MATH 124B: INTRODUCTION TO PDES AND FOURIER SERIES

SHO SETO

Contents

1. Fourier Series

Recall the homogeneous wave equation with Dirichlet boundary conditions

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{for } 0 < x < L \\ u(0,t) = 0 \\ u(L,t) = 0 \end{cases} \tag{1}$$

with initial conditions

$$\begin{cases} u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

We look for special solutions to the above by using separation of variables, that is, assume the solution takes the form

$$u(x,t) = F(x)G(t).$$

Plugging into (1), we obtain

$$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}$$

Date: September 5, 2016.

Since the right hand side is independent of the variable x and vice versa, we conclude that both sides are equal to a constant, say $-\lambda := -\beta^2$, $\beta > 0$. λ is positive since

$$F''(x) = -\lambda F(x)$$

and we multiply both side by $F(x)$ and integrate from 0 to L so that

$$\begin{aligned}-\lambda \int_0^L (F(x))^2 dx &= \int_0^L F''(x)F(x)dx \\ &= F'(x)F(x)\Big|_0^L - \int_0^L (F'(x))^2 dx \\ &= -\int_0^L (F'(x))^2 dx \leq 0.\end{aligned}$$

Where we use the Dirichlet boundary condition in the second line. Hence we obtain a pair of ODEs to solve the separated solution. That is

$$\begin{cases} F''(x) + \beta^2 F(x) = 0 \\ G''(t) + c^2\beta^2 G(t) = 0. \end{cases}$$

These can be solved by finding the roots to the characteristic polynomials so that

$$\begin{aligned} F(x) &= A\cos(\beta x) + B\sin(\beta x) \\ G(t) &= C\cos(\beta ct) + D\sin(\beta ct), \end{aligned}$$

where A, B, C, D are constants. Now we plug in the Dirichlet conditions:

$$F(0) = A = 0$$

and

$$F(L) = B\sin(\beta L) = 0.$$

Assuming $B \neq 0$, we have that $\beta = \frac{n\pi}{L}$ for $n \in \mathbb{Z}$. Thus

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

are the possible eigenvalues and

$$F_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

are solutions. Note that each sine function may be multiplied by an arbitrary constant and remains a solution to the ODE. For each n, we obtain a solution

$$u_n(x,t) = \left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi}{L}x\right),$$

where A_n and B_n are arbitrary constants Since the PDE is linear, any finite sum

$$u(x,t) = \sum_n u_n(x,t)$$

is also a solution. If we can choose A_n and B_n such that

$$\phi(x) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right) \tag{2}$$

and

$$\psi(x) = \sum_n \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi}{L}x\right),$$

then we get a solution with the given initial conditions. The question now is, when is it possible to write ϕ and ψ in the given form? Infinite series of the form (2) are called **Fourier sine series** on $(0, L)$.

To obtain the coefficients, we use the following property for sine functions:

Lemma 1.1.

$$\frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq 0. \end{cases}$$

Proof. Using the trig identity

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B)).$$

Then when

$$\frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \frac{1}{L}\int_0^L \cos\left(\frac{(n-m)\pi}{L}x\right) - \cos\left(\frac{(m+n)\pi}{L}x\right)dx$$

Evaluating gives the result. $\square$

Remark 1.1. A similar proof shows orthogonality with the cosine functions as well.

Therefore, integrating against ϕ, all but one term vanishes. This gives us a formula for the Fourier coefficients of the sine series

$$A_m = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{m\pi}{L}x\right)dx.$$

We also define the **Fourier cosine series** as

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right),$$

with the coefficient formula given by

$$A_m = \frac{2}{L}\int_0^L \phi(x)\cos\left(\frac{m\pi}{L}x\right)dx.$$

Finally, summing the sine and cosine series, we obtain the full **Fourier series** of $\phi(x)$ defined on $-L < x < L$.

