MATH 122B: INTRODUCTION TO THEORY OF COMPLEX VARIABLES

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1. Review of Math 122A

Convention. We use i to denote the scalar such that $i^2 = -1$. Then

$$\mathbb{C} = \{z := x + iy \mid x, y \in \mathbb{R}\}$$

and the conjugate $\bar{z} = x - iy$.

1.1. Notions of regularity.

Definition 1.1. Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be a function defined in a neighborhood of a point $z_0 \in \mathbb{C}$. Then, the **derivative** of f at z_0 is given by

$$\frac{df}{dz}(z_0) = f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

provided this limit exists.

Remark 1.1. Note that unlike the case for $\mathbb{R}$, h tends to zero from possibly a complex direction as well.

Definition 1.2. A complex-valued function f continuous in an open set $D \subset \mathbb{C}$ is called **holomorphic** if it has a derivative at every point of D.

Lets first consider an example of a *non-holomorphic* function.

Example 1.1. The function $f(z) = \overline{z}$ is continuous for all $z \in \mathbb{C}$. However this function is nowhere differentiable. Without loss of generality, let $z_0 = 0$. Then for $h \in \mathbb{R}$

$$\lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

and

$$\lim_{h \to 0} \frac{f(ih)}{ih} = \lim_{h \to 0} \frac{-ih}{ih} = -1.$$

Since we have found two paths with different limits, f is not differentiable.

Let us now investigate the conditions required for (complex) differentiability. Since f is differentiable, say at z_0, the limit of the difference quotient must be equal for any path. Let $f(z) = f(x+iy) = u(x,y) - iv(x,y)$ and $h \in \mathbb{R}$ such that $h \to 0$. Then

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{f(z_0 + ih) - f(z_0)}{ih}.$$

So that

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Matching the real and imaginary parts, we obtain a set of equations, called the **Cauchy-Riemann Equations**.

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0. \end{cases}$$

We see that a (complex) differentiable function necessarily satisfies the Cauchy-Riemann equation. It turns out that these are also sufficient conditions for differentiability.

Theorem 1.1. *Let $f(z) = u(x,y) + iv(x,y)$ be defined in a neighborhood of the point $z_0 = x_0 + iy_0 \in \mathbb{C}$. Then, f is differentiable at the point z_0 if and only if both the real and imaginary part of f are differentiable at the point (x_0, y_0) and satisfy the Cauchy-Riemann equations*

Proof. We already showed the necessary direction. Assume now that u and v are differentiable at the point (x_0, y_0) and satisfy the Cauchy-Riemann equations. By Taylor expansion,

$$u(x_0 + h, y_0 + k) - u(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0)h + \frac{\partial u}{\partial y}(x_0, y_0)k + \varepsilon_1(h,k)$$

and

$$v(x_0 + h, y_0 + k) - v(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0)h + \frac{\partial v}{\partial y}(x_0, y_0)k + \varepsilon_2(h,k)$$

where ε_i is a function such that $\lim\limits_{(h,k)\to(0,0)} \frac{\varepsilon_i(h,k)}{\sqrt{h^2+k^2}} = 0$ for $i = 1, 2$. Then

$$\begin{aligned} \lim_{h+ik \to 0} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0)}{h + ik} &= \lim_{h+ik \to 0} \frac{\frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k + i(\frac{\partial v}{\partial x}h + \frac{\partial v}{\partial y}k) + \varepsilon_1 + i\varepsilon_2}{h + ik} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \lim_{h+ik \to 0} \frac{\varepsilon_1 + i\varepsilon_2}{h + ik} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{aligned}$$

where we used the Cauchy-Riemann equations in the second line. □

Example 1.2. Consider the function $f(z) = \frac{1}{z}$. This function is differentiable on $\mathbb{C} - \{0\}$. We compute the real and imaginary parts:

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} := u + iv.$$

The Cauchy-Riemann equations are easily seen to be satisfied by symmetry.

Definition 1.3. A continuous function f which is defined in an open set $D \subset \mathbb{C}$ is called **analytic** if it admits at every point of D a power series expansion: For every $z_0 \in D$, there exists $R > 0$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad \forall z \in B(z_0, R).$$

The largest $R > 0$ for which the power series converges is called the **radius of convergence** and is given by the formula

$$R = \frac{1}{\limsup\limits_{n\to\infty} |a_n|^{\frac{1}{n}}}$$

We will see later that holomorphic functions are in fact analytic, hence we can use the terms interchangeably in this course.

An alternative characterization for holomorphic can be given by first defining the differential operators

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \end{aligned}$$

Note that for $z = x + iy$, we have

$$\frac{\partial}{\partial z}(z) = 1, \quad \frac{\partial}{\partial \bar{z}}(z) = 0,$$

and similarly for $\bar{z}$. Then

Proposition 1.1. The Cauchy-Riemann equations hold if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Proof. Let $f = u + iv$. Then

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) \\ &= \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0. \end{aligned}$$

$\square$

1.2. **Integration.** Given a curve $C \in \mathbb{C}$ and a function f whose domain contains C, the **contour integral** of f along C can be evaluated by taking a parametrized curve $z(t) : [a,b] \to \mathbb{C}$ whose image is C:

$$\oint_C f dz = \int_a^b f(z(t))z'(t)dt.$$

The contour integral is invariant under reparametrization, which follows from the change of variables theorem.

Example 1.3. Let $z_0 \in \mathbb{C}$, we compute the contour integral of $f(z) = \frac{1}{z-z_0}$ along the boundary of the ball $B(z_0, R)$ for some $R > 0$. The boundary of the ball can be parametrized by the curve

$$z(t) = z_0 + Re^{it}, \quad 0 \leq t \leq 2\pi,$$

so that

$$z'(t) = iRe^{it}dt$$

Then

$$\begin{aligned} \oint_{\partial B(z_0,R)} \frac{dz}{z-z_0} &= \int_0^{2\pi} \frac{iRe^{it}}{Re^{it}} dt \\ &= 2\pi i. \end{aligned}$$

Note that the value does not depend on R.

In practice, when we want to evaluate certain contour integrals, we apply the fundamental theorem of line integrals

Theorem 1.2. Let $D \subset \mathbb{C}$ be an open connected set, and let C be a smooth path with parametrization $z(t)$, $a \leq t \leq b$. Let f be holomorphic in D and f' be continuous on D. Then

$$\int_C f'(z)dz = f(\gamma(b)) - f(\gamma(a)).$$

Remark 1.2.

In fact, the following theorem holds

Theorem 1.3. Let $D \subset \mathbb{C}$ be open and connected and let g be continuous in D. A necessary and sufficient condition for g to have a primitive in D, i.e. G such that $G' = g$ is that

$$\oint_C g(z)dz = 0$$

holds for every closed path C in D.

