}|D^2\varphi|} \int_{B(z_0,\frac{1}{\sqrt{k}})} |\tilde{s}(z)|^2 e^{-k\varphi(z_0)-k\varphi_z(z_0)(z-z_0)-k\varphi_{\bar{z}}(z_0)\overline{(z-z_0)}} dV_E \\
&= \frac{1}{C_2} e^{-k\varphi(z_0)} \int_{B(z_0,\frac{1}{\sqrt{k}})} |\tilde{s}(z) e^{-k\varphi_z(z_0)(z-z_0)}|^2 dV_E.
\end{aligned}
$$

Since $\tilde{s}(z)e^{-k\varphi_z(z_0)(z-z_0)}$ is holomorphic, by the mean-value inequality we have

$$
\begin{aligned}
&\frac{1}{C_2} e^{-k\varphi(z_0)} \int_{B(z_0,\frac{1}{\sqrt{k}})} |\tilde{s}(z) e^{-k\varphi_z(z_0)(z-z_0)}|^2 dV_E \\
&\geq \frac{1}{C_3 k^n} e^{-k\varphi(z_0)} |\tilde{s}(z_0)|^2.
\end{aligned}
$$

So we have

$$
e^{-k\varphi(z_0)} |\tilde{s}(z_0)|^2 \leq C_3 k^n,
$$

where C_3 is uniform for any $z_0 \in B(0,1)$ and any $s \in H^0(M, L^k)$. Taking the supremum over all such s and a standard finite cover argument yields the desired result. $\square$

3.1.2 Bergman kernel on the complex projective space

As a concrete example, we will compute the Bergman kernel on the polarized manifold $(\mathbb{CP}^n, \mathcal{O}(1))$.

Consider the open set $U_0 = \{Z_0 \neq 0\}$ of $\mathbb{CP}^n$. The homogeneous coordinates $[1 : z_1 : \cdots : z_n]$ give local coordinates $(z_1, \cdots, z_n)$ for U_0. Since $\mathbb{CP}^N$ is a symmetric space, we only need to consider the Bergman kernel at one point, say $[1 : 0 : \ldots : 0]$. By considering the hyperplane line bundle $\mathcal{O}(1)$ with Hermitian metric $h = \frac{1}{1+|z|^2}$, we consider $\mathbb{CP}^n$ with the Fubini-Study metric, which on U_0 is represented by

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(1 + |z|^2).$$

The tensor powers of the hyperplane line bundle is denoted $\mathcal{O}(k) = \mathcal{O}(1)^k$ and sections of the $\mathcal{O}(k)$ bundle correspond to k-th degree homogeneous polynomials.

To compute the Bergman kernel, we need to find an orthonormal basis for $H^0(\mathbb{CP}^n, \mathcal{O}(m))$. Let $\rho := 1 + |z|^2$ and $dV_0 := \prod_{i=1}^n dz^i \wedge d\overline{z}^i$, where the product is the wedge product. We compute the volume form of the Fubini-Study metric

$$\begin{aligned}
\omega_g^n &= \left(\frac{\sqrt{-1}}{2\pi}\right)^n \left(\frac{\sum dz^i \wedge d\overline{z}^i}{\rho} - \frac{\partial\rho \wedge \overline{\partial}\rho}{\rho^2}\right)^n \\
&= \frac{n!dV_0}{\rho^n} - \frac{n!|z|^2 dV_0}{\rho^{n+1}} \\
&= \left(\frac{\sqrt{-1}}{2\pi}\right)^n \frac{n!dV_0}{\rho^{n+1}}
\end{aligned}$$

where we use the fact that $(\partial\rho \wedge \overline{\partial}\rho)^k = 0$ for $k > 1$.

Next the Hermitian metric h^k on U_0 can be expressed locally as $\left(\frac{1}{\rho}\right)^k$. Let e_L be a frame on H^k, write $z^P = z_0^{p_0} \cdots z_n^{p_n} \otimes e_L$, $P = (p_0, p_1, \ldots p_n) \in \mathbb{Z}^{n+1}$ such that $|P| = \sum p_i = k$, we have

$$
\begin{aligned}
(z^P, z^Q) &= \int_{\mathbb{CP}^n} \frac{z^P \overline{z}^Q}{(\sum |z_i|^2)^{k+n+1}} dV_0 \\
&= \int_{U_0} \frac{z_1^{p_1} \overline{z}_1^{q_1} \cdots z_n^{p_n} \overline{z}_n^{q_n}}{\rho^{k+n+1}} dV_0
\end{aligned}
$$

where $dV_0 = \left(\frac{\sqrt{-1}}{2\pi} \right)^n dz^1 \wedge d\overline{z}^1 \cdots dz^n \wedge d\overline{z}^n$.

By using polar coordinates, we can see that the set $\{z^P\}$ is orthogonal under the $\mathcal{L}^2$ norm. Hence we compute the norm of each monomial and normalize to form an orthonormal set.

Proposition 3.1.1.

$$
\int_{\mathbb{C}^n} \frac{|z_1|^{2p_1} \cdots |z_n|^{2p_n}}{(1 + |z_1|^2 + \ldots + |z_n|^2)^{k+n+1}} dV_0 = \frac{p_0! p_1! \cdots p_n!}{(n+k)!}
$$

Proof. Changing to polar coordinates, $z_i = r_i e^{\sqrt{-1}\theta_i}$, we obtain

$$
\int_{\mathbb{C}^n} \frac{|z_1|^{2p_1} \cdots |z_n|^{2p_n}}{(1 + |z_1|^2 + \ldots + |z_n|^2)^{k+n+1}} dV_0 = \int_0^\infty \cdots \int_0^\infty \frac{r_1^{2p_1+1} \cdots r_n^{2p_n+1}}{(1 + r_1^2 + \cdots + r_n^2)^{(k+n+1)}} dr \cdot \left(\int_0^{2\pi} d\theta_i \right)^n
$$

For convenience, we change variables $s_i = r_i^2$ to get

$$
\frac{1}{2^n} \int_0^\infty \cdots \int_0^\infty \frac{s_1^{p_1} \cdots s_n^{p_n}}{(1 + s_1 + \ldots + s_n)^{(k+n+1)}} dV(s)
$$

Now consider the integral

$$
\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \frac{ds_1 \ldots ds_n}{(1 + t_1 s_1 + \ldots + t_n s_n)^{(n+p_0+1)}} \\
&= \frac{p_0!}{(n+p_0)! t_1 \cdots t_n}
\end{aligned}
$$

Then inductively we see that

$$\frac{\partial^{p_i}}{(\partial t_i)^{p_i}}\left(\frac{p_0!}{(n+p_0)!t_1\cdots t_n}\right)=\frac{(-1)^{p_i}p_0!p_i!}{(n+p_0)!t_1\cdots t_i^{p_i+1}\cdots t_n}$$

On the other hand, we have

$$\begin{aligned}&\frac{\partial^{p_i}}{(\partial t_i)^{p_i}}\left(\int_0^\infty\cdots\int_0^\infty\frac{ds_1\dots ds_n}{(1+t_1s_1+\ldots+t_ns_n)^{(n+p_0+1)}}\right)\\&=\int_{R_+^n}\frac{(-1)^{p_i}(n+p_0+1)(n+p_0+2)\cdots(n+p_0+p_i)s_i^{p_i}}{(1+t_1s_1+\ldots+t_ns_n)^{(n+p_0+p_i+1)}}dV(s)\end{aligned}$$

So

$$\begin{aligned}&\int_{R_+^n}\frac{(-1)^{k-p_0}(n+p_0+1)(n+p_0+2)\cdots(n+k)s_1^{p_1}\cdots s_n^{p_n}}{(1+s_1+\ldots s_n)^{(n+k+1)}}dV(s)\\&=\left(\frac{\partial}{\partial t_1}\right)^{p_1}\cdots\left(\frac{\partial}{\partial t_n}\right)^{p_n}\int_0^\infty\cdots\int_0^\infty\frac{ds_1\dots ds_n}{(1+t_1s_1+\ldots+t_ns_n)^{(n+p_0+1)}}\Bigg|_{(t_i=0)}\\&=\frac{(-1)^{(k-p_0)}(p_0)!(p_1)!\cdots(p_n)!}{(n+p_0)!}\end{aligned}$$

and the result follows. □

Using the above, we see that the set

$$S^P=\sqrt{\frac{(k+n)!}{p_0!\cdots p_n!}}z^Pe_L,\quad z^P=z_0^{p_0}\cdots z_n^{p_n},\quad P=(p_0,\dots,p_n),\quad |P|=p_0+\cdots+p_n=k$$

forms an orthonormal basis for $H^0(\mathbb{CP}^n,\mathcal{O}(k))$. The Bergman kernel is then given by

$$B_k(x,y)=\sum_{|P|=k}S^P(x)\otimes\overline{S}^P(y)=\sum_{|P|=k}\frac{(k+n)!}{P!}x^P\overline{y}^Pe_L\otimes\overline{e}_L$$

Computing the Bergman function, we have

$$\mathcal{B}(x) = \sum_{|P|=k} \frac{(k+n)!}{P!} |x^P|^2$$

Which at the origin is $\frac{(k+n)!}{k!}$. Expanding out the terms in term of the power of k, we have

$$\frac{(k+n)!}{k!} = k^n \left(1 + \frac{1}{2k} n(n+1) + O(\frac{1}{k^2})\right)$$

where we see that the second coefficient is indeed $1/2$ of the scalar curvature of $\mathbb{CP}^n$.

3.2 Asymptotic expansion

In this section, we discuss the asymptotic expansion of the Bergman kernel. We begin with the following theorem

Theorem 3.2.1 (Zelditch [33], Catlin [5])**.** There is a complete asymptotic expansion:

$$\sum_{\alpha=0}^{N} \|S_\alpha^k(x)\|_{h^k}^2 = k^n \left(a_0(x) + \frac{a_1(x)}{k} + \frac{a_2(x)}{k^2} + \cdots\right)$$

for smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely, for any m

$$\left\| \sum_{\alpha=0}^{N} \|S_\alpha^k(x)\|_{h^m}^2 - \sum_{j<R} a_j(x) m^{n-j} \right\|_{C^m} \leq C_{R,m} k^{n-R}$$

where $C_{R,m}$ depends on R, m and the manifold M.

The algorithm to compute the coefficients were given by Lu [19], who also determined the coefficients to be polynomials in the metric and covariant derivatives of the curvature, and computed the first 4 coefficients, and extended his computation with Shiffman to the near-diagonal terms. For the near diagonal expansion, it is given in powers of $\frac{1}{\sqrt{k}}$.

As a corollary of the asymptotic expansion, we give a quick proof of Tian's theorem, Theorem 2.3.1.

Proof of Theorem 2.3.1. Let $\{S_i\}$ be an orthonormal basis of $H^0(M, L^k)$ and let $\varphi : M \to \mathbb{CP}^N$ be the associated Kodaira map, see (2.6). Consider homogeneous coordinates $[Z_0 : \cdots : Z_N]$ and open set $U_0 = \{Z_0 \neq 0\}$. Then the pullback of the Fubini Study metric by φ is given in local coordinates by

$$\begin{aligned}
\varphi^*\omega_{FS} &= \varphi^* \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(1 + \sum_{i=1}^{n} |z_i|^2\right) \\
&= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(\frac{|S_0|^2}{|S_0|^2} + \frac{|S_1|^2}{|S_0|^2} + \cdots + \frac{|S_N|^2}{|S_0|^2}\right) \\
&= -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(|S_0|^2_{h^k}) + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log B_k \\
&= k\omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log B_k
\end{aligned}$$

where the last line is due to the fact that we are considering a polarized manifold. Using the asymptotic expansion, we have

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log B_k = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(k^n\left(1 + \frac{\rho}{k} + O\left(\frac{1}{k^2}\right)\right)\right) = O\left(\frac{1}{k}\right)$$

Combining with the above, we have

$$\frac{1}{k}\varphi^*\omega_{FS} - \omega = O\left(\frac{1}{k^2}\right) \qquad \text{in } C^\infty$$

$\square$

Another immediate corollary that can be obtained is an approximation of the dimension of space $H^0(M, L^k)$ for high tensor powers. More precisely,

Corollary 3.2.1. As $k \to \infty$, we have

$$\dim H^0(M, L^k) = k^n \int_M \frac{\omega^n}{n!} + \frac{k^{n-1}}{2} \int_M \rho \frac{\omega^n}{n!} + O(k^{n-2})$$

Proof.

$$\begin{aligned}
\dim H^0(M, L^k) &= \sum_{i=0}^{N} \int_M |S_i|^2_{h^k} \\
&= \int_M B_k \frac{\omega^n}{n!} \\
&= k^n \int_M \left(1 + \frac{\rho}{2k} + O\left(\frac{1}{k^2}\right)\right) \frac{\omega^n}{n!}
\end{aligned}$$

□

3.3 Peak Sections

We now introduce Tian's peak section method [31] to compute the coefficients of the asymptotic expansion. The peak sections were originally introduced by Tian to prove the following theorem approximating polarized metrics by a sequence of Bergman metrics:

Let $x_0 \in M$. Choose local normal coordinates $(z_i)_{i=1}^n$ centered at x_0 such that the Hermitian matrix $(g_{i\bar{j}})$ satisfies

$$\begin{aligned}
&g_{i\bar{j}}(x_0) = \delta_{ij} \\
&\frac{\partial^{p_1+\cdots+p_n} g_{i\bar{j}}(x_0)}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}} = 0
\end{aligned}$$

for $i, j = 1, \ldots, n$ and any nonnegative integers $p_1, \ldots, p_n$ with $p_1 + \cdots + p_n \neq 0$. Next choose a local holomorphic frame e_L of L at x_0 such that the local representation function $\tilde{h}$ of the

Hermitian metric h has the properties

$$\tilde{h}(x_0) = 1, \quad \frac{\partial^{p_1+\cdots+p_n}\tilde{h}}{\partial z_1^{p_1}\cdots\partial z_n^{p_n}}(x_0) = 0$$

for any nonnegative integers $(p_1, \ldots, p_n)$ with $p_1 + \cdots + p_n \neq 0$.