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi}{L}x\right) + B_n \sin\left(\frac{n\pi}{L}x\right)\right).$$

with the coefficients given by

$$A_n = \frac{1}{L}\int_{-L}^{L} \phi(x)\cos\left(\frac{n\pi}{L}x\right)dx, \quad (n = 0, 1, 2, \ldots)$$
$$B_n = \frac{1}{L}\int_{-L}^{L} \phi(x)\sin\left(\frac{n\pi}{L}x\right)dx, \quad (n = 1, 2, \ldots)$$

Example 1.1. Let $\phi(x) = 1$ in the interval $[0, L]$. Computing the coefficients, we have

$$\begin{aligned} A_m &= \frac{2}{L}\int_0^L \sin\left(\frac{m\pi}{L}x\right)dx \\ &= \frac{2}{m\pi}(1 - (-1)^m). \end{aligned}$$

Therefore,

$$1 = \frac{4}{\pi}\left(\sin\left(\frac{\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{L}\right) + \cdots\right).$$

in $(0, L)$.

Example 1.2. Let $\phi(x) = x$ in the interval $(0, L)$. Its Fourier sine coefficients are

$$\begin{aligned} A_m &= \frac{2}{L}\int_0^L x\sin\left(\frac{m\pi}{L}x\right)dx \\ &= (-1)^{m+1}\frac{2L}{m\pi} \end{aligned}$$

so that

$$x = \frac{2L}{\pi}\left(\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) - \cdots\right).$$

Example 1.3. Next compute the cosine series for $\phi(x) = x$. The coefficients are given by

$$\begin{aligned} A_0 &= \frac{2}{L}\int_0^L x\,dx = L \\ A_m &= \frac{2}{L}\int_0^L x\cos\left(\frac{m\pi x}{L}\right)dx \\ &= \frac{2L}{m^2\pi^2}((-1)^m - 1). \end{aligned}$$

Thus in $(0, L)$, we have

$$x = \frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\frac{\pi x}{L} + \frac{1}{9}\cos\left(\frac{3\pi x}{L}\right) + \frac{1}{25}\cos\left(\frac{5\pi x}{L}\right) + \cdots\right).$$

Example 1.4. Solve the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0, t) = 0 = u(L, t) \\ u(x, 0) = x \\ u_t(x, 0) = 0. \end{cases}$$

The expansion of a separated solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi}{L}x\right)$$

Differentiating this with respect to t yields

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} \left(-A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi}{L}x\right)$$

so that

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right)$$

hence $B_n = 0$ and

$$x = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right).$$

Comparing with the Fourier sine series of x, we have that

$$A_n = (-1)^{n+1}\frac{2L}{n\pi}$$

and so the solution is given by

$$u(x,t) = \frac{2L}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right).$$

1.1. **Even, Odd, Period, and Complex functions.**

Definition 1.1. A function $\phi(x)$ that is defined for $x \in \mathbb{R}$ is called **periodic** if there is a number $p > 0$ such that

$$\phi(x+p) = \phi(x), \qquad \text{for all } x.$$

The number p is called the **period** of $\phi(x)$.

Proposition 1.1. If $\phi(x)$ has period p, then

$$\int_a^{a+p} \phi(x)dx$$

does not depend on a.

Proof. By fundamental theorem of calculus,

$$\begin{aligned}\frac{d}{da}\int_a^{a+p} \phi(x)dx &= \phi(a+p) - \phi(a)\\ &= \phi(a) - \phi(a) = 0.\end{aligned}$$

□

If a function is defined only on an interval of length p, it can be extended to a function of period p by taking its **periodic extension**: Let ϕ be defined on the interval $-L < x < L$. Then define $\phi_{\text{per}}(x)$ by

$$\phi_{\text{per}}(x) = \phi(x - 2Lm) \text{ for } -L + 2Lm < x < L + 2Lm$$

for all integers m. Another symmetry property a function may possess are the following

Definition 1.2. A function is **even** if

$$\phi(x) = \phi(-x)$$

and **odd** if

$$\phi(-x) = -\phi(x).$$

To ensure the definition makes sense, we require the function be defined on an interval $(-L, L)$.