Relating to holomorphic functions, we have that

Theorem 1.4. If f is holomorphic at all points interior to and on a simple closed contour C, then

$$\oint_C f(z)dz = 0.$$

A rigorous proof of the above uses the fact that the Cauchy-Riemann equations are exactly the integrability conditions required for a function to have a primitive, however, here we present a quick and dirty “proof”.

Proof. By Green's Theorem,

$$\begin{aligned}\oint_C f(z)dz &= \oint_C (u+iv)(dx+idy)\\ &= \oint_C (u+iv)dx + (-v+iu)dy\\ &= \iint_D -v_x + iu_x - u_y - iv_y dA = 0\end{aligned}$$

where the last equality follows from the Cauchy-Riemann equations. □

Remark 1.3. The "proof" would only hold for sufficiently nice curves C.

Also useful for computations is the following independence of path theorem

Theorem 1.5. Let γ_1 and γ_2 be positively oriented, simple, closed contours with γ_2 interior to γ_1. If f is holomorphic on the closed region containing γ_1 and γ_2 and points between them, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Sketch of proof. We simply connect the two contours by two lines to create two closed contours. The integral along this closed contour vanishes. By splitting the integral into pieces, we can solve for "half" of the integral of γ_1 in terms of the lines and "half" of the integral of γ_2. □

The above theorems lead to the central theorem of complex analysis,

Theorem 1.6 (Cauchy's Integral Formula)**.** Let γ be a simple, closed, positively oriented contour. If f is analytic in some simply connected domain D containing γ and z_0 is any point inside γ, then

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}dz.$$

Proof. The function $\frac{f(z)}{z-z_0}$ is analytic everywhere in S except at the point z_0, hence we can deform the contour to a circle centered at z_0 with small radius R i.e.,

$$\int_\gamma \frac{f(z)}{z-z_0}dz = \int_{\partial B_R} \frac{f(z)}{z-z_0}dz.$$

Now

$$\frac{1}{2\pi i}\int_{\partial B_R} \frac{1}{z-z_0}dz = 1$$

hence

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial B_R} \frac{f(z_0)}{z-z_0}dz.$$

Since the values of the integral are invariant under change of radius R, we will show that

$$\lim_{R\to 0}\int_{\partial B_R} \frac{f(z)-f(z_0)}{z-z_0}dz = 0.$$

Let $\varepsilon > 0$ be fixed. Since f is holomorphic, in particular, it is continuous, hence there is some $\delta > 0$ such that $|z-z_0| < \delta$ implies that $|f(z)-f(z_0)| < \varepsilon$. Choose R such that $2R < \delta$, then

$$\left|\int_{\partial B_R} \frac{f(z)-f(z_0)}{z-z_0}dz\right| \le \varepsilon\int_{\partial B_R} \frac{1}{|z-z_0|}dz \le \varepsilon\frac{2\pi R}{R} = 2\pi\varepsilon.$$

□

We can extend the Cauchy integral formula to derivatives of holomorphic functions as well

Theorem 1.7. Let f be holomorphic in and on a simple closed contour C, with positive orientation. Then

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw,$$

for z contained in the region interior of the contour.

Proof. Formally, this is essentially moving the derivative inside the integral and differentiating with respect to the z variable. However, we need to justify this step and so we will show that f' exists and is equal to the expression given.

By Cauchy's integral formula, we have

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{1}{h}\int_C \left(\frac{1}{w-(z+h)} - \frac{1}{w-z}\right) f(w)dw \\ &= \int_C \frac{f(w)}{(w-z-h)(w-z)}dw.\end{aligned}$$

Here we choose h such that $|h|$ is sufficiently small. Then

$$\int_C \frac{f(w)}{(w-z-h)(w-z)}dw - \int_C \frac{f(w)}{(w-z)^2}dw = h\int_C \frac{f(w)}{(w-z-h)(w-z)^2}dw.$$

Let $M = \max_C |f(w)|$ and let $d = \mathrm{dist}(z, C)$ so that $|w-z| \geq d > 0$ for all w on C. Also assume that $|h| < \frac{d}{2}$. Then

$$|w-z-h| \geq |w-z| - |h| \geq \frac{d}{2}$$

for all w on C, hence

$$\left|\frac{f(w)}{(w-z-h)(w-z)^2}\right| \leq \frac{2M}{d^3}$$

hence

$$\left|h\int_C \frac{f(w)}{(w-z-h)(w-z)^2}dw\right| \leq \frac{2hML(C)}{d^3}$$

where $L(C)$ is the length of the contour. Let $h \to 0$ and we obtain our result. □

By induction and similar technique, we obtain the general derivative formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)dw}{(w-z)^{n+1}}.$$

An immediate consequence of this theorem is the following

Theorem 1.8. If f is holomorphic at z_0, then it is infinitely differentiable and its derivatives are also holomorphic at z_0.

Hence if f is holomorphic, the extra assumption f' is continuous may be dropped.

1.3. **Power series.** Now we are ready to show that holomorphic functions are analytic.

Theorem 1.9. Suppose that a function f is holomorphic on a ball $B(z_0, R)$. Then $f(z)$ is analytic in $B(z_0, R)$.

Proof. We need to show that $f(z)$ has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for $z \in B(z_0, R)$. Without loss of generality, let $z_0 = 0$. Now, By Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw$$

where C is the contour of $\partial B(0, R)$. By finite geometric series expansion, we have

$$\frac{1}{w - z} = \sum_{n=0}^{N-1} \frac{1}{s^{n+1}} z^n + z^N \frac{1}{(w - z)w^N}.$$

Hence

$$\int_C \frac{f(w)}{w - z} dw = \sum_{n=0}^{N-1} \int_C \frac{f(w)}{w^{n+1}} dw z^n + z^N \int_C \frac{f(w)}{(w - z)w^N}.$$

By the Cauchy derivative formula, we know that

$$\frac{1}{2\pi i} \int_C \frac{f(w)}{w^{n+1}} dw = \frac{f^{(n)}(0)}{n!}$$

so that

$$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \frac{z^N}{2\pi i} \int_C \frac{f(w)}{(w - z)z^N} dw.$$

Now we bound the remainder term, let M be the maximum of f on C. We have

$$|w - z| \geq R - |z| > 0$$

and so

$$\left| \frac{z^N}{2\pi i} \int_C \frac{f(w)}{(w - z)z^N} dw \right| \leq \frac{MR}{R - |z|} \left(\frac{|z|}{R} \right)^N \to 0$$

as $N \to \infty$. □

Suppose f is not holomorphic at a point z_0 but is holomorphic in its neighborhood. While we cannot directly find a Taylor expansion, we may be able to still find a series expansion with singularities i.e. one with positive and negative powers. Such is series is called a **Laurent Series**.