Lemma 3.3.1 (Peak Sections)**.** Let c_k be a sequence such that

$$\lim_{k\to\infty} \frac{c_k}{\log(k)^\alpha} = 1, \quad \alpha > 1$$

Suppose $\mathrm{Ric}(g) \geq -K\omega_g$, where $K > 0$ is a constant. For $P \in \mathbb{Z}_+^n$ and an integer $p' > |P|$. Then there is a $k > 0$ and a holomorphic section $S_{P,k} \in H^0(M, L^k)$, satisfying

$$\left| \int_M ||S_{P,k}||^2_{h^k} dV_g - 1 \right| \leq C e^{-\frac{1}{8}c_k} \tag{3.1}$$

Moreover, $S_{P,k}$ can be decomposed as

$$S_{P,k} = \tilde{S}_{P,k} - u_{P,k}$$

such that

$$\tilde{S}_{P,k}(x) = \lambda_P \eta\left(\frac{k|z|^2}{c_k}\right) z^P e_L^k = \begin{cases} \lambda_P z^P e_L^k & x \in \{|z| \leq \sqrt{\frac{c_k}{2k}} \\ 0 & x \in M \setminus \{|z| \leq \sqrt{\frac{c_k}{k}}\} \end{cases}$$

and

$$\int_M ||u_{P,k}||^2_{h_k} dV_g \leq C e^{-\frac{1}{8}c_k}$$

where η is a smoothly cut-off function

$$\eta(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq \frac{1}{2} \\ \\ 0 & \text{for } t \geq 1 \end{cases}$$

satisfying $-4 \geq \eta'(t) \geq 0$ and $|\eta''(t)| \leq 8$ and

$$\lambda_P^{-2} = \int_{|z| \leq \sqrt{\frac{c_k}{k}}} |z^P|^2 a^k dV_g$$

Proof. Define the weight function

$$\Psi(z) = (n + 2p)\eta \left(\frac{k|z|^2}{c_k} \right) \log \left(\frac{k|z|^2}{c_k} \right)$$

Where $\eta : \mathbb{R} \to \mathbb{R}$ is a smooth cut-off function such that

$$\eta(t) = \begin{cases} 1 & \text{for } t < \frac{1}{2} \\ \\ 0 & \text{for } t \geq 1 \end{cases}$$

satisfying $-4 \geq \eta'(t) \geq 0$ and $|\eta''(t)| \leq 8$ and $|z|^2 = |z_1|^2 + \ldots + |z_n|^2$. Directly computing we have

$$\begin{aligned} \sqrt{-1}\partial\overline{\partial}\Psi = \sqrt{-1}(n + 2p) \Bigg\{ \Bigg[& \eta'' \left(\frac{k|z|^2}{c_k} \right) \frac{k^2}{c_k^2} \partial|z|^2 \wedge \overline{\partial}|z|^2 \\ & + \eta' \left(\frac{k|z|^2}{c_k} \right) \frac{k}{c_k} \partial\overline{\partial}|z|^2 \Bigg] \log \left(\frac{k|z|^2}{c_k} \right) \\ & + 2 \operatorname{Re} \left[\eta' \left(\frac{k|z|^2}{c_k} \right) \frac{k}{c_k} \partial|z|^2 \wedge \overline{\partial} \log |z|^2 \right] + \eta \left(\frac{k|z|^2}{c_k} \right) \partial\overline{\partial} \log |z|^2 \Bigg\} \end{aligned}$$

To obtain a lower bound, we first note that $\sqrt{-1}\partial\overline{\partial} \log(|z|^2)$ is positive definite, hence can be dropped. According to the support of the cut-off function η and its derivatives, we restrict our attention to the interval $\frac{c_k}{2k} \leq |z|^2 \leq \frac{c_k}{k}$. In that interval, $\log\left(\frac{k|z|^2}{c_k}\right) \leq 0$, hence

$$\eta'\left(\frac{k|z|^2}{c_k}\right)\log\left(\frac{k|z|^2}{c_k}\right)\sqrt{-1}\partial\overline{\partial}|z|^2 \geq 0,$$

thus can be dropped. Continuing the computation, we have

$$\begin{aligned}
\sqrt{-1}\partial\overline{\partial}\Psi(z) &\geq -\frac{8k(n+2p)}{c_k}\sqrt{-1}\left(\frac{k}{c_k}\partial|z|^2 \wedge \overline{\partial}|z|^2 + \partial|z|^2 \wedge \overline{\partial}\log|z|^2\right) \\
&\geq -\frac{24k^2(n+2p)}{c_k^2}\sqrt{-1}\partial|z|^2 \wedge \overline{\partial}|z|^2 \\
&\geq -\frac{24k(n+2p)}{c_k}\sqrt{-1}dz^i \wedge d\overline{z}^j \\
&= -\frac{48\pi k(n+2p)}{c_k}\omega_g.
\end{aligned}$$

Using the above, we have for any unit vector $v \in T^{1,0}M$ and any point $p \in M$,

$$\left\langle \partial\overline{\partial}\Psi + \frac{2\pi}{\sqrt{-1}}\left(\mathrm{Ric}(h^k) + \mathrm{Ric}(g)\right), v \wedge \overline{v}\right\rangle \geq \frac{k}{4}\|v\|_g^2$$

For k sufficiently large.

Define the following 1-form

$$w_P = \frac{1}{4}\overline{\partial}(\eta\left(\frac{k|z|^2}{c_k}\right))z_1^{p_1}\cdots z_n^{p_n}e_L^k$$

which will serve as the main portion of the peak section. By solving the equation $\overline{\partial}u_P = w_P$, we obtain an L^k-valued section u_P such that

$$\int_M \|u_P\|_{h^k}^2 e^{-\Psi}dV_g \leq \frac{4}{k}\int_M \|w_P\|_{h^k}^2 e^{-\Psi}$$

Computing out the right hand side to get a more explicit form of the bound, we have

$$
\begin{aligned}
\frac{4}{k}\int_M ||w_P||^2_{h^k} e^{-\Psi} dV_g &= \frac{4}{k}\int_M ||\overline{\partial}\eta\left(\frac{k|z|^2}{c_k}\right) z^P e_L^k||^2_{h^k} e^{-\Psi} dV_g \\
&\leq \frac{4}{k}\int_M ||\eta'\left(\frac{k|z|^2}{c_k}\right)\frac{kz_i}{c_k} z^P d\overline{z}^i e_L^k||^2 e^{-\Psi} dV_g \\
&\leq \frac{16k}{c_k^2}\int_{\frac{c_k}{2k}\leq |z|^2 \leq \frac{c_k}{k}} |z|^{2p+2} h^k e^{-\Psi} dV_g \\
&\leq C\frac{c_k^{p-1}}{k^p}\int_{\frac{c_k}{2k}\leq |z|^2 \leq \frac{c_k}{k}} h^k dV_0.
\end{aligned}
$$

For k large, we expand h^k in K-coordinates

$$h^k = e^{-k\varphi} = e^{-k(|z|^2 + O(|z|^4))}$$

so that

$$\int_M \|u_P\|^2_{h^k} e^{-\Psi} dV_g \leq C\frac{c_k^{p-1+n}}{k^{p+n}} e^{-\frac{c_k}{2}}.$$

Since $c_k \sim (\log k)^\alpha$, with $\alpha > 1$, for sufficiently large k, we can obtain the order $O\left(\frac{1}{k^p}\right)$ for any desired p. $\square$

Using these peak sections, Tian proved a convergence theorem for Bergman metrics. In computing the coefficient of the Bergman kernel, we require an orthonormal basis. However, if we only want to compute up to a certain order, we only require that the sections be "almost orthogonal". In that direction, we have a lemma by Tian [31] and generalized by Ruan [26]

Lemma 3.3.2. Let $x_0 \in M$ be fixed. Let S_P be a peak section and $T \in H^0(M, L^k)$. Locally, $T = fe_L^k$ for a holomorphic function f.

1. If z^P is not in the Taylor expansion of f at x_0, then

$$(S_P, T)_{h^k} = O\left(\frac{1}{k}\right) \|T\|_{\mathcal{L}^2}.$$

2. If f contains no terms z^Q such that $q < p + \sigma$ in its Taylor expansion at p, then

$$(S_P, T)_{h^k} = O\left(\frac{1}{k^{1+\sigma/2}}\right) \|T\|_{\mathcal{L}^2}.$$

3.4 Reduction of the Problem

The following reduction used to compute the asymptotic expansion has been given in [19], and is included here for completeness.

Let $\{S_0, \ldots, S_d\}$ be a basis for $H^0(M, L^k)$ such that at $x_0 \in M$,

$$\begin{cases} S_0(x_0) \neq 0 \\ S_i(x_0) = 0 \quad \text{for } i = 1, \ldots d \end{cases} \tag{3.2}$$

Let $S = (S_0, \ldots, S_d)$ and define the Gram matrix $F = S^\dagger S$, where the entries are

$$F_{ij} = \langle S_i, S_j \rangle_{\mathcal{L}^2} \tag{3.3}$$

It is positive definite, since $xFx^\dagger = \|Sx^\dagger\|^2$, hence admits a decomposition

$$F = GG^\dagger \tag{3.4}$$

Let $H = G^{-1}$. Then the entries of SH, i.e.

$$\{\sum_{j=0}^{d} H_{ij} S_j\}, \quad i = 0, \ldots, d \tag{3.5}$$

form an orthonormal basis of $H^0(M, L^k)$. Using this orthonormal basis, the Bergman kernel at x_0 can be reduced to

$$\sum_i \| \sum_j H_{ij} S_j(x_0)\|^2_{h^k} = \sum_i |H_{i0}|^2 \|S_0(x_0)\|^2_{h^k} \tag{3.6}$$

Furthermore, if $I = F^{-1}$, then the $(0,0)$th entry is given by

$$I_{00} = \sum_i |H_{i0}|^2. \tag{3.7}$$

Hence to compute the asymptotic expansion, we only need to estimate I_{00} and $\|S_0(x_0)\|^2_{h^k}$ to the desired order.

3.5 Computation up to first order

With the above considerations, we will compute the diagonal asymptotic expansion up to first order. It was shown in [19] that $I_{00} = 1 + O\left(\frac{1}{k^3}\right)$ hence to compute up to the first order, we only need to compute λ_0^2. We will need the following integral identity:

Lemma 3.5.1 (Lemma 4.1 [19]). Let A be a symmetric function on $\{1, \ldots, n\}^p \times \{1, \ldots, n\}^p$. Then for any $p' > 0$,

$$\begin{aligned}
&\sum_{I,J} \int_{|z| \le \frac{\log k}{\sqrt{k}}} A_{I,J} z_{i_1} \cdots z_{i_p} \overline{z_{j_1} \cdots z_{j_p}} |z|^{2q} e^{-k|z|^2} dV_0 \\
&\quad = \left(\sum_I A_{I,\bar{I}} \right) \frac{p!(n+p+q-1)!}{(p+n-1)! m^{n+p+q}} + O\left(\frac{1}{m^{p'}}\right),
\end{aligned}$$

where $I = (i_1, \ldots, i_p), J = (j_1, \ldots, j_p)$ and $1 \leq i_1, \ldots, i_p, j_1, \ldots, j_p \leq n$.

Under normal coordinates and frame,

$$\begin{aligned}
\|S_0\|^2_{h^k}(x_0) = \lambda^2 &= \int_{|z|^2 \leq \frac{c_k}{k}} h^k dV_g \\
&= \int_{|z|^2 \leq \frac{c_k}{k}} e^{-k\varphi} \det(g) dV_0 \\
&= \int_{|z|^2 \leq \frac{c_k}{k}} e^{-k\varphi} e^{\log\det(g)} dV_0
\end{aligned}$$

Taking the Taylor expansions with normal coordinates gives

$$\varphi(z) = |z|^2 - \frac{\mathrm{Rm}_{i\bar{j}k\bar{l}}}{4} z^i z^k \overline{z}^j \overline{z}^l + O(|z|^5)$$

and

$$\log\det g = -\operatorname{Ric}_{i\bar{j}} z^i \overline{z}^j + O(|z|^3).$$

Therefore

$$e^{-k\varphi(z)} = e^{-k|z|^2} e^{\frac{k}{4}\mathrm{Rm}_{i\bar{j}k\bar{l}} z^i z^k \overline{z}^j \overline{z}^l} = e^{-k|z|^2}(1 + \frac{k}{4}\mathrm{Rm}_{i\bar{j}k\bar{l}} z^i z^k \overline{z}^j \overline{z}^l) + O(|z|^5)),$$

and

$$e^{\log\det g} = 1 - \operatorname{Ric}_{i\bar{j}} z^i \overline{z}^j + O(|z|^3).$$

Applying Lemma 3.5.1 and computing up to $\frac{1}{k}$ order,

$$\begin{aligned}&\int_{|z|\leq \frac{\log k}{\sqrt{k}}} (1 - \mathrm{Ric}_{i\bar{j}}\, z^i \overline{z}^l + \frac{k}{4}\,\mathrm{Rm}_{i\bar{j}k\bar{l}}\, z^i z^k \overline{z}^j \overline{z}^l) e^{-k|z|^2} \\ &\quad = \frac{1}{k^n}(1 - \frac{\rho}{2k}) + O\left(\frac{1}{k^{n+2}}\right).\end{aligned}$$

Inverting the above we have that

$$\lambda_0^2 = k^n \left(1 + \frac{\rho}{2k} + O\left(\frac{1}{k^2}\right)\right).$$

Chapter 4

Near-Diagonal Expansion

In this chapter, we provide a proof of the near-diagonal expansion of the Bergman kernel via a perturbation method given in [14]. We use the observation that the Bergman kernel is concentrated in the near-diagonal. The first section will review the calculus of the Bargmann-Fock space. In the second section, we begin by establishing the local setting. There we will construct our candidate local kernel. The third section will show that the difference between the candidate kernel with the asymptotic expansion and the global kernel differ by an decaying term. In the last section, we show the smooth convergence of the asymptotic expansion.

4.1 Bargmann-Fock Model

The *Bargmann-Fock space* is the space of entire functions that satisfy the weighted square integrability condition:

$$\int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} dV < \infty.$$

The space $\mathcal{F}$ is precisely $H^0(\mathbb{C}^n, |z|^2)$, and is thus a closed linear subspace of the space $\mathcal{L}^2(\mathbb{C}^n, |z|^2)$ with inner product given by

$$\langle f, g \rangle_{\mathcal{F}} := \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-|z|^2} dV,$$

and thus is a Hilbert space. In fact, it is a *reproducing kernel Hilbert space* on $\mathbb{C}^n$, with reproducing kernel

$$\mathcal{R}_{\mathbb{C}^n}(u, v) := e^{u \cdot \overline{v}}.$$

We first show that this kernel has the reproducing property on $\mathbb{C}$ and then extend this argument to $\mathbb{C}^n$.

Lemma 4.1.1. On $\mathbb{C}$, the Bargmann-Fock kernel is given by

$$\mathcal{R}_{\mathbb{C}}(u, v) := e^{u\overline{v}}.$$

Proof. Taking some $f \in H^0(\mathbb{C})$, we consider the inner product against $\mathcal{R}_{\mathbb{C}}$. We convert the resulting integral to polar coordinates and then apply the Cauchy Integral Formula to obtain

$$\begin{aligned}
\langle f(v), \mathcal{R}_{\mathbb{C}} \rangle_{\mathcal{F}} &= \sqrt{-1} \int_{\mathbb{C}} f(v) e^{u\overline{v} - |v|^2} \frac{dv \wedge d\overline{v}}{2\pi} \\
&= -\frac{1}{\pi} \int_0^\infty \int_0^{2\pi} f(u + re^{i\theta}) e^{u(\overline{u} + re^{-i\theta}) - |u + re^{i\theta}|^2} \frac{r}{2} d\theta dr \\
&= -\frac{1}{\pi} \int_0^\infty re^{-r^2} \int_0^{2\pi} f(u + re^{i\theta}) e^{-\overline{u}re^{i\theta}} d\theta dr \\
&= -f(u) \int_0^\infty 2re^{-r^2} dr \\
&= f(u).
\end{aligned}$$

The result follows. ∎

Corollary 4.1.1. On $\mathbb{C}^n$, the Bargmann-Fock kernel is given by

$$\mathcal{R}_{\mathbb{C}^n}(u,v) := e^{u\cdot\overline{v}}.$$

Proof. Let $u, v \in \mathbb{Z}_+^n$ with $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$. Observe that

$$e^{u\cdot\overline{v}} = \prod_{i=1}^{n} e^{u_i\overline{v_i} - |v_i|^2}.$$

To demonstrate the reproducing property, we consider $f \in H^0(\mathbb{C}^n)$ and decompose the integrand of the resulting inner product agains $\mathcal{R}_{\mathbb{C}^n}$. Applying Lemma **??** to each dimensional component, we have

$$\begin{aligned}
\langle f(v), \mathcal{R}_{\mathbb{C}^n}\rangle_{\mathcal{F}} &= \int_{\mathbb{C}^n} f(v) e^{u\cdot v - |v|^2} dV \\
&= \int_{\mathbb{C}^n} f(v_1, \ldots, v_n) \left(\prod_{i=i}^{n} e^{u_i\overline{v_i} - |v_i|^2} \right) dV \\
&= f(u).
\end{aligned}$$

The result follows. □

The following lemmas demonstrate the Bargmann-Fock projection of monomials of different variables.