We define the **even extension** of a function defined on $(0, L)$ to be

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & \text{for } 0 < x < L \\ \phi(-x) & \text{for } -L < x < 0. \end{cases}$$

and its **odd extension** is

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for } 0 < x < L \\ -\phi(-x) & \text{for } -L < x < 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Since $\sin(n\pi x/L)$ is odd, the Fourier sine series can be regarded as an expansion of an arbitrary function that is odd and has period $2L$ defined on the whole line $\mathbb{R}$. Likewise, since $\cos(n\pi x/L)$ is even so the Fourier cosine series can be regarded an an expansion of an arbitrary function which is even and has period $2L$ defined on the whole line $\mathbb{R}$. Note that the Dirichlet boundary condition corresponds to an odd extension, the Neumann to an even extension, and periodic boundary conditions correspond to the periodic extension.

Since we can express the sine and cosine functions as complex exponential functions, we have a way to express the Fourier series in complex form, namely

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

with coefficients given by

$$c_n = \frac{1}{2L} \int_{-L}^{L} \phi(x) e^{-in\pi x/L}.$$

1.2. **Orthogonality and General Fourier Series.** Now we consider a more general setting.

Definition 1.3. For two real valued functions f and g defined on an interval (a, b), define their (L^2)**-inner product** to be

$$(f, g) = \int_a^b f(x)g(x)dx.$$

We also have the L^2 **norm**

$$\|f\|_{L^2}^2 = (f, f) = \int_a^b f^2 dx.$$

f and g are **orthogonal** (with respect to the inner product) if $(f, g) = 0$.

The key property we use for Fourier series is that **every eigenfunction is orthogonal to every other eigenfunction**.

1.3. Convergence and completeness.

Definition 1.4. We say that an infinite series $\sum f_n(x)$ converges to $f(x)$ **pointwise** in (a, b) if it converges to $f(x)$ for each $a < x < b$, that is

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0, \qquad \text{as } N \to \infty.$$

Note that the rate of convergence depends on x.

Definition 1.5. We say that the series **converges uniformly** to f in $[a, b]$ if

$$\max_{[a,b]} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \qquad \text{as } N \to \infty.$$

Note that the convergence is independent of the point x.

Definition 1.6. We say the series **converges in** L^2 to f if

$$\int_a^b \left| f(x) - \sum_{n=1}^{N} f_n(x) \right|^2 dx \to 0 \qquad \text{as } N \to \infty.$$

In norm notation:

$$\|f - \sum_{n=1}^{N} f_n\|_{L^2} \to 0 \qquad \text{as } N \to \infty.$$

Example 1.5. Let $f_n(x) = (1 - x)x^{n-1}$ on $0 < x < 1$. Then

$$\sum_{n=1}^{N} f_n(x) = \sum_{1}^{N} (x^{n-1} - x^n) = 1 - x^N.$$

Since $0 < x < 1$, $x^N \to 0$ as $N \to \infty$. The convergence depends on x however so it is only pointwise. However, it does converge in the L^2 sense since

$$\int_0^1 |x^N|^2 dx = \frac{1}{2N + 1} \to 0.$$

Example 1.6. Let

$$f_n(x) = \frac{n}{1 + n^2x^2} - \frac{n - 1}{1 + (n - 1)^2x^2}$$

in the interval $0 < x < L$. Then its partial sum is given by

$$\sum_{n=1}^{N} f_n(x) = \frac{N}{1 + N^2x^2} \to 0$$

as $N \to \infty$ when $x > 0$, hence converges pointwise. However in the L^2 sense,

$$\int_0^L \left(\sum_{n=1}^{N} f_n(x) \right)^2 dx = N \int_0^{NL} \frac{1}{(1 + y^2)^2} dy \to \infty.$$

Furthermore, it does not converge uniformly since

$$\max_{(0,L)} \frac{N}{1 + N^2x^2} = N$$

Now we present 3 theorems on convergence of Fourier series.

Theorem 1.1 (Uniform Convergence)**.** The Fourier series $\sum A_n X_n$ converges to $f(x)$ uniformly on $[a,b]$ provided that

(1) $f(x), f'(x)$ and $f''(x)$ exist and are continuous for $a \le x \le b$.