Theorem 1.10. Let $f(z)$ be analytic in an annulus domain $A = \{z \mid R_1 < |z - z_0| < R_2\}$. Then $f(z)$ can be represented by the Laurent series

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j}, \quad z \in A,$$

where the coefficients can be computed by

$$a_j = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{j+1}}dw, \quad j \in \mathbb{Z}$$

and

$$b_j = \frac{1}{2\pi i}\int_C f(w)(w-z_0)^{j-1}dw, \quad j \in \mathbb{N},$$

and C is any positively oriented, simple closed contour around z_0 lying in A.

Proof. Let $z \in A$. Let C_1 and C_2 be two circles, positively oriented, inside A such that z in contained between them and assume C_1 is the "outer" circle. By Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}dw$$

$w \in C_1$, we have that $|z-z_0| < |w-z_0|$, hence

$$\frac{1}{w-z} = \frac{1}{w-z_0}\frac{1}{1-\left(\frac{z-z_0}{w-z_0}\right)} = \frac{1}{w-z_0}\sum_{j=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^j$$

uniformly on C_1, hence we can change the integral and the summation so that

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}dw = \sum_{j=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-z_0)^{j+1}}dw\right)(z-z_0)^j.$$

and similarly, for $w \in C_2$, we have $|w-z_0| < |z-z_0|$ and so

$$\begin{aligned} -\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}dw &= \frac{1}{2\pi i}\int_{C_2} f(w)\sum_{j=0}^{\infty}\frac{(w-z_0)^j}{(z-z_0)^{j+1}}dw \\ &= \sum_{j=1}\left(\frac{1}{2\pi i}\int_{C_2} f(w)(w-z_0))^{j-1}dw\right)\frac{1}{(z-z_0)^j}. \end{aligned}$$

□

1.4. **Classification of singularities.** A point z_0 is called **singular point** of the function $f(z)$ if $f(z)$ is not analytic/holomorphic at z_0 but is analytic at some point in $B(z_0, r)$ for all $r > 0$. It is an **isolated** singularity if there exists $R > 0$ such that $f(z)$ is analytic on some punctured open disk $0 < |z - z_0| < R$.

Suppose f is holomorphic in a punctured neighborhood $N(z_0)$ of an isolated singular point. Isolated singularities are further classified as follows:

(1) **Poles** If $f(z)$ has the form

$$f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j + \frac{a_{-1}}{z-z_0} + \ldots + \frac{a_{-n}}{(z-z_0)^n},$$

with $a_{-n} \neq 0$, then z_0 is a **pole of order** n. If $n = 1$, it is called a **simple pole**.

(2) **Removable singularity** If a single-valued function $f(z)$ is not defined at $z = z_0$ but $\lim_{z \to z_0} f(z)$ exists, then $z = z_0$ is called **removable singularity**.

(3) **Essential singularity** If $f(z)$ is single valued, then any singularity that is not a pole or a removable singularity is called an **essential singularity**.

We will investigate these singularities later.

2. Cauchy's Residue Theorem

Suppose f is a holomorphic function in a punctured neighborhood of some isolated singular point z_0. Then it has a Laurent series expansion $\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$. The coefficient a_{-1} is called the **residue** of $f(z)$ at z_0 and is denoted by $\operatorname{Res}(f, z_0)$.

Example 2.1. Since

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots$$

we have $\operatorname{Res}(e^{1/z}, 1) = 1$.

Example 2.2. To compute $\operatorname{Res}(e^{z+1/z}, 0)$, we have

$$e^{z+1/z} = e^z e^{1/z} = \left(\sum_{j=0}^{\infty} \frac{z^j}{j!}\right)\left(\sum_{k=0}^{\infty} \frac{1}{k! z^k}\right)$$

and the coefficient of $1/z$ is when $k = j+1$ hence $\sum_{j=0}^{\infty} \frac{1}{j!(j+1)!}$.

We can see that instead of explicitly computing the integral formula for a_{-1}, we can expand simpler functions by their series and look at the coefficients.

Theorem 2.1. If $f(z)$ has a removable singularity at z_0, then $\operatorname{Res}(f, z_0) = 0$

Proof. Since the coefficients of the negative powers of $z - z_0$ in its Laurent expansion are zero, the residue is zero. □

Theorem 2.2. If $f(z)$ has a pole of order m at z_0, then

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m f(z)).$$

Proof. Since z_0 is a pole of order m, the Laurent series for $f(z)$ around z_0 is given by

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots \frac{a_{-1}}{(z-z_0)} + \sum_{j=0}^{\infty} a_j(z-z_0)^j.$$

So that

$$(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + \cdots$$

Differentiating $m-1$ times, we have

$$\frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m f(z)) = (m-1)! a_{-1} + m! a_0(z-z_0) + \cdots$$

hence $z \to z_0$ gives the result. □

Corollary 2.1. If $f(z) = \frac{P(z)}{Q(z)}$ where P and Q are both holomorphic at z_0 and $Q(z)$ has a simple zero at z_0, i.e. $Q(z_0) = 0$ and $Q'(z_0) \neq 0$, and $P(z_0) \neq 0$, then

$$\operatorname{Res}(f, z_0) = \frac{P(z_0)}{Q'(z_0)}.$$

Proof. $f(z)$ has a simple pole at z_0 hence

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0)\frac{P(z)}{Q(z)} = \lim_{z \to z_0} (z - z_0)\frac{P(z)}{Q(z) - Q(z_0)}$$

□

Example 2.3. For $f(z) = \dfrac{z}{z^n - 1}$, there is a simple pole at each n-th root of unity, $\alpha_k = e^{2i\pi k/n}$. Hence

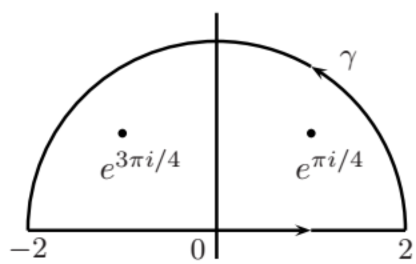
$$\operatorname{Res}(\frac{z}{z^n - 1}, \alpha_k) = \frac{\alpha_k}{n\alpha_k^{n-1}}.$$

The residues are useful when computing contour integrals:

Theorem 2.3. If C is a postively oriented simple closed contour and f is holomorphic inside and on C except at the points $z_1, z_2, \ldots z_n$ inside C, then

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_j).$$

Example 2.4. Evaluate $\displaystyle\int_C \frac{dz}{z^4 + 1}$, where C is the contour



2.1. **Residue at infinity.** Now suppose $f(z)$ is holomorphic in a punctured neighborhood of $z = \infty$, i.e., $f(z) = \displaystyle\sum_{j=-\infty}^{\infty} a_j z^j$ holds for $R < |z| < \infty$. We can compute by considering a change of variables $z = 1/w$. Then its Laurent series changes by

$$\cdots + a_1 z + a_0 + \frac{a_{-1}}{z} + \cdots = \cdots \frac{a_1}{w} + a_0 + a_{-1}w + \cdots$$

hence to obtain the a_{-1} coefficient, we multiply by $-1/w^2$ so that

$$\operatorname{Res}(f, \infty) = -\operatorname{Res}(\frac{1}{w^2} f\left(\frac{1}{w}\right), 0).$$