Lemma 4.1.2. Given some multiindex $m \in \mathbb{Z}_+^n$ the following equality holds.

$$\int_{\mathbb{C}^n} \overline{v}^m e^{u\cdot\overline{v} - |v|^2} dV = 0.$$

Proof. By manipulation and an application of Dominated Convergence Theorem,

$$\begin{aligned}\int_{\mathbb{C}^n} \overline{v}^m e^{u\cdot\overline{v}-|v|^2} dV &= \int_{\mathbb{C}^n} \partial_u^{(m)} \left[e^{u\cdot\overline{v}-|v|^2}\right] dV \\ &= \partial_u^{(m)} \left[\int_{\mathbb{C}^n} e^{u\cdot\overline{v}-|v|^2} dV\right] \\ &= 0.\end{aligned}$$

Note that the integral is constant with respect to u, hence the derivative vanishes. The result follows. $\square$

Lemma 4.1.3. The following equality holds, for $p, q \in \mathbb{Z}_+$ with $p \leq q$.

$$\int_{\mathbb{C}^n} \overline{v}^p v^q e^{u\cdot\overline{v}-|v|^2} dV = \begin{cases} 0 & \text{if } p > q, \\ \frac{q!}{(q-p)!} u^{q-p} & \text{if } p \leq q. \end{cases}$$

Proof. Again by manipulation and an application of Dominated Convergence Theorem,

$$\begin{aligned}\int_{\mathbb{C}^n} \overline{v}^p v^q e^{u.\overline{v}-|v|^2} dV &= \int_{\mathbb{C}^n} \partial_u^{(p)} \left[v^q e^{u\cdot\overline{v}-|v|^2}\right] dV \\ &= \partial_u^{(p)} \left[\int_{\mathbb{C}^n} v^q e^{u\cdot\overline{v}-|v|^2} dV\right] \\ &= \partial_u^{(p)} \left[u^q\right],\end{aligned}$$

therefore

$$\int_{\mathbb{C}^n} \overline{v}^p v^q e^{u\cdot\overline{v}-|v|^2} dV = \begin{cases} 0 & \text{if } p > q, \\ \frac{q!}{(q-p)!} u^{q-p} & \text{if } p \leq q. \end{cases}$$

Result follows. $\square$

4.2 Local Construction

The idea is to first construct a local kernel, then use Hörmander's L^2-estimates to compare the local "candidate" Bergman kernel, which by construction will be in the form of an asymptotic expansion, with the global Bergman kernel. To make precise what we mean by local construction, we define what we mean by *local reproducing property* (modulo $k^{-\frac{N+1}{2}}$).

Definition 4.2.1 (Local Reproducing Property)**.** A function $Q_N(x,y)$ on $U_{x_0} \times U_{x_0}$ which is holomorphic in x, antiholomorphic in y, is a *local reproducing kernel modulo* $k^{-\frac{N+1}{2}}$ on U_{x_0}, if it satisfies the following *local reproducing property* (modulo $k^{-\frac{N+1}{2}}$)

$$f(x) = \langle \chi_k(y) f(y), Q_N(y,x) \rangle_{\mathcal{L}^2(U_x, k\varphi)} + O\left(k^{n-\frac{N+1}{2}}\right) \|f\|_{\mathcal{L}^2(U_x,k\varphi)}, \ \ f \in H^0(U_{x_0}),$$

where χ_k is a cutoff function supported in a scale ball of radius $k^{-\frac{1}{4}-\epsilon}$ for some $\varepsilon > 0$

The key here is to show that it is possible to construct such kernel for any $N > 0$ to be able to say that Bergman kernel has an asymptotic expansion. To construct such a local reproducing kernel, we need to consider Böchner coordinates and frames,

Definition 4.2.2 (Böchner coordinates)**.** Let $x_0 \in M$ and U_{x_0} be a sufficiently small neighborhood which admits a local holomorphic frame e_L of L. Define the local Kähler potential φ by

$$h(e_L, e_L) = e^{-\varphi}.$$

Böchner coordinates $(z_i)_{i=1}^n$ centered at x_0 are special coordinates in which φ admits the following form

$$\varphi(z) = |z|^2 + R(z), \qquad R(z) = O(|z|^4) \tag{4.1}$$

On a Kähler manifold, we can always find holomorphic Böchner coordinates.

To construct the local reproducing kernel modulo N at a point $x_0 \in M$, we begin by choosing Böchner coordinates and a local trivialization of the bundle. Our cutoff function χ_k is chosen to have shrinking support on $B(k^{-\frac{1}{4}-\epsilon})$, so that the inner product is localized near the diagonal and also to ensure that the local rescaled Kähler potential admits an asymptotic expansion of the form

$$\varphi\left(\tfrac{v}{\sqrt{k}}\right) := \left|\tfrac{v}{\sqrt{k}}\right|^2 \left(1 + \sum_{j=2}^{\infty} \tfrac{a_j(v,\bar{v})}{\sqrt{k}^j}\right), \ k \to \infty.$$

This expression is an asymptotic perturbation of $\left|\frac{v}{\sqrt{k}}\right|^2$, hence we propose that the local Bergman kernel admits an asymptotic expansion of the form

$$K^{loc}\left(\tfrac{u}{\sqrt{k}}, \tfrac{v}{\sqrt{k}}\right) = k^n e^{u\bar{v}} \left(1 + \sum_{j=2}^{\infty} \tfrac{c_j(u,\bar{v})}{\sqrt{k}^j}\right), \tag{4.2}$$

where the c_j's depends on a_j and satisfy $c_j(u,\bar{v}) = \overline{c_j(v,\bar{u})}$. The reason we propose such an expansion is that if $\phi(x) = |x|^2$, then $a_j = 0$ for all $j \geq 2$ and hence $c_j = 0$ for all $j \geq 2$ yielding $K^{loc}(\frac{u}{\sqrt{k}}, \frac{v}{\sqrt{k}}) = k^n e^{u \cdot \bar{v}}$, which is precisely the rescaling of the kernel of Bargmann-Fock model $k|z|^2$.

4.2.1 Local Kernel

Existence of coefficients

We first establish some notations. For convenience, we write the volume form as

$$\frac{\omega^n}{n!} = \Omega dV$$

Let $a_m^{r,s}$ be the coefficients in the formal power series expansion of the product

$$e^{-kR\left(\frac{v}{\sqrt{k}}\right)}\Omega\left(\frac{v}{\sqrt{k}}\right) = \sum_{m=0}^{\infty}\sum_{r+s=0}^{2m}\frac{a_m^{r,s}v^r\overline{v}^s}{\sqrt{k^m}}, \tag{4.3}$$

where $R(z)$ was defined in (4.1). The following was shown by the author, with Hezari, Kelleher, and Xu in [14]

Proposition 4.2.1 (Existence of coefficients (Proposition 2.1 [14]))**.** There exist unique coefficients $c_j^{p,q} \in \mathbb{C}$ depending only on the Kähler potential φ such that for any polynomial F and any $N \geq 0$,

$$F\left(\frac{u}{\sqrt{k}}\right) = \int_{\mathbb{C}^n} F\left(\frac{v}{\sqrt{k}}\right) e^{u\cdot\overline{v}-|v|^2}\left(\sum_{t=0}^{N}\sum_{m+j=t}\sum_{p,q}\sum_{r,s}\frac{c_j^{p,q}a_m^{r,s}}{\sqrt{k^t}}u^p\overline{v}^q v^r\overline{v}^s\right)dV. \tag{4.4}$$

Furthermore, the coefficients $c_j^{p,q}$ have the following finiteness and "parity property"

1. $c_m^{p,q} = 0$ when $|p+q| > 2m$,
2. $c_m^{p,q} = 0$ when $|p| + |q| \not\equiv_2 m$.

Remark. When $N = 0$ the equation reduces to the reproducing property of the Bargmann-Fock kernel. Also note that the expression in the parenthesis is precisely the truncation up to $k^{-N/2}$ of the product

$$\left(\sum_{j=0}^{\infty}\frac{c_j(u,\overline{v})}{\sqrt{k^j}}\right)e^{-kR\left(\frac{v}{\sqrt{k}}\right)}\Omega\left(\frac{v}{\sqrt{k}}\right).$$

The key point is that there exists coefficients that reproduce the polynomials each time we increase the accuracy of the expansion of the volume form. The full proof is provided in [14] and is purely algebraic relying heavily on the identity given in Lemma 4.1.3.

4.2.2 Estimates of local kernel

In this section, we show a series of estimates necessary to show that the coefficients constructed perturbing the Bargmann-Fock kernel satisfies the local reproducing property.

Proposition 4.2.2 (Local reproducing property)**.** Let $f \in H^0(B)$, and c_j be the quantities as found in Proposition 4.2.1. Then for $u \in B$,

$$f\left(\frac{u}{\sqrt{k}}\right) = \left\langle \chi_k\left(\frac{v}{\sqrt{k}}\right) f\left(\frac{v}{\sqrt{k}}\right), e^{\overline{u}\cdot v}\left(\sum_j^N \frac{c_j\left(v,\bar{u}\right)}{\sqrt{k^j}}\right)\right\rangle_{\mathcal{L}^2\left(B(\sqrt{k}),k\varphi_k(v)\right)}$$
$$+ O\left(k^{n-\frac{N+1}{2}}\right)\|f\|_{\mathcal{L}^2(B,k\varphi)}.$$

To show the above, we need to show that we can approximate each term by its series expansion, and the error term depending on the $\mathcal{L}^2$ norm of the function being reproduced.

Lemma 4.2.1 (Remainder of the exponential term)**.** Let $M = [\frac{N+1}{2\varepsilon}] + 1$. Then for $N \geq 0$ and any $f \in H^0(B)$,

$$\int_{B(\sqrt{k})} \chi_k\left(\tfrac{v}{\sqrt{k}}\right) f\left(\tfrac{v}{\sqrt{k}}\right) e^{u\overline{v}-|v|^2}\left(\sum_{j=0}^N \tfrac{c_j(u,\overline{v})}{\sqrt{k^j}}\right)\left(e^{-kR\left(\frac{u}{\sqrt{k}}\right)} - \left(e^{-kR_{2N+5}\left(\frac{u}{\sqrt{k}}\right)}\right)_M\right)\Omega\left(\tfrac{v}{\sqrt{k}}\right)dV$$
$$= \|f\|_{\mathcal{L}^2(B,k\varphi)} O\left(k^{n-\frac{N+1}{2}}\right).$$

Lemma 4.2.2 (Remainder of determinant)**.** Let $M = [\frac{N+1}{2\epsilon}] + 1$. Then

$$\left|\int_{B(\sqrt{k})} \chi_k\left(\tfrac{v}{\sqrt{k}}\right) f\left(\tfrac{v}{\sqrt{k}}\right) e^{u\overline{v}-|v|^2}\left(\sum_{j=0}^N \tfrac{c_j(u,\overline{v})}{\sqrt{k^j}}\right)\left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_M (\Omega - \Omega_{2N+1})\left(\tfrac{v}{\sqrt{k}}\right)dV\right|$$
$$= \|f\|_{\mathcal{L}^2(B,k\varphi)} O\left(k^{n-\frac{N+1}{2}}\right). \tag{4.5}$$

Lemma 4.2.3 (Estimate outside the ball)**.** Let F be a holomorphic polynomial. Then the following estimate holds:

$$\int_{\mathbb{C}^n} \left(1 - \chi_k\left(\frac{v}{\sqrt{k}}\right)\right) F\left(\frac{v}{\sqrt{k}}\right) e^{u\overline{v}-|v|^2} \left(\sum_{t=0}^{N} \sum_{m+j=t} \frac{c_j(u,\overline{v})a_m(v,\overline{v})}{\sqrt{k^t}}\right) dV$$
$$\leq C_N k^n \|F\|_{\mathcal{L}^2(B,k\varphi)} e^{-\frac{1}{32}k^{\frac{1}{2}-2\varepsilon}}.$$

Given the results above, we prove Proposition 4.2.2.

Proof of Proposition 4.2.2. By Proposition 4.2.1,

$$F\left(\frac{u}{\sqrt{k}}\right) = \int_{\mathbb{C}^n} F\left(\frac{v}{\sqrt{k}}\right) e^{u\cdot\overline{v}-|v|^2} \left(\sum_{t=0}^{N} \sum_{m+j=t} \frac{c_j(u,\overline{v})a_m(v,\overline{v})}{\sqrt{k^t}}\right) dV,$$

for all $N \geq 0$ and all holomorphic polynomials F. We then split the above to two pieces.

$$F\left(\frac{u}{\sqrt{k}}\right) = \int_{\mathbb{C}^n} \left(1 - \chi_k\left(\frac{v}{\sqrt{k}}\right)\right) F\left(\frac{v}{\sqrt{k}}\right) e^{u\cdot\overline{v}-|v|^2} \left(\sum_{t=0}^{N} \sum_{m+j=t} \frac{c_j(u,\overline{v})a_m(v,\overline{v})}{\sqrt{k^t}}\right) dV$$
$$+ \int_{\mathbb{C}^n} \chi_k\left(\frac{v}{\sqrt{k}}\right) F\left(\frac{v}{\sqrt{k}}\right) e^{u\cdot\overline{v}-|v|^2} \left(\sum_{t=0}^{N} \sum_{m+j=t} \frac{c_j(u,\overline{v})a_m(v,\overline{v})}{\sqrt{k^t}}\right) dV.$$

The first integral is bounded above by $C_N k^n \|f\|_{\mathcal{L}^2(B,k\varphi)} e^{-\frac{1}{32}k^{\frac{1}{2}-\varepsilon}}$ from Lemma 4.2.3. For the second integral, we note that since $M = [\frac{N+1}{2\varepsilon}] + 1 > N/4$, and $|u| < 1$, we have

$$\left|\left(\sum_{t=0}^{N} \sum_{m+j=t} \frac{c_j(u,\overline{v})a_m(v,\overline{v})}{\sqrt{k^t}}\right) - \left(\sum_{j=0}^{N} \frac{c_j(u,\overline{v})}{\sqrt{k^j}}\right) \left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_M \Omega_{2N+1}\right|$$
$$\leq C_N k^{-\frac{N+1}{2}} |v|^{4(M+N+1)}.$$

Then by applying this to the second integral, and using the Cauchy-Schwarz inequality we get

$$
\begin{aligned}
&C_N k^{-\frac{N+1}{2}} \left| \int_{\mathbb{C}^n} \chi_k\left(\tfrac{v}{\sqrt{k}}\right) F\left(\tfrac{v}{\sqrt{k}}\right) |v|^{4(M+N+1)} e^{u\bar{v}-|v|^2} dV \right| \\
&\le C_N k^{-\frac{N+1}{2}} \left(\int_{\mathbb{C}^n} \chi_k\left(\tfrac{v}{\sqrt{k}}\right) \left| F\left(\tfrac{v}{\sqrt{k}}\right)\right|^2 e^{-|v|^2} dV \right)^{\frac{1}{2}} \left(\int_{\mathbb{C}^n} \chi_k\left(\tfrac{v}{\sqrt{k}}\right) \left| |v|^{4(M+N+1)} e^{u\bar{v}-\frac{|v|^2}{2}} \right|^2 dV \right)^{\frac{1}{2}} \\
&\le C_N \|F\|_{\mathcal{L}^2(B,k\varphi)} k^{n-\frac{N+1}{2}}.
\end{aligned}
$$

Hence we obtain the estimate,

$$
\begin{aligned}
F\left(\frac{u}{\sqrt{k}}\right) = &\int_{\mathbb{C}^n} \chi_k\left(\frac{v}{\sqrt{k}}\right) F\left(\frac{v}{\sqrt{k}}\right) e^{u\cdot\bar{v}-|v|^2} \left(\sum_{j=0}^{N} \frac{c_j(u,\bar{v})}{\sqrt{k}}\right) \left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_M \Omega_{2N+1} dV \\
&+ O(k^{n-\frac{N+1}{2}})\|F\|_{L^2(B,k\varphi)}.
\end{aligned}
$$

Now by applying Lemma 4.2.2 and Lemma 4.2.1, we have

$$
\begin{aligned}
&\int_{\mathbb{C}^n} \chi_k\left(\tfrac{v}{\sqrt{k}}\right) F\left(\tfrac{v}{\sqrt{k}}\right) e^{u\bar{v}-|v|^2} \left(\sum_{j=0}^{N} \tfrac{c_j(u,\bar{v})}{\sqrt{k^j}}\right) \left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_M \Omega_{2N+1}\left(\tfrac{v}{\sqrt{k}}\right) dV \\
&= \left\langle \chi_k\left(\tfrac{v}{\sqrt{k}}\right) F\left(\tfrac{v}{\sqrt{k}}\right), e^{\bar{u}\cdot v} \left(\sum_{j}^{N} \tfrac{c_j(\bar{u},v)}{\sqrt{k^j}}\right) \right\rangle_{\mathcal{L}^2(B(\sqrt{k}),k\varphi(\frac{v}{\sqrt{k}}))} + \|F\|_{\mathcal{L}^2(B,k\varphi)} O\left(k^{n-\frac{N+1}{2}}\right).
\end{aligned}
$$

We can extend to arbitrary $f \in H^0(B)$ by putting $F = f_L$, letting $L \to \infty$, and using the uniform convergence of f_L. The result follows. □

We end this section with the proofs of Lemma 4.2.1, Lemma 4.2.2, and Lemma 4.2.3.