(2) $f(x)$ satisfies the appropriate boundary conditions.

In fact, for the classical Fourier series, f'' is not required to exist.

Theorem 1.2 (L^2 Convergence)**.** The Fourier series converges to $f(x)$ in the L^2 sense in (a,b) if $f(x) \in L^2(a,b)$.

Theorem 1.3 (Pointwise convergence for classical Fourier series)**.** (1) The classical Fourier series converges to $f(x)$ pointwise on (a,b) provided that $f(x)$ is continuous on $a \le x \le b$ and $f'(x)$ is piecewise continuous on $a \le x \le b$, for example $f(x) = |x|$ on $[-1,1]$.

(2) More generally, if $f(x)$ is only piecewise continuous on $a \le b \le b$ and $f'(x)$ is also piecewise continuous on $a \le x \le b$, then the classical Fourier series converges (as an infinite sum) at every point x. The sum is given by

$$\sum_n A_n X_n(x) = \frac{1}{2}(\lim_{t\to x^+} f(t) + \lim_{t\to x^-} f(t))$$

for all $a < x < b$. Outside the interval, it is given by the corresponding extensions of f.

Example 1.7. The Fourier sine series of $f(x) = 1$ on $(0,\pi)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin((2n+1)x).$$

It does not converge uniformly since the end points are zero. Notice that this does not contradict the convergence theorem since the sine series requires the Dirichlet boundary conditions. Note that this example shows that the terms cannot in general be differentiated term by term, since the derivative is $f' = 0$ however

$$\frac{4}{\pi}\sum_{n=0}^{\infty} \cos((2n+1)x)$$

is not a convergent series.

Theorem 1.4 (Least square approximation)**.** Let $\{X_n\}$ be any (L^2) orthogonal set of functions. Let $\|f\| < \infty$. Let N be a fixed positive integer. Among all possible choices of $c_1, \dots c_N$, the choice that minimizes

$$\|f - \sum_{n=1}^{N} c_N X_n\|$$

is given by the Fourier coefficients $c_i = A_i$.

Proof. For simplicity, assume we are working with real valued functions f and X_n. We want to minimize

$$E_N = \|f - \sum_{n\le N} c_n X_n\|^2 = (f,f) - 2\sum_{n\le N} c_n (f, X_n) + \sum_{n,m} c_n c_m (X_n, X_m).$$

By orthogonality, $(X_n, X_m) = 0$ except when $n = m$. Hence,

$$E_N = \|f\|^2 - 2\sum_{n\le N} c_n(f, X_n) + \sum_{n\le N} c_n^2\|X_n\|^2.$$

We compute the square in c_n so that

$$E_N = \sum_{n\le N} \|X_n\|^2 \left(c_n - \frac{(f, X_n)}{\|X_n\|^2}\right)^2 + \|f\|^2 - \sum_{n\le N} \frac{(f, X_n)^2}{\|X_n\|^2}.$$

Then the term containing c_n is minimized when it is zero since it is nonnegative, i.e.

$$c_n = \frac{(f, X_n)}{\|X_n\|} = A_n.$$

□

We can say a little more. We know that $E_N \ge 0$, hence

$$0 \le E_N = \|f\|^2 - \sum_{n\le N} A_n^2\|X_n\|^2.$$

Since this holds for all N, writing out explicitly the L^2 norm, we obtain **Bessel's inequality**

$$\sum_{n=1}^{\infty} A_n^2 \int_a^b |X_n(x)|^2 dx \le \int_a^b |f(x)|^2 dx.$$

Furthermore, if $E_N \to 0$, we obtain the following:

Theorem 1.5 (Parseval's equality)**.** The Fourier series of $f(x)$ converges in L^2 if and only if

$$\sum_{n=1}^{\infty} |A_n|^2 \int_a^b |X_n(x)|^2 dx = \int_a^b |f(x)|^2 dx$$

Definition 1.7. The infinite orthogonal set $\{X_i(x)\}$ is called **complete** if Parseval's equality holds for all $\|f\| < \infty$.