Let $C_r = \{z \mid |z| = r\}$, $r > R$. We can still apply the residue theorem for singularities at $z = \infty$ by considering a positively oriented curve around ∞ as a one with the orientation reversed so that

$$\int_{C_r} f(z)dz = \int_{C_r} \frac{a_{-1}}{z} dz = -2\pi i a_{-1}.$$

More precisely, if we use the change of variables $w = \frac{1}{z}$, then

$$f(w) = \sum_{j=-\infty}^{\infty} a_j w^{-j}$$

and the singularity is now at $w = 0$ so

$$\int_{C_r} f(z)dz = \int_{C_{1/r}} f(w)d(1/w) = -\int_{C_{1/r}} \frac{f(w)}{w^2} dw = -\int_{C_{1/r}} \frac{a_{-1}w}{w^2} dw = -2\pi i a_{-1}$$

This leads to the following theorem

Theorem 2.4. Let the function $f(z)$ be holomorphic in the extended complex plane, except at isolated singular points. Then, the sum of all residues of $f(z)$ is equal to zero.

Proof. The function $f(z)$ can only have a finite number of singularities otherwise, there would be a limit point, possibly at infinity, which will be a nonisolated singularity. Thus there exists a positively oriented circle C_R such that all the finite singularities, $z_1, \ldots, z_n$ are contained in C_R. Then by the residue theorem,

$$\int_{C_R} f(z)dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_j)$$

and applying the residue theorem at infinity, we have

$$-\int_{C_R} f(z)dz = 2\pi i \operatorname{Res}(f, \infty).$$

$\square$

Example 2.5. Consider

$$\int_C \frac{dz}{(z-7)(z^{23}-1)}$$

where C is given by the positively oriented curve $|z| = 3$. By the residue theorem, we have

$$\int_C \frac{dz}{(z-7)(z^{23}-1)} = 2\pi i \sum_{j=1}^{2} 3 \operatorname{Res}(f, \omega_j)$$

where ω_j are the 23 roots of unity. This is difficult to compute however since the sum of the residues is zero, we have

$$2\pi i \sum_{j=1}^{2} 3 \operatorname{Res}(f, \omega_j) = -2\pi i(\operatorname{Res}(f, \infty) + \operatorname{Res}(f, 7) = \frac{-2\pi i}{7^2 3 - 1}.$$

2.2. **Evaluation of real integrals by contour integration.** We can apply the Residue theorem to evaluate real integrals.

2.2.1. *Trigonometric Integrals.* First we discuss how to integrate integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta,$$

where $R(x, y)$ is a rational function with real coefficients and whose denominator does not vanish on $[0, 2\pi]$. Let C be a positively oriented unit circle $|z| = 1$. This can be parametrized by $z = e^{i\theta}$ and so

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right), \text{ and } \sin(\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Also, by change of variables,

$$d\theta = -i\frac{dz}{z}.$$

Hence we can change the real integral into a contour integral

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta = \int_C R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right)\frac{dz}{iz}.$$

Example 2.6. Evaluate $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta}$, $0 < |a| < 1$. By the transformation formula above, we get

$$I = \int_C \frac{2}{az^2 + 2iz - a}dz = \frac{2}{a}\int_C \frac{dz}{[z + i(1 + \sqrt{1 - a^2})/a][z + i(1 - \sqrt{1 - a^2})/a]}.$$

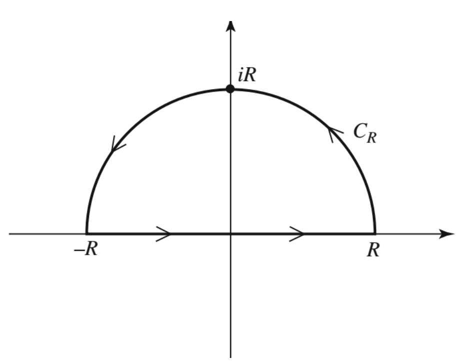
Since $|(1 + \sqrt{1 - a^2})/a| > 1$ and $|(1 - \sqrt{1 - a^2})/a| < 1$, by residue theorem we have

$$I = \frac{2\pi}{\sqrt{1 - a^2}}.$$

2.2.2. *Rational Improper.* Next we evaluate improper integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx,$$

where P and Q are polynomials. We must make further assumptions that $Q(x) \neq 0$ for $x \in \mathbb{R}$ and $\deg Q \geq 2 + \deg P$, which is natural if we want to ensure that the integral converges. Let C_R be the closed contour consisting of the real line segment from $-R$ to R and the upper semi-circle centered at the origin of radius R.



Assume that R is large enough so that it contains all the complex zeroes of $Q(x)$ in the upper half plane.. Then by the residue theorem,

$$\int_{C_R} \frac{P(z)}{Q(z)} dz = 2\pi i \sum_k \mathrm{Res}(\frac{P}{Q}, z_k)$$

where z_k are the complex zeroes of Q. We can split the contour integral as

$$\int_{C_R} \frac{P}{Q} = \int_{-R}^{R} \frac{P}{Q} + \int_{\Gamma_R} \frac{P}{Q}$$

where Γ_R is the upper half of the circle C_R. By our assumption, for large R, we have

$$\int_{\Gamma_R} \frac{P}{Q} \leq \frac{A}{R}$$

for some constant A, hence letting $R \to \infty$, we obtain

$$\int_{-\infty}^{\infty} \frac{P}{Q} = 2\pi i \sum_k \mathrm{Res}(\frac{P}{Q}, z_k).$$

Example 2.7.

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\pi i \,\mathrm{Res}(\frac{1}{z^4+1}, z_k).$$

Of the four 4-th roots of unity, $z_1 = e^{i\pi/4}$ and $z_2 = e^{i3\pi/4}$ are in the upper half plane, hence

$$\mathrm{Res}(\frac{1}{z^4+1}, e^{i\pi/4}) = \frac{1}{4z_1^3} = \frac{-z_1}{4} = -\frac{1}{8}(\sqrt{2} + i\sqrt{2}).$$

and

$$\mathrm{Res}(\frac{1}{z^4+1}, e^{3i\pi/4}) = \frac{1}{8}(\sqrt{2} - i\sqrt{2}),$$

hence

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{2}.$$

2.2.3. *Rational function times a trigonometric function.* Next we compute integrals of the form

$$\int_{-\infty}^{\infty} R(x)\cos(x)dx, \quad \int_{-\infty}^{\infty} R(x)\sin(x)dx$$

where

$$R(x) = \frac{P(x)}{Q(x)}$$

such that P and Q are polynomials and $Q(x) \neq 0$. The above converges as long as $\deg Q > \deg P$. We will show that

$$\int_{\Gamma_R} R(z)e^{iz}dz \to 0$$

so that

$$\int_{C_R} R(z)e^{iz}dz \to \int_{-\infty}^{\infty} R(x)e^{ix}dx.$$

Then its real and imaginary parts are the trigonometric functions we wanted to evaluate. Now consider the two regions of Γ_R:

$$\begin{aligned} A &= \{z \in \Gamma_R \mid \operatorname{Im} z \geq h\} \\ B &= \{z \in \Gamma_R \mid \operatorname{Im} z < h\}. \end{aligned}$$