Proof of Lemma 4.2.1. First note that since $|v| \le k^{\frac{1}{4}-\varepsilon}$ we have

$$
kR\left(\frac{v}{\sqrt{k}}\right) = O(k^{-\varepsilon}).
$$

We regroup the quantity

$$e^{-kR} - e^{-kR_{2N+5}} = e^{-kR}\left(1 - e^{k(R-R_{2N+5})}\right). \tag{4.6}$$

By Taylor expansion

$$\begin{aligned}
k\left|(R - R_{2N+5})\left(\frac{v}{\sqrt{k}}\right)\right| &\le k \sup_{\substack{|\alpha|=2N+6\\ |\xi|\le \frac{|v|}{\sqrt{k}}}} \left|\frac{D^\alpha R(\xi)}{(\alpha)!}\right| \left|\frac{v}{\sqrt{k}}\right|^\alpha \\
&\le C_N k \left(\frac{|v|}{\sqrt{k}}\right)^{2N+6} \\
&\le C_N k^{-\frac{N+1}{2}}.
\end{aligned}$$

Applying the above to (4.6), we have

$$\left|e^{-kR\left(\frac{v}{\sqrt{k}}\right)} - e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right| \le C_N k^{-\frac{N+1}{2}}. \tag{4.7}$$

Next we consider the difference

$$\left|e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)} - \left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_M\right|, \tag{4.8}$$

where M is a fixed constant such that $M \ge \frac{N+1}{2\varepsilon}$.

By $\left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)}\right)_M$ we mean to truncate as

$$\sum_{j=0}^{M} \frac{1}{j!}\left(-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)\right)^j.$$

Hence we have an estimate

$$
\begin{aligned}
\left| e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)} - \left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)} \right)_M \right| &\leq \sup_{|x| \leq |-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)|} \frac{|x|^{M+1}}{(M+1)!} \\
&\leq Ck^{-\varepsilon M+1} \\
&\leq Ck^{-\frac{N+1}{2}}.
\end{aligned}
$$

Combining (4.7) and (4.8), we have

$$
\left| e^{-kR\left(\frac{v}{\sqrt{k}}\right)} - \left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)} \right)_M \right| \leq C_N k^{-\frac{N+1}{2}}.
$$

Applying our estimate directly to the integral,

$$
\begin{aligned}
&\left| \int_{\mathbb{C}^n} \chi_k \left(\frac{v}{\sqrt{k}}\right) f\left(\frac{v}{\sqrt{k}}\right) e^{u\bar{v}-|v|^2} \left(\sum_{j=0}^{N} \frac{c_j(u,\bar{v})}{\sqrt{k^j}} \right) \left(e^{-kR\left(\frac{v}{\sqrt{k}}\right)} - \left(e^{-kR_{2N+5}\left(\frac{v}{\sqrt{k}}\right)} \right)_M \right) \Omega\left(\frac{v}{\sqrt{k}}\right) dV \right| \\
&\leq C_N k^{-\frac{N+1}{2}} \int_{\mathbb{C}^n} \left| \chi_k \left(\frac{v}{\sqrt{k}}\right) f\left(\frac{v}{\sqrt{k}}\right) e^{-\frac{|v|^2}{2}} \left(\sum_{j=0}^{N} \frac{c_j(u,\bar{v})}{\sqrt{k^j}} \right) \Omega\left(\frac{v}{\sqrt{k}}\right) e^{u\bar{v}-\frac{|v|^2}{2}} \right| dV \\
&\leq C_N k^{-\frac{N+1}{2}} \left(\int_{\mathbb{C}^n} \chi_k \left(\frac{v}{\sqrt{k}}\right) \left| f\left(\frac{v}{\sqrt{k}}\right) \right|^2 e^{-|v|^2} dV \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\mathbb{C}^n} \chi_k \left(\frac{v}{\sqrt{k}}\right) \left| \left(\sum_{j=0}^{N} \frac{c_j(u,\bar{v})}{\sqrt{k^j}} \right) \Omega\left(\frac{v}{\sqrt{k}}\right) e^{u\bar{v}-\frac{|v|^2}{2}} \right|^2 dV \right)^{\frac{1}{2}} \\
&\leq C_N k^{-\frac{N+1}{2}} \left(\int_{\mathbb{C}^n} \chi_k \left(\frac{v}{\sqrt{k}}\right) \left| f\left(\frac{v}{\sqrt{k}}\right) \right|^2 e^{-k\varphi\left(\frac{v}{\sqrt{k}}\right)} dV \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\mathbb{C}^n} \chi_k \left(\frac{v}{\sqrt{k}}\right) \left| \left(\sum_{j=0}^{N} \frac{c_j(u,\bar{v})}{\sqrt{k^j}} \right) \Omega\left(\frac{v}{\sqrt{k}}\right) e^{u\bar{v}-\frac{|v|^2}{2}} \right|^2 dV \right)^{\frac{1}{2}} \\
&\leq C_N \|f\|_{\mathcal{L}^2(B,k\varphi)} k^{n-\frac{N+1}{2}}.
\end{aligned}
$$

The result follows. $\square$

Proof of Lemma 4.2.2. We first observe the following estimate

$$\left|(\Omega-\Omega_{2N+1})\left(\frac{v}{\sqrt{k}}\right)\right| \leq \sup_{\substack{|\alpha|=2N+2\\ |\xi|\leq\left|\frac{v}{\sqrt{k}}\right|}}\left|\frac{D^{\alpha}\Omega(\xi)}{\alpha!}\right|\left|\frac{v}{\sqrt{k}}\right|^{2N+2} \leq C_N k^{-\frac{N+1}{2}}.$$

Using the above estimate with a similar manipulation as Lemma 4.2.1 we conclude (4.5). □

Proof of Lemma 4.2.3. First note that since $|u|\leq 1$ and $|v|\geq \frac{1}{2}k^{\frac{1}{4}-\varepsilon}$, we have $|u-v|\geq \frac{1}{4}k^{\frac{1}{4}-\varepsilon}$. Next we use the identity

$$\overline{\partial}\left(\sum_i e^{\bar{v}(u-v)}\frac{1}{u^i-v^i}d\widehat{\overline{V}^i}\right) = -ne^{\bar{v}(u-v)}dV,$$

where $d\widehat{\overline{V}^i} := \left(\frac{\sqrt{-1}}{2\pi}\right)^n dv^1\wedge d\overline{v}^1\wedge\ldots\wedge dv^i\wedge\widehat{d\overline{v}^i}\wedge\ldots\wedge dv^n\wedge d\overline{v}^n$. Integrating by parts, we have

$$\begin{aligned}
&\int_{\mathbb{C}^n}\left(1-\chi_k\left(\frac{v}{\sqrt{k}}\right)\right)F\left(\frac{v}{\sqrt{k}}\right)e^{\bar{v}(u-v)}\left(\sum_{t=0}^{N}\sum_{m+j=t}\frac{c_j(u,\overline{v})a_m(v,\overline{v})}{\sqrt{k^t}}\right)dV\\
&=-\frac{1}{n}\int_{\mathbb{C}^n}\left(1-\chi_k\left(\frac{v}{\sqrt{k}}\right)\right)F\left(\frac{v}{\sqrt{k}}\right)\left(\sum_{t=0}^{N}\sum_{m+j=t}\frac{c_j(u,\overline{v})a_m(v,\overline{v})}{\sqrt{k^t}}\right)\overline{\partial}\left(\sum_i e^{\bar{v}(u-v)}\frac{1}{u^i-v^i}d\widehat{\overline{V}^i}\right)\\
&=-\frac{1}{n}\int_{\mathbb{C}^n}F\left(\frac{v}{\sqrt{k}}\right)\sum_i\overline{\partial}_i\left(\left(1-\chi_k\left(\frac{v}{\sqrt{k}}\right)\right)\left(\sum_{t=0}^{N}\sum_{m+j=t}\frac{c_j(u,\overline{v})a_m(v,\overline{v})}{\sqrt{k^t}}\right)\right)e^{\bar{v}(u-v)}\frac{1}{u^i-v^i}dV.
\end{aligned}$$

Iterating the above integration by parts $2N$ times we obtain

$$=\frac{(-1)^{2N+1}}{n^{2N+1}}\int_{\mathbb{C}^n}F\left(\frac{v}{\sqrt{k}}\right)\sum_{\substack{I=(i_1,\ldots,i_{2N+1})\\ |I|=2N+1}}\partial_{\bar{I}}\left(\left(1-\chi_k\left(\frac{v}{\sqrt{k}}\right)\right)\sum_{t=0}^{N}\sum_{m+j=t}\frac{c_j(u,\overline{v})a_m(v,\overline{v})}{\sqrt{k^t}}\right)\frac{e^{\bar{v}(u-v)}}{(u-v)^I}dV.$$

Since the degrees of a_m and c_j are $2m$ and $2j$ respectively, always one differentiation is applied to $1-\chi_k$. Therefore, the integrand is supported on the annulus $\frac{1}{2}k^{\frac{1}{4}-\varepsilon}\leq|v|\leq k^{\frac{1}{4}-\varepsilon}$.

The above integral is then bounded above by

$$\left(\int_{\frac{1}{2}k^{\frac{1}{4}-\varepsilon}\le|v|\le k^{\frac{1}{4}-\varepsilon}}\left|F\left(\frac{v}{\sqrt{k}}\right)\right|^2 e^{-|v|^2}dV\right.$$
$$\left.\times\int_{\frac{1}{2}k^{\frac{1}{4}-\varepsilon}\le|v|\le k^{\frac{1}{4}-\varepsilon}}\left|\sum_{\substack{I=(i_1,\ldots,i_{2N+1})\\|I|=2N+1}}\partial_{\bar{I}}\left(\left(1-\chi_k\left(\frac{v}{\sqrt{k}}\right)\right)\sum_{t=0}^{N}\sum_{m+j=t}\frac{c_j(u,\bar{v})a_m(v,\bar{v})}{\sqrt{k^t}}\right)\frac{e^{\frac{|u|^2-|u-v|^2}{2}}}{(u-v)^I}\right|^2 dV\right)^{\frac{1}{2}}$$
$$\le C_N k^n\|F\|_{L^2(B,k\varphi)}e^{-\frac{1}{32}k^{\frac{1}{2}-2\varepsilon}}.$$

The result follows □

4.3 Local to Global

Now that we have constructed the local kernel, we will use Hörmander's estimate to construct a global holomorphic section such that the contribution of the terms outside the local neighborhood is negligible to the asymptotic expansion. This will prove the existence of the asymptotic expansion. Recall that the norm of B_k as a section of the bundle $L^k\otimes\bar{L}^k$ is the *Bergman function* $\mathcal{B}$ (Definition 3.1.2). Hence in local coordinates

$$\mathcal{B}(x)=|B_k(x,x)|_{h^k}=|\tilde{B}_k(x,x)|e^{-k\varphi(x)},$$

where $\tilde{B}_k(x,x)$ is the coefficient function of the Bergman kernel with respect to the frame $e_L^k\otimes\bar{e_L}^k$. We also have an extremal characterization of the Bergman function given by

$$\mathcal{B}(x)=\sup_{\|s\|_{\mathcal{L}^2}\le 1}|s(x)|^2_{h^k},\tag{4.9}$$

where $s\in H^0(M,L^k)$.

Let $B_k(x,y)=B_{k,y}(x)$ be the global Bergman kernel of $H^0(M,L^k)$. We view $B_{k,y}(x)$ as a section of $L^k \otimes \overline{L}_y^k$. We shall use $\tilde{B}_k(x,y)$ for the local representation of $B_k(x,y)$ with respect to the frame $e_L(x)^k \otimes \bar{e_L}(y)^k$.

Theorem 4.3.1 (Local to global)**.** The following equality relates the truncated local Bergman kernel

$$B^{loc}_{k,N}\left(\frac{u}{\sqrt{k}},\frac{v}{\sqrt{k}}\right) = k^n e^{u\cdot\bar{v}} \sum_{j=0}^{N} \frac{c_j(u,\overline{v})}{\sqrt{k}^j}$$

to the global Berman kernel $\tilde{B}_k$.

$$\tilde{B}_k\left(\frac{u}{\sqrt{k}},\frac{v}{\sqrt{k}}\right) = B^{loc}_{k,N}\left(\frac{u}{\sqrt{k}},\frac{v}{\sqrt{k}}\right) + O\left(k^{2n-\frac{N+1}{2}}\right).$$

Proof. Fix $u, v \in B$. We apply the local reproducing property to the global Bergman kernel $f(w) = \tilde{B}_{k,\frac{u}{\sqrt{k}}}(w) = \tilde{B}_k(w, \frac{u}{\sqrt{k}})$

$$\tilde{B}_k\left(\frac{v}{\sqrt{k}},\frac{u}{\sqrt{k}}\right) = \left\langle \chi_k(w)\tilde{B}_k\left(w,\frac{u}{\sqrt{k}}\right), B^{loc}_{k,N}\left(w,\frac{v}{\sqrt{k}}\right)\right\rangle_{\mathcal{L}^2(B,k\varphi(w))}$$
$$+ O\left(k^{n-\frac{N+1}{2}}\right)\|\tilde{B}_{k,\frac{u}{\sqrt{k}}}\|_{\mathcal{L}^2(B,k\varphi)}.$$

By the reproducing property, we obtain from Lemma 3.1.2,

$$\|\tilde{B}_{k,\frac{u}{\sqrt{k}}}\|^2_{\mathcal{L}^2(B,k\varphi)} \le \|B_{k,\frac{u}{\sqrt{k}}}\|^2_{\mathcal{L}^2} = \tilde{B}_k\left(\frac{u}{\sqrt{k}},\frac{u}{\sqrt{k}}\right) = \mathcal{B}\left(\frac{u}{\sqrt{k}}\right)e^{k\varphi_k(u)} \le Ck^n,$$

where $B_{k,\frac{u}{\sqrt{k}}}(w)$ means section with respect to w and local coefficient function with respect to u. Thus we have

$$\tilde{B}_k\left(\frac{v}{\sqrt{k}},\frac{u}{\sqrt{k}}\right) = \left\langle \chi_k(w)\tilde{B}_k\left(w,\frac{u}{\sqrt{k}}\right), B^{loc}_{k,N}\left(w,\frac{v}{\sqrt{k}}\right)\right\rangle_{\mathcal{L}^2(B,k\varphi(w))} + O\left(k^{2n-\frac{N+1}{2}}\right).$$

We next estimate the difference of the local Bergman kernel with the projection of the local kernel.

$$
\begin{aligned}
& g_{k,v}\left(\frac{w}{\sqrt{k}}\right) \\
& := \chi_k\left(\frac{w}{\sqrt{k}}\right) B^{loc}_{k,N}\left(\frac{w}{\sqrt{k}}, \frac{v}{\sqrt{k}}\right) - \overline{\left\langle \chi_k(x)\tilde{B}_k\left(x, \frac{w}{\sqrt{k}}\right), B^{loc}_{k,N}\left(x, \frac{v}{\sqrt{k}}\right)\right\rangle}_{\mathcal{L}^2(B, k\varphi(x))} \\
& = \chi_k\left(\frac{w}{\sqrt{k}}\right) B^{loc}_{k,N}\left(\frac{w}{\sqrt{k}}, \frac{v}{\sqrt{k}}\right) - \left\langle \chi_k(x) B^{loc}_{k,N}\left(x, \frac{v}{\sqrt{k}}\right), \tilde{B}_k\left(x, \frac{w}{\sqrt{k}}\right)\right\rangle_{\mathcal{L}^2(B, k\varphi(x))}
\end{aligned}
$$