Example 1.8. By applying Parseval's identity to the sine series for $f(x) = 1$, we have

$$\sum_{n \text{odd}} \left(\frac{4}{n\pi}\right)^2 \int_0^{\pi} \sin^2(nx) dx = \int_0^{\pi} 1 dx$$

Integrating, we have

$$\sum_{n \text{odd}} \left(\frac{4}{n\pi}\right)^2 \frac{\pi}{2} = \pi$$

which is another proof for obtaining the Basel sum ($\sum_n \frac{1}{n^2}$).

1.4. **Proof of convergence.** For convenience, let $L = \pi$. For C^1 function f which is 2π periodic. Then the Fourier series is

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos(nx) + B_n \sin(nx))$$

with coefficients given by

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos(ny)dy$$

and

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin(ny)dy.$$

Therefore the N-th partial sum, $S_N(x)$ is given by

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1 + 2\sum_{n=1}^{N}(\cos(ny)\cos(nx) + \sin(ny)\sin(nx))\right) f(y)dy.$$

Using the cosine summation formula $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$, we rewrite the above as

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x-y)f(y)dy$$

where

$$K_N(\theta) := 1 + 2\sum_{n=1}^{N}\cos(n\theta).$$

The function $K_N(\theta)$ is called the **Dirichlet kernel**. Note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)d\theta = 1$$

and in fact, can be expressed as

$$K_N(\theta) = \frac{\sin((N+\frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)}$$

To see this, we rewrite as a finite geometric series

$$\begin{aligned}
K_N(\theta) &= 1 + \sum_{n=1}^{N}(e^{in\theta} + e^{-in\theta}) \\
&= \sum_{n=-N}^{N} e^{in\theta} \\
&= \frac{e^{-iN\theta} - e^{i(N+1)\theta}}{1 - e^{i\theta}} \\
&= \frac{e^{-i(N+\frac{1}{2})\theta} - e^{i(N+\frac{1}{2})\theta}}{-e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}}} \\
&= \frac{\sin((N+\frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)}.
\end{aligned}$$

Proof of pointwise convergence of classical Fourier series. By changing variables to $\theta = y - x$ and using the evenness of K_N, we have

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta) f(x+\theta)d\theta.$$

Using the fact that the integral of the Dirichlet kernel is 1, we have

$$\begin{aligned} S_N(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)(f(x+\theta) - f(x))d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\sin((N+\frac{1}{2})\theta)d\theta, \end{aligned}$$

where

$$g(\theta) = \frac{f(x+\theta) - f(x)}{\sin(\frac{1}{2}\theta)}.$$

It remains to show that the integral tends to zero as $N \to \infty$. Since the functions

$$\phi_N(\theta) = \sin((N+\frac{1}{2})\theta), \quad N = 1, 2, 3, \ldots$$

form an orthogonal set on the interval $(0, \pi)$, and therefore on $(-\pi, \pi)$, Bessel's inequality holds:

$$\sum_{N=1}^{\infty} \frac{|(g, \phi_N)|^2}{\|\phi_N\|^2} \le \|g\|^2.$$

Since $\|\phi_N\|^2 = \pi$, if $\|g\| < \infty$, then the infinite sum converges, i.e. $(g, \phi_N) \to 0$ as $N \to \infty$. To check that $\|g\|$ is finite, we have

$$\|g\|^2 = \int_{-\pi}^{\pi} \frac{(f(x+\theta) - f(x))^2}{\sin^2(\frac{1}{2}\theta)} d\theta.$$