Since $\deg Q > \deg P$, we have $|R(z)| \leq \frac{K}{|z|}$ for R sufficiently large. Also note $|e^z| = e^{\operatorname{Re} z}$, hence

$$\left| \int_A R(z) e^{iz} dz \right| \leq C_1 e^{-h}.$$

Also

$$\left| \int_B R(z) e^{iz} dz \right| \leq \frac{K}{R} \int_B dz \leq C_2 \frac{h}{R}.$$

Hence, combining the two, we have

$$\left| \int_{\Gamma_R} R(z) e^{iz} dz \right| \leq C_1 e^{-h} + C_2 \frac{h}{R}.$$

Choosing $h = \sqrt{R}$, we have

$$\int_{C_R} R(z) e^{iz} dz \leq C_1 e^{-\sqrt{R}} + \frac{C_2}{\sqrt{R}} \to 0$$

Example 2.8. Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx$$

This is the imaginary part of

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx,$$

which has a pole at $x = 0$, hence we need to consider

$$\operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{ix} - 1}{x} dx.$$

Note that

$$0 = \int_{C_R} \frac{e^{iz} - 1}{z} dz = \int_{-R}^{R} \frac{e^{ix} - 1}{x} dx + \int_{\Gamma_R} \frac{e^{iz} - 1}{z} dz$$

where the first equality is from Cauchy integral theorem, since $z = 0$ is a removable singularity. Therefore,

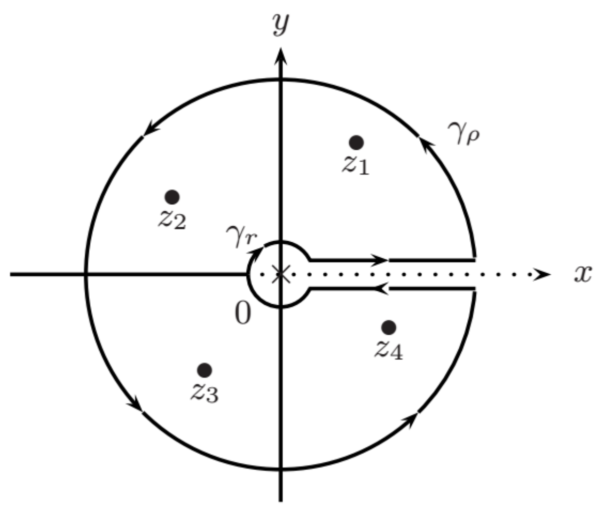
$$\begin{aligned} \int_{-R}^{R} \frac{e^{ix} - 1}{x} dx = \int_{\Gamma_R} \frac{1 - e^{iz}}{z} dz &= \int_{\Gamma_R} \frac{1}{z} dz - \int_{\Gamma_R} \frac{e^{iz}}{z} dz \\ &= \pi i - \int_{\Gamma_R} \frac{e^{iz}}{z} dz \to \pi i. \end{aligned}$$

2.2.4. *Contour integrals involving multi-valued functions.* We will investigate how to evaluate integrals of the form

$$I = \int_0^\infty x^{a-1} f(x)dx, \quad 0 < a < 1,$$

where $f(z)$ is a single-valued analytic function, except for a finite number of isolated singularities not on the positive real axis, has a removable singularity at $z = 0$ and f has a zero of order at least one at $z = \infty$.

Consider the domain $S : 0 < \arg z < 2\pi$, which is the z-plane cut along the positive real axis. Then z^{a-1} is single valued in S. Consider the contour C given by



where γ_ρ is a circle large enough to contain all the singularities and γ_r is small so that it contains no singularities.

By residue theorem, we have

$$\begin{aligned} 2\pi i \sum_k \operatorname{Res}(z^{a-1} f(z), z_k) &= \int_C z^{a-1} f(z) \\ &= \int_r^\rho x^{a-1} f(x)dx + \int_{\gamma_\rho} z^{a-1} f(z)dz \\ &\quad + \int_\rho^r z^{a-1} f(z)dz - \int_{\gamma_r} z^{a-1} f(z)dz. \end{aligned}$$

Since the zero at ∞ as order at least one, we have that

$$\left| \int_{\gamma_\rho} z^{a-1} f(z)dz \right| \leq 2\pi M \rho^{a-1} \to 0$$

for some M as $\rho \to \infty$. Since $z = 0$ is a removable singularity, it is bounded near $z = 0$ as well hence

$$\left| \int_{\gamma_r} z^{a-1} f(z)dz \right| \leq C r^{a-1} 2\pi r \to 0$$

as $r \to 0$. Since $\arg z = 2\pi$ for the ray on the bottom of the real axis, we parametrize $z = xe^{2\pi i}$ so

$$\int_\rho^r z^{a-1} f(z)dz = -e^{2\pi i(a-1)} \int_r^\rho x^{a-1} f(x)dx$$

Combining these, we get

$$\int_0^\infty x^{a-1} f(x)dx = \frac{2\pi i}{1-e^{2\pi a i}} \sum \text{Res}(z^{a-1}f(z), z_j).$$

Example 2.9. When evaluating the integral $\int_0^\infty \frac{\sqrt{x}}{x^3+1}dx$, we rewrite in the above form so that

$$\begin{aligned}\int_0^\infty \frac{\sqrt{x}}{x^3+1}dx &= \int_0^\infty x^{1/2-1}\frac{x}{x^3+1}dx \\ &= \frac{2\pi i}{1-e^{\pi i}} \sum_{k=1}^{3} \text{Res}(\frac{\sqrt{z}}{z^3+1}, \alpha_k)\end{aligned}$$

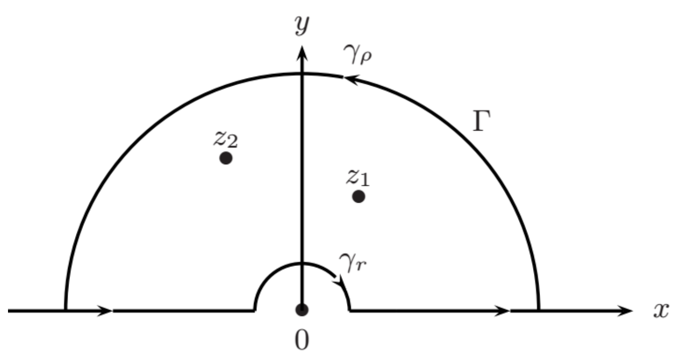
where α_k are the third roots of unity.