We can regard $g_{k,v}$ as a global section of L^k because of the cut-off function χ_k. Since

$$
\left\langle \chi_k B^{loc}_{k,N,\frac{v}{\sqrt{k}}}, B_{k,\frac{u}{\sqrt{k}}}\right\rangle_{\mathcal{L}^2} = \mathcal{P}_{H^0}\left(\chi_k B^{loc}_{k,N,\frac{v}{\sqrt{k}}}\right),
$$

where $\mathcal{P}_{H^0}$ is the Bergman projection and $g_{k,v}$ is the $\mathcal{L}^2$-minimal solution to

$$
\overline{\partial} g_{k,v} = \overline{\partial}\left(\chi_k B^{loc}_{k,N,\frac{v}{\sqrt{k}}}\right).
$$

Now we estimate $\overline{\partial}(\chi_k B^{loc}_{k,N,\frac{v}{\sqrt{k}}})$.

$$
\begin{aligned}
\overline{\partial}\left(\chi_k B^{loc}_{k,N,\frac{v}{\sqrt{k}}}\right)\Big|_{\frac{w}{\sqrt{k}}} &= \left(\overline{\partial}\left(\chi_k\right) B^{loc}_{k,N,\frac{v}{\sqrt{k}}} + \chi_k \overline{\partial}\left(B^{loc}_{k,N,\frac{v}{\sqrt{k}}}\right)\right)\Big|_{\frac{w}{\sqrt{k}}} \\
&= \overline{\partial}\left(\chi_k\right) B^{loc}_{k,N,\frac{v}{\sqrt{k}}}\Big|_{\frac{w}{\sqrt{k}}}.
\end{aligned}
$$

We note that $\overline{\partial}\left(B^{loc}_{k,N,\frac{v}{\sqrt{k}}}\right) = 0$ because $B^{loc}_{k,N}$ is holomorphic. The term $\overline{\partial}\left(\chi_k\right)$ on the right hand side ensures $|w-v| \geq \frac{1}{4}k^{\frac{1}{4}-\varepsilon}$, and $|w| \leq k^{\frac{1}{4}-\varepsilon}$. Furthermore, since $B^{loc}_{k,N,\frac{v}{\sqrt{k}}}(\frac{w}{\sqrt{k}}) = O\left(e^{w\bar{v}}|vw|^{2N}\right)$, and because

$$
|e^{w\bar{v}}|^2 e^{-|w|^2} = e^{2\operatorname{Re} w\bar{v} - |w|^2} = e^{-|w-v|^2 + |v|^2} \leq Ce^{-\frac{1}{16}k^{\frac{1}{2}-2\varepsilon}},
$$

we obtain

$$\left\|\overline{\partial}(\chi_k)B^{loc}_{k,N,\frac{v}{\sqrt{k}}}\right\|_{\mathcal{L}^2(M,L^k)} \leq Ce^{-\frac{1}{32}k^{\frac{1}{2}-2\varepsilon}}.$$

So by the Hormänder's $\mathcal{L}^2$-estimate, the following inequality holds uniformly for $v \in B$,

$$\|g_{k,v}\|_{\mathcal{L}^2(M,L^k)} \leq Ce^{-\frac{1}{32}k^{\frac{1}{2}-2\varepsilon}}. \tag{4.10}$$

By the same argument as in the Lemma 3.1.2 above, for all $u \in B$ we obtain the uniform estimate

$$\left|g_{k,v}\left(\frac{u}{\sqrt{k}}\right)\right| \leq Ce^{-\frac{1}{64}k^{\frac{1}{2}-2\varepsilon}}.$$

This concludes the estimate

$$\left|B^{loc}_{k,N}\left(\frac{u}{\sqrt{k}}, \frac{v}{\sqrt{k}}\right) - B_k\left(\frac{u}{\sqrt{k}}, \frac{v}{\sqrt{k}}\right)\right| \leq Ck^{2n-\frac{N+1}{2}},$$

uniformly for all $u, v \in B$. □

4.4 Higher Order Convergence

As the C^m norms depend on the choice of coordinates, we must give some care when discussing the convergence in higher order. The local kernel that we have constructed is an expansion at one point $p \in M$. We now show the regularity of the local kernel depending on the point p.

We have shown that at a point $p \in M$

$$|B_k(p+z, p+w) - B^{loc}_{k,N}(p, z, w)| \leq \frac{C_{p,N}}{k^{N+1-2n}}, \qquad d(z,w) < \frac{1}{\sqrt{k}}.$$

In fact, the $C_{p,N}$ depends on the local potential, that is,

$$C_{p,N} \leq \sup_{\substack{|\alpha| \leq \alpha(N) \\ x \in B_p(2\delta)}} |D^\alpha \varphi(x)|$$

We first would like to show that given a point $q \in B_p(\delta)$, the constant $C_{p,N}$ is uniform in that neighborhood, i.e.

$$|B_k(q+z, q+w) - B^{loc}_{k,N}(q, z, w)| \leq \frac{C_{p,N}}{k^{N+1-2n}}, \qquad d(z,w) < \frac{1}{\sqrt{k}}.$$

Consider a smooth family of Böchner coordinates. The existence of such a coordinate is given, for example in [18]. Then consider a finite cover of M by $B_p(2\delta)$ of fixed radius. Then for $q \in B_p(2\delta)$, we have

$$\sup_{B_q(\delta)} |D^\alpha \varphi| \leq C \sup_{B_p(2\delta)} |D^\alpha \varphi|,$$

where C is independent of q, and the derivatives D^α on the left correspond to the Böchner coordinates centered at p and the right corresponds to the Böchner coordinates centered at q.

To show the convergence for higher order derivatives with respect to the variable p, we first apply the Böchner-Martinelli formula to the difference of the local kernel and the global kernel.

We recall that

Lemma 4.4.1 (Böchner-Martinelli kernel)**.** For $w, z \in \mathbb{C}^n$, we define the Bochner-Martinelli kernel, $M(w,z)$

$$M(w,z) = \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \frac{1}{|z-w|^{2n}} \sum_{1 \le j \le n} (\overline{w}^j - \overline{z}^j) d\overline{w}^1 \wedge dw^1 \wedge \cdots \wedge dw^j \wedge \cdots \wedge d\overline{w}^n \wedge dw^n$$

Suppose that $f \in C^\infty(D)$ where D is a domain in $\mathbb{C}^n$ with piecewise smooth boundary. Then for $z \in D$,

$$f(z) = \int_{\partial D} f(w) M(w,z) - \int_D \overline{\partial} f(w) \wedge M(w,z).$$

Now let $p \in M$ and consider Böchner coordinates $(z^1, \cdots, z^n)$ centered at p. The Bergman kernel and the local kernel are both objects that depend on the base point and two arguments,i.e.

$$K_k(p,z,w) := K_k(p+z, p+w)$$

By polarizing in the p variable and considering the almost holomorphic extension, we may view the kernel as

$$B_k(p,q,z,w) := B_k(p+z, q+w)$$

Let

$$f_k(p,q,z,w) = B_k(p,q,z,w) - B^{loc}_{k,N}(p,q,z,w)$$

be the difference between the global and local kernel. Note that f_k is defined for $q, p+z, q+$

$w \in B_p(\frac{1}{\sqrt{k}})$. From our previous result, we have

$$|f_k(p,p,z,w)| \leq \frac{C_{p,N}}{k^{N+1-2n}}, \qquad d(z,w) < \frac{1}{\sqrt{k}}.$$

We want to estimate

$$|\partial_p^\alpha f_k(p,q)|$$

for $q = p$, where we suppress the z, w variable because it it not essential to the argument. Applying Lemma 4.4.1 to $\partial_p^\alpha f_k(p,q)$ with $D = B_p(\frac{1}{\sqrt{k}}) \times B_q(\frac{1}{\sqrt{k}})$, we obtain

$$\partial_p^\alpha f_k(p,q) = \int_{\partial D} f_k(p',q')\partial_p^\alpha M(p',q',p,q) - \int_D \overline{\partial} f_k(p',q') \wedge \partial_p^\alpha M(p',q',p,q).$$

The boundary integral term can be bounded by the $\mathcal{L}^\infty$-norm of f_k multiplied by $\sqrt{k}^{-|\alpha|}$. By using the fact that f_k is an almost holomorphic extension, $\overline{\partial}\partial_p^\alpha f_k$ in the second integral is bounded by $O_\alpha(|q'-p'|^\infty)$. When $p = q$, we have $d(p',q') < \frac{1}{\sqrt{k}}$, and therefore the second integral is of order $O(k^{-\infty})$.

Now we show the higher order convergence with respect to the z, w variable. We rescale $z \mapsto \frac{u}{\sqrt{k}}$ and $w \mapsto \frac{v}{\sqrt{k}}$ to match the notation as in the statement of our theorem. Since the local kernel $B_{k,N}^{loc}$ and the global Bergman kernel are holomorphic in u and anti-holomorphic in v, the derivatives can be bounded by the L^∞-norms using Cauchy estimates. More precisely, let D_x be any first order differential operator of x. By using the Cauchy estimates on $B_{k,N}^{loc}(x,y)$ and $\tilde{B}_k(x,y)$ on the ball of radius $\frac{1}{\sqrt{k}}$, we obtain

$$\begin{aligned}\left|D_x(B_{k,N}^{loc}(x,\tfrac{v}{\sqrt{k}}) - \tilde{B}_k(x,\tfrac{v}{\sqrt{k}}))\right| &\leq C\sqrt{k}\|B_{k,N}^{loc}(\cdot,\tfrac{v}{\sqrt{k}}) - \tilde{B}_k(\cdot,\tfrac{v}{\sqrt{k}})\|_{\mathcal{L}^\infty(B(k^{-(1/2)}))} \\ &= O(k^{2n+\frac{1}{2}-\frac{N+1}{2}}).\end{aligned}$$

The above holds for $x \in B(\frac{1}{2}k^{-1/2})$, hence we have

$$\left| D_x(B^{loc}_{k,N}(\tfrac{u}{\sqrt{k}}, \tfrac{v}{\sqrt{k}}) - \tilde{B}_k(\tfrac{u}{\sqrt{k}}, \tfrac{v}{\sqrt{k}})) \right| = O(k^{2n+\frac{1}{2}-\frac{N+1}{2}}).$$

By similar argument, we can obtain the same estimates for the holomorphic variables $\bar{y}$.

Now let D^α be any α-th degree differential operator with respect to x or $\bar{y}$. By iterating the previous argument, we obtain the following

$$|D^\alpha(B^{loc}_{k,N} - B_k)| \leq O(k^{\frac{|\alpha|}{2}+2n-\frac{N+1}{2}}).$$

Hence we obtain the smooth convergence of the Bergman kernel asymptotics.

Chapter 5

Off-Diagonal

In this section we prove the rapid decay of the Bergman kernel on the off-diagonal. We will prove the following We prove the off-diagonal estimate of the Bergman kernel.

Theorem 5.0.1. Let $(L,h) \to (M,\omega)$ be a positive line bundle over a complete (not necessarily compact) Kähler manifold of dimension n. Assume the Ricci curvature is bounded from below, $\mathrm{Ric}(g) \geq -K\omega$ and assume the metric is only C^2. Let $B_k(x,y)$ be the Bergman kernel for $H^0_{\mathcal{L}^2}(M,L^k)$. For $d(x,y) > 1$, there exists $\varepsilon_0 > 0$ such that the following off-diagonal decay holds:

$$\|B_k(x,y)\| \leq C \operatorname{vol}(B_x(1))^{-1} \operatorname{vol}(B_y(1))^{-1} e^{-\varepsilon_0\sqrt{k}d(x,y)}.$$

We actually only require the distance $d(x,y)$ to be greater than some positive constant independent of k.

The study of the off-diagonal behavior has gained interest due to a conjecture by Zelditch, which relates the decay rate of the Bergman kernel with the regularity of the metric.

Conjecture 1 ([7] page 2)**.** Let $d(x,y) \geq \delta > 0$. The Bergman kernel decays at a rate

$O(e^{-ck})$ for some $c = c(\delta) > 0$ if and only if the metric potential φ is real analytic.

Results regarding the off-diagonal Agmon estimates in the compact setting can be found in [7], [8], [9], [22], [25].

5.1 Preliminary Discussion

5.1.1 Comparison between Bergman and Green kernel

For compact manifolds by the Hodge decomposition, any smooth section $f \in \Gamma(M, L^k)$ can be decomposed into its harmonic and $\overline{\partial}^*$ exact component,

$$f = f_h + \overline{\partial}^* f_1.$$

Define the $\overline{\partial}$-Laplacian on L^N-valued 0-forms (sections) and $(0,1)$-forms by

$$\Delta_0 := \overline{\partial}^* \overline{\partial}$$
$$\Delta_1 := \overline{\partial}^* \overline{\partial} + \overline{\partial}\overline{\partial}^*.$$

This gives us a decomposition of the identity operator into

$$I - \mathcal{P}_k = \Delta_0 G_0 = \overline{\partial}^* \overline{\partial} G_0,$$

where $\mathcal{P}_k$ is the Bergman projection and G_0 is the Green operator of Δ_0 for sections $\Gamma(M, L^k)$.

Now we assume that M is noncompact. By restricting ourselves to $\Gamma_{\mathcal{L}^2}(M, L^k)$, we can obtain the same Hodge decomposition as above by the spectral theorem.

We will need the following identity to commute the $\overline{\partial}$ operator and the resolvent of Δ_i in our estimates.

Lemma 5.1.1. Let Δ_0 and Δ_1 be the Laplacian for $\Gamma_{\mathcal{L}^2}(M, L^k)$ and $\Gamma_{\mathcal{L}^2}(M, \Lambda^{0,1}(M) \otimes L^k)$ respectively. For $\alpha > 0$ and $s > 0$, we have

$$\overline{\partial}(\Delta_0 + \alpha)^{-s} = (\Delta_1 + \alpha)^{-s}\overline{\partial},$$

on the algebra $C_0^\infty(M)$ of smooth function with compact support.