Next we evaluate integrals of the form

$$I = \int_0^\infty f(x)\,\text{Log}(x)dx$$

where f is an even function and holomorphic on the upper half plane and assume for sufficiently large $|z|$, we have $|f(z)| \leq \frac{M}{|z|^2}$ for some M. Here $\text{Log}(x)$ is the real log and $\text{Log}(z) = \text{Log}\,|z| + i\,\text{Arg}\,z$ is the principal (complex) log.

Let Γ be the closed contour



We then split the integral into the following pieces:

$$\begin{aligned}\int_\Gamma f(z)\,\text{Log}\,z dz &= 2\pi i \sum \text{Res}(f(z)\,\text{Log}\,z, z_j) \\ &= \int_r^\rho f(x)\,\text{Log}\,x dx + \int_{\gamma_\rho} f(z)\,\text{Log}\,z dz \\ &\quad + \int_r^\rho f(x)(\text{Log}\,x + \pi i)dx - \int_{\gamma_r} f(z)\,\text{Log}\,z dz.\end{aligned}$$

For the upper half circle, we have

$$\begin{aligned}\left|\int_{\gamma_\rho} f(z)\,\text{Log}\,z dz\right| &\leq \frac{M}{\rho^2}\int_0^\pi |\,\text{Log}\,\rho e^{i\theta}||\rho e^{i\theta}|d\theta \\ &\leq \frac{M}{\rho}\int_0^\pi |\,\text{Log}\,\rho + i\theta|d\theta \leq \frac{M\pi}{\rho}\sqrt{\text{Log}^2\,\rho + \pi^2} \to 0\end{aligned}$$

as $\rho \to \infty$. Similar bounds from the previous show that the integral over the smaller circle γ_r goes to 0 as $r \to 0$. By residue theorem, we can conclude that

$$\int_0^\infty f(x) \operatorname{Log} x dx = \operatorname{Re} \pi \sum \operatorname{Res}(f(z)(\operatorname{Log} z), z_j)$$

Example 2.10.

$$\int_0^\infty \frac{\operatorname{Log} x}{(x^2+1)^2} dx = \operatorname{Re} \pi i \operatorname{Res}(\frac{\operatorname{Log} z}{(z^2+1)^2}, i) = -\frac{\pi}{4}.$$

2.3. **Summation of series.**

Theorem 2.5. Let $f(z)$ be holomorphic on $\mathbb{C}$ except at finitely many points $z_1, z_2, \dots z_k$, none of which is a real integer. Furthermore, suppose there exists some M such that $|z^2 f(z)| \leq M$ for $|z| > \rho$, for some $\rho > 0$. Consider the functions

$$\begin{aligned} g(z) &= \pi \frac{\cos(\pi z)}{\sin(\pi z)} f(z) \\ h(z) &= \frac{\pi}{\sin(\pi z)} f(z). \end{aligned}$$

Then the following holds:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} f(n) &= -\sum_{j=1}^{k} \operatorname{Res}(g, z_j) \\ \sum_{n=-\infty}^{\infty} (-1)^n f(n) &= -\sum_{j=1}^{k} \operatorname{Res}(h, z_j). \end{aligned}$$

Proof. We will show this for $g(z)$. Suppose first that $f(n) \neq 0$ for all n. Then $g(z)$ has simple poles at each $n \in \mathbb{Z}$ and singularities at $z_1, z_2, \dots z_k$. Let γ be a large rectangle containing all singularities $z_1, z_2, \dots z_k$ of f and integers $-n, \dots -1, 0, 1, \dots n$ and not passing through any integers. By residue theorem,

$$\int_\gamma g(z) dz = 2\pi i \left(\sum_{j=1}^{k} \operatorname{Res}(g, z_j) + \sum_{m=-n}^{n} \operatorname{Res}(g, m) \right).$$

Computing the residues, we have

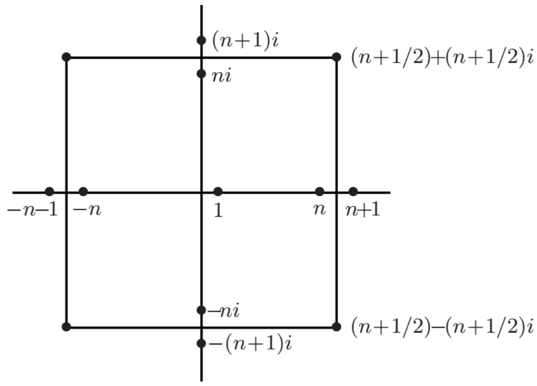
$$\operatorname{Res}(g, m) = \lim_{z \to m} (z-m) g(z) = \lim_{z \to m} f(z) \frac{\pi(z-m)}{\sin(\pi z)} \cos(\pi z) = f(m).$$

Therefore,

$$\int_\gamma g(z) dz = 2\pi i \left(\sum_{j=1}^{k} \operatorname{Res}(g, z_j) + \sum_{m=-n}^{n} f(m) \right).$$

If $f(m) = 0$ for some m, then $g(z)$ has a removable singularity at m, hence has no contribution to $\int_\gamma g(z) dz$ and $f(m)$ has no contribution as well.

Now we show that the contour integral goes to zero as the rectangle grows larger. Let R_n be the rectangle



For large n, we have $|f(z)| \leq \dfrac{M}{n^2}$ for some M.

Next we have

$$\left|\frac{\cos(\pi z)}{\sin(\pi z)}\right| = \left|\frac{e^{\pi ix-\pi y}+e^{-\pi ix+\pi y}}{e^{\pi ix-\pi y}-e^{-\pi ix+\pi y}}\right| \leq \frac{|e^{\pi ix-\pi y}|+|e^{-\pi ix+\pi y}|}{||e^{-\pi ix+\pi y}|-|e^{\pi ix-\pi y}||} \leq \frac{e^{-\pi y}+e^{\pi y}}{e^{\pi y}-e^{-\pi y}} = \frac{1+e^{-2\pi y}}{1-e^{-2\pi y}}.$$

For $y \geq \frac{1}{2}$, we have

$$\frac{1+e^{-2\pi y}}{1-e^{-2\pi y}} \leq \frac{1+e^{-\pi}}{1-e^{-\pi}}.$$

We can do a similar estimate by switching the terms in the denominator so that for $y \leq -\frac{1}{2}$,

$$\left|\frac{\cos(\pi z)}{\sin(\pi z)}\right| \leq \frac{e^{-\pi y}+e^{\pi y}}{e^{-\pi y}-e^{\pi y}} \leq \frac{1+e^{-\pi}}{1-e^{-\pi}}.$$

For $|y| < \frac{1}{2}$ and $z = (N+\frac{1}{2})+iy$, we have

$$\left|\frac{\cos(\pi z)}{\sin(\pi z)}\right| = |\cot(\frac{\pi}{2}+\pi iy)| = |\tanh(\pi y)| \leq \tanh\frac{\pi}{2}$$

and the same bound will hold for $z = -(N+\frac{1}{2})+iy$ so that

$$\left|\int_{R_N} g(z)dz\right| \leq \pi\int_{R_N}\left|\frac{\cos(\pi z)}{\sin(\pi z)}\right||f(z)|dz \leq \frac{\pi AM}{N^2}L(R_N) \leq \frac{\pi AM4(2N+1)}{N^2} \to 0y$$

□

2.4. **Argument principle.**

Theorem 2.6. Let $f(z)$ be meromorphic (ratio of holomorphic functions) inside and on a positively oriented contour γ. Furthermore, let $f(z) \neq 0$ on γ. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = Z_f - P_f$$

where Z_f is the number of zeroes and P_f is the number of poles, both up to multiplicity of f inside γ.