Proof. For any $f \in C_0^\infty(M)$, we define the operator

$$A_t := e^{-t(\Delta_1+\alpha)}\overline{\partial} - \overline{\partial}e^{-t(\Delta_0+\alpha)}.$$

Taking the derivative, we have

$$\frac{dA_t f}{dt} = -(\Delta_1 + \alpha)A_t f,$$

where we use that fact that $\overline{\partial}\Delta_0 = \Delta_1\overline{\partial}$. Since $A_0 f = 0$, since the Ricci curvature is bounded below, by [10] we have by the uniqueness of the solution of the heat equation, $A_t f = 0$ for all t and $f \in C_0^\infty(M)$. Multiplying the equality through by t^{s-1} and integrating both sides we have

$$\int_0^\infty t^{s-1}e^{-t(\Delta_1+\alpha)}\overline{\partial}dt = \int_0^\infty \overline{\partial}e^{-t(\Delta_0+\alpha)}t^{s-1}dt,$$

gives us the result. $\square$

Now on $\Gamma_{\mathcal{L}^2}(M, L^k)$, we have the following decomposition

$$I - \mathcal{P}_k = \Delta G_0 = \overline{\partial}^*\overline{\partial}G_0 = \overline{\partial}^*G_1\overline{\partial}.$$

Applying $f \in \Gamma_{\mathcal{L}^2}(M, L^k)$, we have

$$f(x) = \mathcal{P}_k f(x) + \overline{\partial}^* G_1 \overline{\partial} f(x).$$

and the corresponding integral kernel is given by

$$\int_M \langle f(y), \delta(y,x)\rangle dy = \int_M \langle f(y), B_k(y,x)\rangle dy + \int_M \langle \overline{\partial}_y f(y), \partial_x^* G_1(y,x)\rangle dy$$

From the above argument, we conclude the following relation between the Bergman kernel and Green's operator

Theorem 5.1.1. For $d(x,y) > 1$, we have the following estimate for the Bergman kernel

$$\|B_k(x,y)\| = \|\partial_y^* \overline{\partial}_x^* G_1(x,y)\|$$

5.1.2 Heat kernel for L^k section

Since the Green kernel of the Laplacian is given by the time integral of the heat kernel, we give some initial estimates on the heat kernel for sections $\Gamma(M, L^k)$. An initial computation gives the following heat inequality for the norm of the heat kernel on sections:

Proposition 5.1.1. Let $\tilde{\Delta} := g^{i\bar{j}}\partial_i\partial_{\bar{j}}$ be the (complex) Laplacian on functions. Let h_0 be the heat kernel on sections of L^k. The following heat inequality holds:

$$\left(\frac{d}{dt} - \tilde{\Delta}\right)(\|h_0\|^2) \leq nk\|h_0\|^2 \tag{5.1}$$

Proof. Viewing $h_0 \in \Gamma(L^k)$ we directly calculate:

$$\begin{aligned}\tilde{\Delta}(\|h_0\|^2) &= g^{i\bar{j}}\partial_i\partial_{\bar{j}}\langle h_0, h_0\rangle \\ &= g^{i\bar{j}}\left(\langle \nabla_i\nabla_{\bar{j}}h_0, h_0\rangle + \langle h_0, \nabla_{\bar{i}}\nabla_j h_0\rangle\right) + \|\nabla h_0\|^2 + \|\overline{\nabla} h_0\|^2\end{aligned}$$

The first term, the rough Laplacian is equal to the $\overline{\partial}$-Laplacian,

$$g^{i\bar{j}}\langle \nabla_i\nabla_{\bar{j}}h_0, h_0\rangle = -\langle \Delta_0 h_0, h_0\rangle.$$

For the second term, due to the inner product being Hermitian, we require the use of the Ricci formula to commute the covariant derivatives. This yields

$$[\nabla_i, \nabla_{\bar{j}}]h_0 = -k\,\mathrm{Ric}(h)_{i\bar{j}}h_0.$$

Then

$$g^{i\bar{j}}\langle h_0, \nabla_{\bar{i}}\nabla_j h_0\rangle = \langle h_0, g^{j\bar{i}}\nabla_j\nabla_{\bar{i}}h_0\rangle - nk\|h_0\|^2 = \langle h_0, \Delta_0 h_0\rangle - nk\|h_0\|^2.$$

Combining everything together, we have

$$\begin{aligned}\Delta_0(\|h_0\|^2) &\geq -\langle \Delta_0(h_0), h_0\rangle - \langle h_0, \Delta_0 h_0\rangle - nk\|h_0\|^2 + \|\nabla h_0\|^2 + \|\overline{\nabla} h_0\|^2 \\ &\geq \frac{d}{dt}(\|h_0\|^2) - nk\|h_0\|^2.\end{aligned}$$

□

By rescaling, we get the heat inequality

$$\left(\frac{d}{dt} - \tilde{\Delta}\right)(e^{-nkt}\|h_0\|^2) \leq 0.$$

Hence by a semigroup domination argument [12], we have the estimate

$$\|h_0\| \leq e^{\frac{nkt}{2}} |k_0|. \tag{5.2}$$

We can see that we get an exponential growth with respect to t.

It was shown in [27] that for the heat kernel on functions k_0, there exists constants $\beta_1, C_1 < \infty$ such that

$$k_0(t,x,y) \leq \frac{C_1}{\mathrm{vol}(B_x(\sqrt{t}))^{\frac{1}{2}} \, \mathrm{vol}(B_y(\sqrt{t}))^{\frac{1}{2}}} e^{-\frac{d^2(x,y)}{C_1 4t}} e^{\beta_1 t}$$

Combining with our initial heat kernel estimate, we have

$$\|h_0(t,x,y)\| \leq \frac{C_1}{\mathrm{vol}(B_x(\sqrt{t}))^{\frac{1}{2}} \, \mathrm{vol}(B_y(\sqrt{t}))^{\frac{1}{2}}} e^{-\frac{d^2(x,y)}{C_1 4t}} e^{C_2 nkt}$$

for some $C_2 > 0$.

By a volume comparison argument from (6.6), there exists $\beta_2, C_3 < \infty$ such that

$$\mathrm{vol}(B_x(\sqrt{t}))^{-1/2} \, \mathrm{vol}(B_y(\sqrt{t}))^{-1/2} \leq \frac{C_3}{\mathrm{vol}(B_x(1))} \sup\{t^{-n}, 1\} e^{\beta_2 d(x,y)}$$

By an application of Cauchy's inequality, for any $\gamma \in \mathbb{R}$,

$$e^{-\frac{d(x,y)^2}{4C_4 t}} \leq e^{-\gamma d(x,y)} e^{C_4 \gamma^2 t} \tag{5.3}$$

With the above considerations, we obtain

$$\|h_0(t,x,y)\| \leq \frac{C_3}{\mathrm{vol}(B_x(1))} \sup\{t^{-n}, 1\} e^{(\beta_2 - \gamma) d(x,y)} e^{(C_4 \gamma^2 + C_2 nk)t} \tag{5.4}$$

5.2 Proof of Theorem 5.0.1

5.2.1 Perturbation of the Green Operator

We consider the class of functions

$$\Psi := \{\psi \in \mathcal{C}^1(M) \mid |\nabla\psi|^2 \le c_0^2 k\},$$

where $c_0 > 0$ is a constant to be determined later. The choice of k in the upper bound will also become evident in §5.2.4. Then we have that

$$\inf\{\psi(x) - \psi(y) \mid \psi \in \Psi\} = -c_0\sqrt{k}d(x,y),$$

for any $x, y \in M$. Let

$$\phi(x) = \mathrm{vol}(B_x(1))^{-\frac{1}{2}}.$$

First note that the integral kernel for the operator

$$A := \phi^{-1}e^{\psi}\overline{\partial}^*\Delta_1^{-1}\overline{\partial}e^{\psi}\phi^{-1},$$

is given by

$$A(x,y) = \phi^{-1}(x)e^{\psi(x)}\overline{\partial}_x^*\partial_y^* G(x,y)e^{-\psi(y)}\phi^{-1}(y).$$

We want to show a uniform upper bound for $A(x,y)$ when $d(x,y) > 1$ so that

$$\|\overline{\partial}_x^*\partial_y^* G(x,y)\| \le C\phi(x)\phi(y)e^{\psi(y)-\psi(x)}.$$

Varying over $\psi \in \Psi$ yields

$$\|\overline{\partial}_x^* \partial_y^* G(x,y)\| \leq C\phi(x)\phi(y)e^{-c_0\sqrt{k}d(x,y)} \quad \text{for } d(x,y) > 1,$$

and Theorem 5.1.1 will imply the off-diagonal decay of the Bergman kernel. Note that we allow C to be at most polynomial growth in k, as a proper choice of the constant in the exponent will absorb the polynomial term. In the following, we will use a resolvent identity to obtain a kernel identity and provide the necessary bounds.

5.2.2 Resolving the singularity

We consider the following resolvent and corresponding Green kernel for the operator $(\Delta_i+\alpha)$:

$$\begin{aligned}(\Delta_i + \alpha)^{-s-1} f(x) &= \int_M (G_{\alpha,s}(y,x), f(y))dy \\ &:= \frac{1}{\Gamma(s+1)}(\int_0^\infty h_i(t,y,x)e^{-\alpha t}t^s dt, f(y))dy\end{aligned}$$

$i = 0, 1$. We abuse the notation for the Green kernel as it will be clear from context which we are using. By choosing an appropriately large $\alpha > 0$, we can "push away" the exponential growth in k. The exponent s is used to remove the singularity of the heat kernel when $t = 0$ for $x = y$. To introduce such a term, we use the following resolvent identity:

$$\begin{aligned}\Delta_1^{-1} &= (\Delta_1 + \alpha)^{-1} + \alpha(\Delta_1 + \alpha)^{-1}\Delta_1^{-1} \\ &= (\Delta_1 + \alpha)^{-1} + \alpha(\Delta_1 + \alpha)^{-1/2}\Delta_1^{-1}(\Delta_1 + \alpha)^{-1/2}\end{aligned}$$

By iterating, we obtain

$$\begin{aligned}\Delta_1^{-1} &= (\Delta_1+\alpha)^{-1} + \alpha(\Delta_1+\alpha)^{-2} + \cdots + \alpha^{2n+3}(\Delta_1+\alpha)^{-2n-4} \\ &\quad + \alpha^{2n+4}(\Delta_1+\alpha)^{-n-2}\Delta_1^{-1}(\Delta_1+\alpha)^{-n-2}.\end{aligned} \tag{5.5}$$

Perturbing the operator, we get

$$\begin{aligned}A = \phi^{-1}e^{\psi}\overline{\partial}^*\Delta_1^{-1}\overline{\partial}e^{-\psi}\phi^{-1} &= \sum_{p=0}^{2n+3}\alpha^p\phi^{-1}e^{\psi}\overline{\partial}^*(\Delta_1+\alpha)^{-(p+1)}\overline{\partial}e^{-\psi}\phi^{-1} \\ &\quad + \alpha^{2n+4}\phi^{-1}e^{\psi}\overline{\partial}^*(\Delta_1+\alpha)^{-n-2}\Delta_1^{-1}(\Delta_1+\alpha)^{-n-2}\overline{\partial}e^{-\psi}\phi^{-1}.\end{aligned}$$

We can see that we have two cases to consider

1. $A_s := \phi^{-1}e^{\psi}\overline{\partial}^*(\Delta_1+\alpha)^{-s}\overline{\partial}e^{-\psi}\phi^{-1}$
2. $B := \phi^{-1}e^{\psi}\overline{\partial}^*(\Delta_1+\alpha)^{-n-2}\Delta_1^{-1}(\Delta_1+\alpha)^{-n-2}\overline{\partial}e^{-\psi}\phi^{-1}$

where A_s is singular if $s \leq n$. For the first case, we use Lemma 5.1.1 and we have

$$\begin{aligned}A_s(x,y) &= \phi^{-1}e^{\psi}\overline{\partial}^*(\Delta_1+\alpha)^{-s}\overline{\partial}e^{-\psi}\phi^{-1} \\ &= \phi^{-1}e^{\psi}\overline{\partial}^*\overline{\partial}(\Delta_0+\alpha)^{-s}e^{-\psi}\phi^{-1} \\ &= \phi^{-1}e^{\psi}\Delta_0(\Delta_0+\alpha)^{-s}e^{-\psi}\phi^{-1} \\ &= \phi^{-1}e^{\psi}(\Delta_0+\alpha)^{-s+1}e^{-\psi}\phi^{-1} - \alpha\phi^{-1}e^{\psi}(\Delta_0+\alpha)^{-s}e^{-\psi}\phi^{-1}.\end{aligned}$$

Inserting this into the above identity leads to a cancellation in all the middle terms. The leading term becomes the delta kernel, which vanishes for $d(x,y) > 0$. Hence we have the operator identity

$$A = \phi^{-1}e^{\psi}Ie^{-\psi}\phi^{-1} - \alpha^{2n+4}\phi^{-1}e^{\psi}(\Delta_0+\alpha)^{-2n-4}e^{-\psi}\phi^{-1} + \alpha^{2n+4}B.$$

Let $A(x,y)$ and $B(x,y)$ be the kernel of the operator A and B respectively. We then have the kernel identity

$$A(x,y) = -\alpha^{2n+4}\phi^{-1}(x)e^{\psi(x)}G_{\alpha,2n+3}(x,y)e^{-\psi(y)}\phi^{-1}(y) + \alpha^{2n+4}B(x,y).$$

for $d(x,y) > 0$.

5.2.3 Estimates

Since the α terms will be at most a polynomial in k, we will incorporate it into the exponential decaying term.

Lemma 5.2.1. There exists $\alpha > 0$ and $C \leq \infty$ such that for $\psi \in \Psi$,

$$\sup_{x,y} \|\phi^{-1}(x)e^{\psi(x)}G_{\alpha,2n+3}(x,y)e^{-\psi(y)}\phi^{-1}(y)\| \leq C.$$

Proof. We modify Lemma 6.6 and consider both the volume of x and y so

$$\operatorname{vol}(B_{\sqrt{t}}(x))^{-1/2}\operatorname{vol}(B_{\sqrt{t}}(y))^{-1/2} \leq C_2\phi(x)\phi(y)\sup\{t^{-n},1\},$$

Applying the heat kernel estimates, we have

$$\begin{aligned}
&\|G_{\alpha,2n+3}(x,y)\| \\
&\quad \leq C_3\phi(x)\phi(y)\int_0^\infty \sup\{t^{-n},1\}e^{-\gamma d(x,y)}t^{2n+3}e^{(C_4\gamma^2+C_2nk-\alpha)t} \\
&\quad \leq C_5\phi(x)\phi(y)e^{-\gamma d(x,y)},
\end{aligned}$$

for any γ and $\alpha > C_4\gamma^2 + C_2nk$. □

Using this, we have a bound on the Green function

$$\sup_{x,y} \|\phi^{-1}(x)e^{\psi(x)}G_{\alpha,2n+3}(y,x)e^{-\psi(y)}\phi^{-1}(y)\|$$
$$\leq C \sup_{x,y} e^{-\gamma+c_0\sqrt{k}d(x,y)}$$
$$\leq C.$$

By choosing appropriate γ.

Next we consider the $\mathcal{L}^1 \to \mathcal{L}^\infty$ bound for the B operator. This would imply the same bound on the kernel $B(x,y)$. We first show the following lemma

Lemma 5.2.2. There exists $\alpha > 0$ and $C < \infty$ such that for all $\psi \in \Psi$

$$\sup_{x\in M} \phi^{-2}(x) \int_M \|G_{\alpha,n}(x,y)\|^2 e^{2(\psi(x)-\psi(y))}dy \leq C$$

Proof. We repeat the steps of the previous lemma and applying Lemma 6.6 to obtain

$$\|G_{\alpha,n}(x,y)\|$$
$$\leq C_3\phi^2(x)\int_0^\infty \sup\{t^n,1\}e^{-\gamma d(x,y)}e^{(C_4\gamma^2+C_2nk-\alpha)t}$$
$$\leq C\phi^2(x)e^{-\gamma d(x,y)}.$$

Again, by choosing appropriate α. Then by direct computation,

$$\begin{aligned}
&\phi^{-2}(x)\int_M \|G_{\alpha,s}(x,y)\|^2 e^{2(\psi(x)-\psi(y))}dy \\
&\quad\le C\phi^2(x)\int_M e^{-2(\gamma-\beta_2-c_0\sqrt{k})d(x,y)}dy \\
&\quad\le C\sum_{j=1}^{\infty} e^{-2(\gamma-\beta_2-c_0\sqrt{k})(j-1)}\frac{\mathrm{Vol}(B_j(x))}{\mathrm{Vol}(B_1(x))} \\
&\quad\le C\sum_{j=1}^{\infty} j^n e^{-2(\gamma-\beta_2-c_0\sqrt{k})(j-1)}e^{(n-1)\sqrt{K}j}
\end{aligned}$$

where the second to last inequality is obtained by splitting the integral as $M=\bigcup_{j=1}^{\infty}\{y \mid j-1\le d(x,y)\le j\}$ and the last inequality is by volume comparison. The series is convergent when

$$2(\gamma-\beta_2-c_0\sqrt{k}) > (n-1)\sqrt{K}. \tag{5.6}$$

Hence we choose γ which satisfies the above. $\square$

Decomposing $B = A_{\alpha,2}T_\psi A_{\alpha,1}$ where

$$\begin{aligned}
&A_{\alpha,2} := \phi^{-1}e^{\psi}\overline{\partial}^*(\Delta_1+\alpha)^{-n-1}e^{-\psi} \\
&A_{\alpha,1} := e^{\psi}(\Delta_1+\alpha)^{-n-1}\overline{\partial}e^{-\psi}\phi^{-1} \\
&T_\psi := e^{\psi}\Delta_1^{-1}e^{-\psi}
\end{aligned}$$

Lemma 5.2.3. There exists some $\alpha\ge 0$ and $C>0$ such that

$$\begin{aligned}
&\|A_{\alpha,1}\|_{1,2}\le C \\
&\|A_{\alpha,2}\|_{2,\infty}\le C.
\end{aligned}$$

Proof. Let $G_{\alpha,n+1}(x,y)$ be the Green kernel from $(\Delta_0+\alpha)^{-n-2}$. By direct computation, we have

$$
\begin{aligned}
&\int_M \|\overline{\partial}_x G_{\alpha,n+1}(x,y)\|^2 e^{2(\psi(x)-\psi(y))}dx \\
&\quad \le 8c_0^2 k \int_M e^{2(\psi(x)-\psi(y))}\|G_{\alpha,n+1}(x,y)\|^2 dx \\
&\qquad + \int_M e^{2(\psi(x)-\psi(y))}(G_{\alpha,n+1}(x,y), \overline{\partial}_x^* \overline{\partial}_x G_{\alpha,n+1}(x,y))dx
\end{aligned}
$$

Now using Lemma 5.1.1, we have

$$
\begin{aligned}
&\|e^{\psi}\overline{\partial}(\Delta_0+\alpha)^{-n-2}e^{-\psi}\phi^{-1}\|_{1,2} \\
&\quad = \left(\int_M \|e^{\psi(x)}\overline{\partial}(\Delta_0+\alpha)^{-n-2}(e^{-\psi}\phi^{-1}u)(x)\|^2 dx\right)^{\frac{1}{2}} \\
&\quad = \left(\int_M \left\|e^{\psi(x)}\overline{\partial}_x\left(\int_M (G_{\alpha,n+1}(y,x)e^{-\psi(y)}\phi^{-1}(y), u(y))dy\right)\right\|^2 dx\right)^{\frac{1}{2}} \\
&\quad \le \sup_y \phi^{-1}(y)\left(\int_M \|\overline{\partial} G_{\alpha,n+1}(x,y)\|^2 e^{2(\psi(x)-\psi(y))}dx\right)^{\frac{1}{2}} \\
&\quad \le \sup_y \phi^{-1}(y)\left(\int_M e^{2(\psi(x)-\psi(y))}\|\overline{\partial} G_{\alpha,n+1}(x,y)\|^2 dx\right)^{\frac{1}{2}} \\
&\quad \le \sup_y \phi^{-1}(y)\left(8c_0^2 k\int_M e^{2(\psi(x)-\psi(y))}\|G_{\alpha,n+1}(x,y)\|^2 dx\right. \\
&\qquad \left. + \int_M e^{2(\psi(x)-\psi(y))}(G_{\alpha,n+1}(x,y), \Delta_1 G_{\alpha,n+1}(x,y))dx\right)^{\frac{1}{2}}
\end{aligned}
$$

When the Laplacian acts on the Green kernel, we have

$$
\begin{aligned}
\Delta_{\overline{\partial},x} G_{\alpha,s}(x,y) &= \int_0^\infty \Delta_{\overline{\partial},x} k(t,x,y)e^{-\alpha t}t^s dt \\
&= -\int_0^\infty (\partial_t k(t,x,y))e^{-\alpha t}t^s dt
\end{aligned}
$$

Integrating by parts,

$$
\begin{aligned}
&= \int_0^\infty k(t,x,y) s t^{s-1} e^{-\alpha t} dt - \int_0^\infty \alpha k(t,x,y) e^{-\alpha t} t^s dt \\
&= s G_{\alpha,s-1}(x,y) - \alpha G_{\alpha,s}(x,y).
\end{aligned}
$$

Inserting the identity, we have

$$
\begin{aligned}
&\int_M e^{2(\psi(x)-\psi(y))}(G_{\alpha,n+1}(x,y), \Delta_{\overline{\partial},x} G_{\alpha,n+1}(x,y))dx \\
&\quad = -(n+1)\int_M e^{2(\psi(x)-\psi(y))}(G_{\alpha,n+1}(x,y), G_{\alpha,n}(x,y))dx \\
&\qquad + \alpha \int_M e^{2(\psi(x)-\psi(y))}\|G_{\alpha,n+1}(x,y)\|^2 dx \\
&\quad \leq (\alpha - \frac{n+1}{2}) \int_M e^{2(\psi(x)-\psi(y))}\|G_{\alpha,n+1}(x,y)\|^2 dx \\
&\qquad - \frac{n+1}{2}\int_M e^{2(\psi(x)-\psi(y))}\|G_{\alpha,n}(x,y)\|^2 dx
\end{aligned}
$$

Hence combining we obtain

$$
\begin{aligned}
&\|e^{\psi}\overline{\partial}(\Delta+\alpha)^{-n-1}e^{-\psi}\phi^{-2}\|_{1,2}^2 \\
&\quad \leq \sup_y \phi^{-1}(y)(8c_0^2 k + \alpha - \frac{n+1}{2}) \int_M e^{2(\psi(x)-\psi(y))}\|G_{\alpha,n+1}(x,y)\|^2 dx \\
&\qquad - \frac{n+1}{2}\int_M e^{2(\psi(x)-\psi(y))}\|G_{\alpha,n}(x,y)\|^2 dx \\
&\quad \leq \sup_y \phi^{-2}(y)(8c_0^2 k + \alpha - \frac{n+1}{2}) \int_M e^{2(\psi(x)-\psi(y))}\|G_{\alpha,n+1}(x,y)\|^2 dx \\
&\qquad + \sup_y \phi^{-2}(y)\frac{n+1}{2}\int_M e^{2(\psi(x)-\psi(y))}\|G_{\alpha,n}(x,y)\|^2 dx.
\end{aligned}
$$

Lemma 5.2.2 gives us the upper bound. For the $A_{\alpha,2}$, we note that the dual operator $A_{\alpha,1}^* = \phi^{-1}e^{-\psi}\overline{\partial}^*(\Delta+\alpha)^{-n-1}e^{\psi}$ has the operator bound $\|A_{\alpha,1}^*\|_{2,\infty} \leq C$ for $\psi \in \Psi$, hence varying over the class of functions in Ψ, we obtain the same bound for $A_{\alpha,2}$. □

Next we show the $\mathcal{L}^2 \to \mathcal{L}^2$ bound of T_ψ.

Lemma 5.2.4. Assume that the Ricci curvature of M has a lower bound. Then for sufficiently large k and for $\psi \in \Psi$, we have

$$\|T_\psi\|_{2,2} \le C < \infty.$$

Proof. Define the operator

$$T^{\psi'} := e^{\psi} \Delta_1 e^{-\psi} - \Delta_1.$$

By the Weitzenböck formula, we have

$$\Delta_1 = -\nabla^2 + E = -\nabla^2 + k\,\omega + \mathrm{Ric}(M).$$

We denote $\tilde{\Delta} = -\nabla^2$ to be the rough Laplacian. Then we can write

$$T^{\psi'} = e^{\psi} \tilde{\Delta} e^{-\psi} - \tilde{\Delta}.$$

For any L^k-valued $(0,1)$ form u, we have

$$\begin{aligned}(T^{\psi'} u, u) &= (\overline{\nabla}(e^{-\psi} u), \overline{\nabla}(e^{\psi} u)) - (\overline{\nabla} u, \overline{\nabla} u)\\ &\le c_0^2 k \int_M \|u\|^2 + 2\,\mathrm{Im}\left(\int_M \nabla\psi \nabla\bar{u}\, u\right)\end{aligned}$$

Using the Cauchy inequality, using the fact that $|\nabla\psi| \le b_k$, we have

$$\|(T^{\psi'} u, u)\| \le 4c_0^2 k \int_M \|u\|^2 + \frac{1}{2} \int_M \|\nabla u\|^2.$$

Since the Ricci curvature has a lower bound $-K$, integrating the Weitzenböck formula gives

$$\int_M \|\nabla u\|^2 \leq \int_M \langle \Delta_1 u, u\rangle + K \int_M \|u\|^2 - k \int_X \|u\|^2.$$

In summary, we have

$$|(T^{\psi'} u, u)| \leq \frac{1}{2}(\Delta_1 u, u) + (4c_0^2 k + K - k) \int_M |u|^2,$$

and for $c_0 < \frac{1}{2}$, we have

$$|(T^{\psi'} u, u)| \leq \frac{1}{2}(\Delta_1 u, u)$$

for k sufficiently large.

We have

$$\begin{aligned} |((\Delta_1 + T^{\psi'})u, u)| &\geq |(\Delta_1 u, u)| - |(T^{\psi'} u, u)| \\ &\geq \frac{1}{2}|(\Delta_1 u, u)| \\ &\geq ck(u, u). \end{aligned}$$

A lower bound for the quadratic form of an operator implies an upper bound for its inverse. Here we see that the order $c_0\sqrt{k}$ is necessary in our choice of the class of functions Ψ. $\square$

Combining Lemmas 5.2.1, 5.2.3, and 5.2.4 proves Theorem 5.0.1.

Chapter 6

Appendix

6.1 Useful Formulas

6.1.1 Bound of normalizing constant for peak section

Lemma 6.1.1.

$$\int r^{2p} e^{-mr^2} r dr = -\frac{1}{2m} e^{-mr^2} \sum_{k=0}^{p} \frac{p!}{(p-k)! m^k} r^{2(p-k)} \tag{6.1}$$

Corollary 6.1.1. Let $P \in \mathbb{Z}_+^n$ and use the following notations: $P! = p_1! \cdots p_n!$ and $p = p_1 + \cdots + p_n$

$$\int_{\mathbb{C}^n} |z^P|^2 e^{-m|z|^2} dV_0 = \frac{P!}{m^{n+p}} \tag{6.2}$$

Corollary 6.1.2 (lower bound for integral)**.** Let $c_m \to \infty$

$$\int_{|z| \le \frac{c_m}{\sqrt{m}}} |z^P|^2 e^{-m|z|^2} dV_0 \ge \frac{C}{m^{n+p}} \tag{6.3}$$

Proof. By using (6.1.1), we have

$$\begin{aligned}
&\int_{\frac{c_m}{\sqrt{m}} \geq |z_1|} |z_1|^{2p_1} e^{-m|z_1|^2} dV_0 \\
&= \frac{\pi}{m^{p_1+1}} e^{-c_m} \sum_{k=0}^{p_1} \frac{p_1!}{(p_1-k)!} c_m^{2(p_1-k)} \\
&\leq \frac{p_1!\pi}{m^{p_1+1}} \sum_{k=0}^{p_1} \frac{c_m^{2(p_1-k)}}{e^{c_m^2}} \\
&\leq \frac{(p_1+1)!\pi}{m^{p_1+1}} \frac{c_m^{2p_1}}{e^{c_m^2}}
\end{aligned}$$

For large $t \gg 0$, we have

$$\frac{t^{p_1}}{e^t} < \frac{1}{(p_1+1)\pi} \tag{6.4}$$

So taking m sufficiently large, we have

$$\int_{\frac{c_m}{\sqrt{m}} \geq |z_1|} |z_1|^{2p_1} e^{-m|z_1|^2} dV_0 \leq \frac{C_1}{m^{p_1+1}} \tag{6.5}$$

for $C_1^n < |P|$ Then we have

$$\begin{aligned}
\int_{|z| \leq \frac{c_m}{\sqrt{m}}} |z^P|^2 e^{-m|z|^2} dV_0 &= \int_{\mathbb{C}^n} |z^P|^2 e^{-m|z|^2} dV_0 - \int_{|z| \geq \frac{c_m}{\sqrt{m}}} |z^P|^2 e^{-m|z|^2} dV_0 \\
&\geq \frac{P!}{m^{n+p}} - \frac{C_1}{m^{n+p}} = \frac{C}{m^{n+p}}
\end{aligned}$$

□

6.1.2 Weitzenböck Formula for L^N-valued $(0,1)$-forms

We define the covariant derivative for holomorphic indices of L^N-valued $(0,q)$-forms as

Definition 6.1.1 (Holomorphic covariant derivative)**.**

$$\nabla_\delta(\phi_{\overline{ij}} e_L^N) = h^{-1} \frac{\partial}{\partial z^\delta}(\phi_{\overline{ij}} h) e_L^N$$

We use the notation

$$\nabla_k f_\alpha = f_{\alpha;k}$$

so that for a local frame $\{e_\alpha\}$ of a vector bundle E and section $f = \sum_\alpha f_\alpha e_\alpha$,

$$\begin{aligned}\nabla f &= \sum_{\alpha,i} f_{\alpha;k} e_\alpha \otimes dz^k \\ &= \sum_{\alpha,i} \partial_k f_\alpha + \Gamma(f)\end{aligned}$$

where $\Gamma(f)$ is a zero order operator involving the connection matrix, and depends on the bundle E.

Proposition 6.1.1 (Anti-holomorphic covariant derivative)**.** Let $\phi \in \Omega^{0,1}(L^N)$, locally written $\phi = \phi_{\bar{i}} d\bar{z}^i$.

$$\overline{\partial}(\phi) = \nabla_{\overline{k}} \phi_{\overline{j}} d\overline{z}^k \wedge d\overline{z}^j$$

and rewriting in skew symmetric form,

$$\overline{\partial}\phi = \frac{1}{2}\left(\nabla_{\overline{k}}\phi_{\overline{j}} - \nabla_{\overline{j}}\phi_{\overline{k}}\right) d\overline{z}^k \wedge d\overline{z}^j$$

Proof. We have

$$\overline{\partial}(\phi) = \overline{\partial}(\phi_{\bar{i}} d\overline{z}^i) = \frac{\partial \phi_{\bar{i}}}{\partial \overline{z}^k} d\overline{z}^k \wedge d\overline{z}^i$$

On the other hand, we have by definition:

$$\nabla_{\overline{k}}\phi_{\overline{i}} = \frac{\partial \phi_{\overline{i}}}{\partial \overline{z}^k} - \phi_{\overline{j}}\Gamma^{\overline{j}}_{\overline{k}\overline{i}}$$

So that

$$\frac{\partial \phi_{\overline{i}}}{\partial \overline{z}^k} = \nabla_{\overline{k}}\phi_{\overline{i}} + \phi_{\overline{j}}\Gamma^{\overline{j}}_{\overline{k}\overline{i}}$$

substituting the above to the first equation yields

$$\overline{\partial}(\phi) = (\nabla_{\overline{k}}\phi_{\overline{i}} + \phi_{\overline{j}}\Gamma^{\overline{j}}_{\overline{k}\overline{i}})d\overline{z}^k \wedge d\overline{z}^i$$

By the alternating 2-form and the symmetry of Γ^k_{ij}, we have

$$\Gamma^k_{ij} = 0$$

which gives us the result. □

We now show the formula for the $\overline{\partial}^*$ operator for L^N-valued $(0,1)$ and $(0,2)$-forms.