Lemma 2.1. Suppose that f is continuous and assumes only integer values on a connected domain S. Then $f(z)$ is constant on S.

Theorem 2.7 (Rouché's Theorem)**.** Suppose f and g are meromorphic in a domain S. If $|f(z)| > |g(z)|$ for all z on γ where γ is a simple closed positively oriented contour in S and $f(z)$ and $g(z)$ have no zeroes or poles on γ, then

$$Z_f - P_f = Z_{f+g} - P_{f+g}.$$

Proof. First we claim that $f(z) + g(z)$ has no no zeroes on γ. If $f(z_0) + g(z_0) = 0$, then $|f(z_0)| = |g(z_0)|$ on γ, contradicting $|f| > |g|$ on γ. Now, since $|f(z)| - |g(z)|$ is continuous on γ, there must be some $m > 0$ such that $|f| - |g| \geq m > 0$ on γ. Hence

$$|f(z) + tg(z)| \geq |f(z)| - |g(z)| \geq m > 0$$

for $t \in [0, 1]$ and $z \in \gamma$. Therefore,

$$J(t) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z) + tg'(z)}{f(z) + tg(z)} dz$$

is continuous in t. By the argument principle, $J(t) = Z_{f+tg} - P_{f+tg}$ which is an integer, hence $J(0) = J(1)$, which is what we wanted to show. □

Corollary 2.2. Suppose $f(z)$ and $g(z)$ are holomorphic in D. If $|f| > |g|$ for all $z \in \gamma$, where γ is a simple closed contour in D, then $f(z)$ and $f(z) + g(z)$ have the same number of zeroes inside γ counting multiplicities.

Example 2.11. Consider the function $\phi(z) = 2z^5 - 6z^2 + z + 1$. We shall compute how many zeroes ϕ has in the annulus $1 \leq |z| \leq 2$. Let $f(z) = -6z^2$ and $g(z) = 2z^5 + z + 1$. On $|z| = 1$, we have that $|f| = 6$ and $|g| \leq 2 + 1 + 1 = 4$. Hence $|f| > |g|$ on $|z| = 1$. Since f has two zeroes in $|z| < 1$ (counting multiplicity), by Rouché's theorem, $\phi = f + g$ has two zeroes there as well. Next let $f(z) = 2z^5$ and $g(z) = -6z^2 + z + 1$. On $|z| = 2$, we have $|f| = 64$ and $|g| \leq 24 + 2 + 1 = 27$. Hence $|f| > |g|$. Since f has five zeroes in $|z| < 2$, we have that ϕ must have the same. Then on the annulus, ϕ has 3 zeroes.

Example 2.12. Consider the function $\phi(z) = 2 + z^2 + e^{iz}$. We shall show that this has exactly one zero in the open upper half-plane $y > 0$. Let $f(z) = 2 + z^2$ and $g(z) = e^{iz}$. Let γ be the contour $[-R, R] \cup \{z \mid \operatorname{Im} z \geq 0, |z| = R\}$, $R > \sqrt{3}$. On $[-R, R]$, we have $|f(z)| \geq 2 > 1 = |g(z)|$ and for $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, $|f(z)| \geq R^2 - 2 > 1 \geq e^{-R\sin\theta} = |g(z)|$. Then ϕ has the same number of zeroes as $2 + z^2$ on the upper half plane, which is one.

Using this, we can give a proof of the Fundamental Theorem of Algebra,

Theorem 2.8 (Fundamental Theorem of Algebra)**.** Every nonconstant polynomial

$$P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where a_i are complex and $a_n \neq 0$, has at least one zero in $\mathbb{C}$, and hence by induction, has exactly n zeroes, counting multiplicities.

Proof. Let $f(z) = a_n z^n$ and $g(z) = a_{n-1} z^{n-1} + \cdots + a_0$ on $|z| = R$. Then $|f| = |a_n| R^n$ and $|g(z)| \leq |a_{n-1}| R^{n-1} + \cdots + |a_1| R + |a_0|$. Choosing R large so that

$$\frac{|a_{n-1}|}{|a_n|} + \cdots + \frac{|a_1|}{|a_n|} + \frac{|a_0|}{|a_n|} < R$$

we have that $|f| > |g|$ on $|z| = R$. Then f has n zeroes in $|z| = R$ and so $P_n = f + g$ has n zeroes. □

3. Behavior of mappings

The set of equations

$$T(x,y) = (u(x,y), v(x,y))$$

defines a **transformation** or **mapping** between points in (x,y) and points in (u,v).

Definition 3.1. The **Jacobian** of the transformation T is given by

$$\frac{\partial(u,v)}{\partial(x,y)} = u_x v_y - u_y v_x.$$

A special case that we are interested in is when the transformation is holomorphic, i.e. if u and v are holomorphic. Then for $f(z) = (u(z), v(z))$, we have

$$\frac{\partial(u,v)}{\partial(x,y)} = |f'(z)|^2.$$

Suppose we have two intersecting curves C_1 and C_2 in the (x,y) plane. This curve will be mapped by the transformation f to curves C_1' and C_2' in the (u,v) plane. If the angle and orientation between C_1 and C_2 are the same as the angle and the orientation of C_1' and C_2' at the point (x_0,y_0) and (u_0,v_0) respectively, then the map f is said to be **conformal** at (x_0,y_0).

Theorem 3.1. If f is holomorphic and $f'(z) \neq 0$ in a region R, then the mapping $w = f(z)$ is conformal at all points of R.

Proof. Let $C : z(t) = x(t) + iy(t)$ be a smooth curve with $z(t_0) = z_0$. Then the tangent line at C at z_0 has the direction vector $z'(t_0) = x'(t_0) + iy'(t_0)$ and its angle of inclination with the positive real axis is $\operatorname{Arg} z'(t_0)$. Let $\Gamma = f(C)$ be the image curve. It is parametrized by $w(t) = f(z(t))$ and the angle of inclination of its tangent line at $f(z(t_0))$ is given by

$$\operatorname{Arg} w'(t_0) = \operatorname{Arg}(f'(z_0)z'(t_0)) = \operatorname{Arg} f'(z_0) + \operatorname{Arg} z'(t_0).$$

Hence the angle of inclination increases by $f'(z_0)$, hence is conformal. $\square$

3.1. Elementary Transformations.

First we consider $w = az + b$.

This is a linear map which is a 1-1 holomorphic map of the entire plane onto itself, for $a \neq 0$.

Example 3.1. The mapping $w = (1+i)z + 2$ is given by first a dilation and rotation

$$(1+i) = \sqrt{2}e^{i\pi/4}$$

then translate by 2.

Next we consider the transformation $w = \frac{1}{z}$.

We can think of this transformation as a composition of $\frac{z}{|z|^2}$ and conjugation. In terms of (x,y) to (u,v) coordinates, we have

$$T(x,y) = \frac{1}{z} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = (u,v).$$

3.2. **Bilinear transformation.**

Definition 3.2. The transformation

$$w = \frac{az+b}{cz+d}, \quad ad - cb \neq 0$$

is called a **bilinear** or **fractional transformation**.

It can be shown that a bilinear transformation maps circles and lines into circles and lines. The inverse transformation is given by

$$z = \frac{-dw+b}{cw-a}, \qquad \text{for } (ad - bc \neq 0).$$

There is always a linear fractional transformation that maps three given distinct points z_1, z_2, and z_3 onto three specified distinct points w_1, w_2, and w_3, respectively.

Example 3.2. Suppose we want to compute the map which sends the points

$$z_1 = -1, \quad z_2 = 0, \quad z_3 = 1$$

onto

$$w_1 = -i, \quad w_2 = 1, \quad w_3 = i.$$

Plugging in $z_2 = 0$, we have $1 = \dfrac{b}{d}$ so $b = d$. Plugging in the other two, we get the relation

$$ic - ib = -a + b, \qquad \text{and } ic + ib = a + b.$$

Combining, we get

$$w = \frac{iz+1}{-iz+1}.$$

Example 3.3. Suppose we want to compute the linear fractional transformation that sends the points

$$z_1 = 1, \quad z_2 = 0, \quad z_3 = -1$$

onto

$$w_1 = i, \quad w_2 = \infty, \quad w_3 = 1.$$

Plugging in $z = 0$, we need to set $d = 0$ and $c \neq 0$. Hence

$$w = \frac{az+b}{cz}, \quad (bc \neq 0).$$

Plugging the other two pairs, we have

$$ic = a + b, \quad -c = -a + b;$$

so that

$$2a = (1+i)c, \quad 2b = (i-1)c.$$

Substituting, we have

$$w = \frac{(i+1)z + (i-1)}{2z}.$$

We can also compute by using the cross ratio formula

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

3.3. **Automorphisms of the disk.**

Definition 3.3. A conformal mapping of a region onto itself is called an **automorphism** of that region.

The following lemma is straightforward.

Lemma 3.1. Suppose $f : D_1 \to D_2$ is a conformal mapping. Then

(1) any other conformal mapping $h : D_1 \to D_2$ is of the form $g \circ f$, where g is an automorphism of D_2;

(2) any automorphism h of D_1 is of the form $f^{-1} \circ g \circ f$, where g is an automorphism of D_2.

Here we record some more properties of holomorphic functions

Theorem 3.2. If f is holomorphic in D and $a \in D$, then $f(a)$ is equal to the mean value of f taken around the boundary of any disc centered at a contained in D, i.e.

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} f(a + re^{i\theta}) d\theta.$$

Proof. One simply parametrizes the Cauchy integral formula using polar coordinates to obtain the identity. □

Theorem 3.3 (Maximum modulus principle)**.** Suppose that $f(z)$ is analytic in an open disc centered at z_0 and that the maximum value of $|f(z)|$ over this disk is $|f(z_0)|$. Then, $|f(z)|$ is a constant in the disc.

We now begin classifying automorphisms.

Lemma 3.2 (Schwarz's Lemma)**.** Suppose that f is holomorphic in the unit disc, $|f| < 1$ and $f(0) = 0$. Then

(1) $|f(z)| \leq |z|$

(2) $|f'(0)| \leq 1$

with equality if and only if $f(z) = e^{i\theta} z$.

Proof. Consider the function

$$g(z) = \begin{cases} \frac{f(z)}{z}, & \text{for } 0 < |z| < 1 \\ f'(0), & \text{for } z = 0. \end{cases}$$

Then g is holomorphic. Since $|g| \leq \frac{1}{r}$ for all $r < 1$, hence letting $r \to 1$ and by the maximum modulus principle, we have $|g(z)| \leq 1$ throughout the disc. If $|g(z_0)| = 1$ in the interior, then by maximum modulus principle, then g is a unit constant, i.e., $g = e^{i\theta}$. □

Lemma 3.3. The only automorphisms of the unit disc with $f(0) = 0$ are given by $f(z) = e^{i\theta} z$.

Proof. By Schwarz's lemma, we have

$$|f(z)| \le |z|, \quad \text{for } |z| < 1.$$

Moreover, f^{-1} also maps the disc onto itself with $f^{-1}(0) = 0$, hence

$$|f^{-1}(z)| \le |z|, \quad \text{for } |z| < 1.$$

hence, $f(z) = |z|$ so that $f(z) = e^{i\theta} z$. □

If we want to relax the condition that $f(0) = 0$, we look for automorphism of the unit disc such that $f(a) = 0$ for some $0 < |a| < 1$.

Theorem 3.4. The automorphisms of the unit disc are of the form

$$f(z) = e^{i\theta} \left(\frac{z - a}{1 - \bar{a} z} \right)$$

with $|a| < 1$.

Proof. Let $g(z) = \frac{z-a}{1-\bar{a}z}$. Then $|g| = 1$ for $|z| = 1$. Since $g(a) = 0$, it follows that g is an automorphism of the unit disc. Let f be any other automorphism of the disc such that $f(a) = 0$. Then $h = f \circ g^{-1}$ is an automorphism with $h(0) = 0$, hence $h = e^{i\theta} z$, hence solving for f, we get our conclusion. □

Appendix A. Differentiable functions of two real variables

Definition A.1. A real-valued function $u(x, y)$ defined in a neighborhood of the point $(x_0, y_0) \in \mathbb{R}^2$ is said to be **differentiable** at (x_0, y_0) if there exist real numbers a and b such that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{u(x, y) - u(x_0, y_0) - a(x - x_0) - b(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

It is a necessary but not sufficient condition for differentiability that the first partial derivatives exist at the point (x_0, y_0). However, if they are continuously differentiable, then it is a sufficient condition i.e.,

Theorem A.1. Assume that the function $u(x, y)$ admits partial derivatives in a neighborhood of (x_0, y_0) and that they are continuous at the point (x_0, y_0). Then u is differentiable at the point (x_0, y_0).