Lemma 6.1.2 ($\overline{\partial}^*$ for L^N-valued $(0,1)$-forms)**.**

$$\overline{\partial}^*(\phi_{\overline{i}}d\overline{z}^i \otimes e_L^N) = -g^{j\overline{i}}\frac{\partial}{\partial z^j}(\phi_{\overline{i}})e_L^N - g^{j\overline{i}}\phi_{\overline{i}}\frac{\partial h}{\partial z^j}h^{-1}e_L^N = -g^{j\overline{i}}\nabla_j(\phi_{\overline{i}}e_L^N)$$

Lemma 6.1.3 ($\overline{\partial}^*$ for L^N-valued $(0,2)$-forms)**.**

$$(\overline{\partial}^*\phi)_{\overline{p}}d\overline{z}^p = -2g^{l\overline{i}}(\nabla_{\partial_l}(\phi_{\overline{i}\overline{p}}e_L^N) - \phi_{\overline{i}\overline{j}}g_{k\overline{p}}g^{s\overline{j}}\Gamma^k_{sl}e_L^N)d\overline{z}^p$$

Proof. We use the defining relation $(\overline{\partial}^*(\phi_{\overline{i}\overline{j}}), \eta_{\overline{k}}) = (\phi_{\overline{i}\overline{j}}, \frac{\partial \eta_{\overline{k}}}{\partial \overline{z}^l})$. Expanding, we obtain

$$\begin{aligned}
(\phi_{\overline{i}\overline{j}}, \frac{\partial \eta_{\overline{k}}}{\partial \overline{z}^l}) =& 2\int_M \phi_{\overline{i}\overline{j}} \overline{\frac{\partial \eta_{\overline{k}}}{\partial \overline{z}^l}} g^{l\overline{i}} g^{k\overline{j}} h \det g dV_E \\
&= -2\int_M \overline{\eta_{\overline{k}}} \frac{\partial}{\partial z^l}(\phi_{\overline{i}\overline{j}} g^{l\overline{i}} g^{k\overline{j}} h \det g) dV_E
\end{aligned}$$

and

$$(\overline{\partial}^*(\phi_{\overline{i}\overline{j}}), \eta_{\overline{k}}) = \int_M g^{k\overline{m}} (\overline{\partial}^* \phi)_{\overline{m}} \overline{\eta_{\overline{k}}} h \det g dV_E$$

Comparing coefficients, we have

$$-2 g_{k\overline{m}} \frac{\partial}{\partial z^l}(\phi_{\overline{i}\overline{j}} g^{l\overline{i}} g^{k\overline{j}} h \det g) = (\overline{\partial}^* \phi)_{\overline{m}} h \det g$$

The left hand side splits into

$$\begin{aligned}
-g_{k\overline{m}} \frac{\partial}{\partial z^l}(\phi_{\overline{i}\overline{j}} g^{l\overline{i}} g^{k\overline{j}} h \det g) =& -\frac{\partial \phi_{\overline{i}\overline{j}}}{\partial z^l} g^{l\overline{i}} g_{k\overline{m}} g^{k\overline{j}} h \det g \\
&- \phi_{\overline{i}\overline{j}} g^{k\overline{j}} g_{k\overline{m}} h \frac{\partial}{\partial z^l}(g^{l\overline{i}} \det g) \\
&- g_{k\overline{m}} \phi_{\overline{i}\overline{j}} \frac{\partial g^{k\overline{j}}}{\partial z^l} g^{l\overline{i}} h \det g \\
&- \phi_{\overline{i}\overline{j}} g_{k\overline{m}} g^{l\overline{i}} g^{k\overline{j}} \frac{\partial h}{\partial z^l} \det g
\end{aligned}$$

The second term vanishes due to the Kähler symmetries, and we simplify the remaining terms:

$$\begin{aligned}
-g_{k\overline{m}} \frac{\partial}{\partial z^l}(\phi_{\overline{i}\overline{j}} g^{l\overline{i}} g^{k\overline{j}} h \det g) =& -\frac{\partial \phi_{\overline{i}\overline{m}}}{\partial z^l} g^{l\overline{i}} h \det g \\
&+ g^{l\overline{i}} \phi_{\overline{i}\overline{j}} g_{k\overline{p}} g^{s\overline{j}} \Gamma^k_{sl} \det g \\
&- \phi_{\overline{i}\overline{m}} \partial_l h \det g
\end{aligned}$$

Using the covariant derivative identity, we have

$$(\overline{\partial}^* \phi)_{\overline{m}} = -2g^{l\bar{i}}(\nabla_l(\phi_{\bar{i}\overline{m}} e_L^N) - \phi_{\overline{i}\overline{j}} g_{k\overline{m}} g^{s\bar{j}} \Gamma^k_{sl} e_L^N)$$

$\square$

Lemma 6.1.4. Let $\phi \in \Omega^{0,1}(L^N)$. Under normal coordinates, we have

$$\overline{\partial}\overline{\partial}^*(\phi) = -g^{j\bar{i}} \left(\nabla_{\bar{k}} \nabla_j (\phi_{\bar{i}} e_L^N)\right) d\overline{z}^k = -\sum_{k,j} \left(\nabla_{\bar{j}} \nabla_k (\phi_{\bar{k}} e_L^N)\right) d\overline{z}^j$$

and

$$\overline{\partial}^* \overline{\partial}(\phi) = -\sum_{k,j} \nabla_k \nabla_{\bar{k}} (\phi_{\bar{j}} e_L^N) + \nabla_k \nabla_{\bar{j}} (\phi_{\bar{k}} e_L^N)$$

Lemma 6.1.5 (Ricci Formula)**.** Let ϕ be a L^N-valued $(0,1)$-form, i.e. $\phi = \phi^\alpha_{\bar{l}} d\overline{z}^l \otimes e_\alpha$

$$[\nabla_k, \nabla_{\bar{j}}]\phi^\alpha_{\bar{l}} = \phi_{\bar{i}} R_{i\bar{j}k\bar{l}} + \phi_{\bar{l}} \operatorname{Ric}(h)_{k\bar{j}}$$

Proof. First calculate in normal coordinates

$$\begin{aligned}
\nabla_k \nabla_{\bar{j}}(\phi_{\bar{l}}) &= \nabla_k\left(\frac{\partial \phi_{\bar{l}}}{\partial \overline{z}^j} - \phi_{\overline{m}} \Gamma^{\overline{m}}_{\bar{j}\bar{l}}\right) \\
&= \frac{\partial^2 \phi_{\bar{l}}}{\partial z^k \partial \overline{z}^j} - \phi_{\overline{m}} \frac{\partial \Gamma^{\overline{m}}_{\bar{j}\bar{l}}}{\partial z^k}
\end{aligned}$$

Next we calculate the opposite term

$$\begin{aligned}
\nabla_{\bar{j}} \nabla_k(\phi_{\bar{l}}) &= \nabla_{\bar{j}}\left(\frac{\partial \phi_{\bar{l}}}{\partial z^k} + \phi_{\bar{l}} d\overline{z}^l \frac{\partial h}{\partial z^k} h^{-1}\right) \\
&= \frac{\partial^2 \phi_{\bar{l}}}{\partial \overline{z}^j \partial z^k} + \phi_{\bar{l}} \frac{\partial^2 h}{\partial z^k \partial \overline{z}^j} h^{-1}.
\end{aligned}$$

Combining the terms yields

$$[\nabla_k, \nabla_{\bar{j}}](\phi_{\bar{l}}) = \phi_{\bar{i}} R_{i\bar{j}k\bar{l}} + \phi_{\bar{l}} \operatorname{Ric}(h)_{k\bar{j}}$$

□

Proposition 6.1.2 (Weitzenböck Formula for L^N-valued $(0,1)$-forms.)**.**

$$\begin{aligned}
\Delta_{\overline{\partial}}(\phi) &= \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}(\phi) \\
&= \sum_{k,j,i} -\nabla_k \nabla_{\bar{k}}(\phi_{\bar{j}}) + \nabla_k \nabla_{\bar{j}}(\phi_{\bar{k}}) - \nabla_{\bar{j}} \nabla_k(\phi_{\bar{k}}) \\
&= \sum_{k,j,i} \left(-\nabla_k \nabla_{\bar{k}}(\phi_{\bar{j}}) + \phi_{\bar{i}} \operatorname{Ric}(g)_{i\bar{j}} + \phi_{\bar{k}} \operatorname{Ric}(h)_{k\bar{j}}\right) d\bar{z}^j \otimes e_L^N
\end{aligned}$$

6.1.3 $\overline{\partial}$ estimate

The following Hörmander's $\mathcal{L}^2$-estimate for the $\overline{\partial}$-equation is used:

Theorem 6.1.1. Let (M, ω) be a complete Kähler manifold, and let $L \to M$ be Hermitian line bundle with Hermitian metric h. Assume that the curvature $\operatorname{Ric}(h) \geq C\omega$ is positive for some $C > 0$. Then there is an integer N_0 depending on M, L and h such that for any $N \geq N_0$, the following holds: for any $g \in \mathcal{L}^2(M, \bigwedge^{0,1} M \otimes L^m)$ satisfying $\overline{\partial} g = 0$, there exists $f \in \mathcal{L}^2(M, L^m)$ such that $\overline{\partial} f = g$ and

$$\int_M |f|^2_{h^m} dV_g \leq \frac{1}{C} \int_M |g|^2_{h^m} dV_g$$

where C is the same constant as in the hypothesis.

6.1.4 Hodge Theory and Kähler identities

Theorem 6.1.2 (Hodge Theorem)**.** Let E be a Hermitian vector bundle over a compact Hermitian manifold X. For $p, q \geq 0$, let

$$H^{p,q}(X, E) = \{\phi \in \Gamma(X, \Omega^{p,q}(E)) \mid \Delta_{\overline{\partial}}\phi = 0\}$$

i.e. the **harmonic** (p, q) forms. Then

1. $\dim H^{p,q}(X, E) < \infty$;
2. For any $\eta \in \Gamma(X, \Omega^{p,q}(E))$, we have

   $$\eta = \eta_h + \overline{\partial}\eta_1 + \overline{\partial}^*\eta_2,$$

   where η_h is harmonic.

On Kähler manifolds, we have the equivalence between the different Laplacians.

$$\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$$

6.1.5 Volume Comparison

Proposition 6.1.3 (Bishop Volume Comparison)**.** Let M be a complete n-dimensional Riemannian manifold such that $\mathrm{Ric} \geq -K$. Let $B_K(r)$ be a ball in the Riemannian space form of constant sectional curvature $-\frac{K}{n-1}$. Then

$$\mathrm{Vol}(B(r)) \leq \mathrm{Vol}(B_K(r))$$

We have

$$\omega_n s^n \le \mathrm{Vol}(B_K(s)) \le \omega_n s^n e^{\sqrt{(n-1)K}s}.$$

For any ball $B_x(r) \subset M$, we have

$$\frac{\mathrm{Vol}(B_x(s))}{\mathrm{Vol}(B_x(s'))} \le \frac{\mathrm{Vol}(B_K(s))}{\mathrm{Vol}(B_K(s'))}, \qquad 0 < s' < s < 2r.$$

Hence we deduce that

$$\mathrm{Vol}(B_x(s)) \le \mathrm{Vol}(B_x(s')) \left(\frac{s}{s'}\right)^n e^{\sqrt{(n-1)K}s}, \qquad 0 < s' < s < 2r.$$

Setting $d = d(x,y)$, we can compare the volume of balls centered at points as follows,

$$\begin{aligned}\mathrm{Vol}(B_y(s)) &\le \mathrm{Vol}(B_x(d+s))\\ &\le \mathrm{Vol}(B_x(s'))\left(\frac{d+s}{s'}\right)^n e^{\sqrt{(n-1)K}(d+s)}, \qquad 0 < s' < s < r, \quad y \in B\end{aligned}$$

Let $\phi(x) = \mathrm{Vol}(B_x(1))^{-\frac{1}{2}}$. Combining the above, there exists $\beta_2, C_2 < \infty$ such that

$$\mathrm{Vol}(B_x(\sqrt{t}))^{-1/2}\,\mathrm{Vol}(B_y(\sqrt{t}))^{-1/2} \le C_2\phi^2(x)\sup\{t^{\frac{-n}{2}}, 1\}e^{\beta_2 d(x,y)} \tag{6.6}$$

Bibliography

[1] K. Adachi. *Several complex variables and integral formulas*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[2] S. Alexakis and K. Hirachi. Integral Kähler invariants and the Bergman kernel asymptotics for line bundles. 2015.

[3] R. Berman, B. Berndtsson, and J. Sjöstrand. A direct approach to Bergman kernel asymptotics for positive line bundles. *Ark. Mat.*, 46(2):197–217, 2008.

[4] T. Bouche. Asymptotic results for Hermitian line bundles over complex manifolds: the heat kernel approach. In *Higher-dimensional complex varieties (Trento, 1994)*, pages 67–81. de Gruyter, Berlin, 1996.

[5] D. Catlin. The Bergman kernel and a theorem of Tian. In *Analysis and geometry in several complex variables (Katata, 1997)*, Trends Math., pages 1–23. Birkhäuser Boston, Boston, MA, 1999.

[6] M. Christ. Slow off-diagonal decay for Szegö kernels associated to smooth Hermitian line bundles. In *Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001)*, volume 320 of *Contemp. Math.*, pages 77–89. Amer. Math. Soc., Providence, RI, 2003.

[7] M. Christ. Off-diagonal decay of Bergman kernels: On a conjecture of Zelditch. 2013.

[8] M. Christ. Upper bounds for Bergman kernels associated to positive line bundles with smooth Hermitian metrics. 2013.

[9] X. Dai, K. Liu, and X. Ma. On the asymptotic expansion of Bergman kernel. *J. Differential Geom.*, 72(1):1–41, 2006.

[10] J. Dodziuk. Maximum principle for parabolic inequalities and the heat flow on open manifolds. *Indiana Univ. Math. J.*, 32(5):703–716, 1983.

[11] S. K. Donaldson. Scalar curvature and projective embeddings. I. *J. Differential Geom.*, 59(3):479–522, 2001.

[12] H. Donnelly and P. Li. Lower bounds for the eigenvalues of Riemannian manifolds. *Michigan Math. J.*, 29(2):149–161, 1982.

[13] R. Hempel and J. Voigt. The spectrum of a Schrödinger operator in $L_p(\mathbf{R}^\nu)$ is p-independent. *Comm. Math. Phys.*, 104(2):243–250, 1986.

[14] H. Hezari, C. Kelleher, S. Seto, and H. Xu. Asymptotic expansion of the Bergman kernel via perturbation of the Bargmann-Fock model. 2014.

[15] S. G. Krantz. *Geometric analysis of the Bergman kernel and metric*, volume 268 of *Graduate Texts in Mathematics*. Springer, New York, 2013.

[16] C.-j. Liu. The asymptotic Tian-Yau-Zelditch expansion on Riemann surfaces with constant curvature. *Taiwanese J. Math.*, 14(4):1665–1675, 2010.

[17] C.-J. Liu and Z. Lu. Asymptotic Tian-Yau-Zelditch expansions on singular Riemann surfaces. *J. Fixed Point Theory Appl.*, 10(2):327–338, 2011.

[18] C.-J. Liu and Z. Lu. Abstract Bergman kernel expansion and its applications. 2014.

[19] Z. Lu. On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch. *Amer. J. Math.*, 122(2):235–273, 2000.

[20] Z. Lu and G. Tian. The log term of the Szegő kernel. *Duke Math. J.*, 125(2):351–387, 2004.

[21] Z. Lu and S. Zelditch. Szegő kernels and Poincaré series. 2014.

[22] X. Ma and G. Marinescu. The Spin^c Dirac operator on high tensor powers of a line bundle. *Math. Z.*, 240(3):651–664, 2002.

[23] X. Ma and G. Marinescu. Generalized Bergman kernels on symplectic manifolds. *C. R. Math. Acad. Sci. Paris*, 339(7):493–498, 2004.

[24] X. Ma and G. Marinescu. The first coefficients of the asymptotic expansion of the Bergman kernel of the Spin^c Dirac operator. *Internat. J. Math.*, 17(6):737–759, 2006.

[25] X. Ma and G. Marinescu. *Holomorphic Morse inequalities and Bergman kernels*, volume 254 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

[26] W.-D. Ruan. Canonical coordinates and Bergmann [Bergman] metrics. *Comm. Anal. Geom.*, 6(3):589–631, 1998.

[27] L. Saloff-Coste. Uniformly elliptic operators on Riemannian manifolds. *J. Differential Geom.*, 36(2):417–450, 1992.

[28] B. Shiffman and S. Zelditch. Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds. *J. Reine Angew. Math.*, 544:181–222, 2002.

[29] K.-T. Sturm. On the L^p-spectrum of uniformly elliptic operators on Riemannian manifolds. *J. Funct. Anal.*, 118(2):442–453, 1993.

[30] G. Székelyhidi. *An introduction to extremal Kähler metrics*, volume 152 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2014.

[31] G. Tian. On a set of polarized Kähler metrics on algebraic manifolds. *J. Differential Geom.*, 32(1):99–130, 1990.

[32] H. Xu. A closed formula for the asymptotic expansion of the Bergman kernel. *Comm. Math. Phys.*, 314(3):555–585, 2012.

[33] S. Zelditch. Szegő kernels and a theorem of Tian. *Internat. Math. Res. Notices*, (6):317–331, 1998.