Since

$$\lim_{\theta\to 0} g(\theta) = 2f'(x)$$

as long as f' exists. □

Proof of uniform convergence. Assume that f and $f'(x)$ are continuous with period 2π. Let A_n and B_n be the Fourier coefficients for $f(x)$ and A_n' and B_n' be the Fourier coefficients of $f'(x)$. They are related by

$$\begin{cases} A_n = -\frac{1}{n}B_n' \\ B_n = \frac{1}{n}A_n'. \end{cases}$$

It can be shown by Bessel's inequality that

$$\sum_{n=1}^{\infty}(|A_n'|^2 + |B_n'|^2) \le \frac{\|f'\|^2}{\pi} < \infty$$

Therefore

$$\begin{aligned}\sum_{n=1}^{\infty}(|A_n\cos(nx)|+|B_n\sin(nx)|) &\leq \sum_{n=1}^{\infty}(|A_n|+|B_n|)\\ &= \sum_{n=1}^{\infty}\frac{1}{n}(|B_n'|+|A_n'|)\\ &\leq \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}2(|A_n'|^2+|B_n'|^2)\right)^{1/2} < \infty.\end{aligned}$$

Therefore, from pointwise convergence, we have

$$\max|f(x)-S_N(x)| \leq \max\sum_{N+1}^{\infty}|A_n\cos(nx)+B_n\sin(nx)| \leq \sum_{N+1}^{\infty}(|A_n|+|B_n|) \to 0$$

as $N \to \infty$. ☐

2. Harmonic functions

2.1. Laplace's equation.

The stationary heat or wave equation

$$u_{xx} = 0,$$

or in higher dimensions

$$\Delta u := \operatorname{div}(\nabla u) = \sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2} = 0$$

is called the **Laplace equation**, and its solutions are called **harmonic functions**. The inhomogeneous version

$$\Delta u = f$$

for a given function f is called **Poisson's equation**. Just like the heat and wave equations, we can give Laplace equations The first thing we will prove is the maximum principle.

Theorem 2.1 (Maximum principle)**.** Let D be a connected bounded open set in $\mathbb{R}^n$. Let u be harmonic in D and continuous on $\overline{D}$. Then the maximum and minimum values of u are attained on ∂D, the boundary of D and nowhere inside unless u is a constant.

Proof. Let $\varepsilon > 0$ and define

$$v_\varepsilon(x) = u(x) - \varepsilon\|x\|^2, \quad x \in \mathbb{R}^n.$$

Then

$$\Delta v_\varepsilon = -2n\varepsilon < 0.$$

Suppose $x_0 \in D$ is a local minimum. Then $\Delta v_\varepsilon \geq 0$, a contradiction, hence there is no local minimum in the interior of D. Let $x_0 \in \partial D$ be the minimum for v_ε. Since

$$u(x) \geq v(x) \geq v_\varepsilon(x_0) = u(x_0) - \varepsilon\|x_0\|^2$$

holds for all $x \in D$ and for all ε, we have

$$u(x) \geq \min_{\partial D} u.$$

Repeating the argument for $v_\varepsilon = u + \varepsilon\|x\|^2$ gives us the maximum. ☐

2.1.1. *Uniqueness.* The maximum principle can be used to prove the uniqueness of solutions for the Poisson equation with Dirichlet condition

$$\begin{cases} \Delta u = f & \text{in } D \\ u = h & \text{on } \partial D. \end{cases}$$

Suppose that v is another solution to the Poisson equation with same boundary conditions. Then $u - v$ is harmonic and 0 on the boundary, hence by the maximum principle $u = v$ on D.

2.1.2. *Invariance.* The Laplace equation is invariant under rigid motion. We show this explicitly in 2 and 3 dimensions. For two dimensions, translation is obvious. Consider a rotation in the plane by angle θ. It can be given by the transformation

$$(x', y') = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta).$$

Changing variables, we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} \\ &= u_{x'}\cos\theta - u_{y'}\sin\theta. \end{aligned}$$

and

$$\frac{\partial^2 u}{\partial x^2} = (u_{x'}\cos\theta - u_{\tilde{y}'}\sin\theta)_{x'}\cos\theta - (u_{x'}\cos\theta - u_{\tilde{y}'}\sin\theta)_{y'}\sin\theta.$$

by a similar computation, we obtain

$$u_{xx} + u_{yy} = (u_{x'x'} + u_{y'y'})(\cos^2\theta + \sin^2\theta) = u_{x'x'} + u_{y'y'}$$

therefore, the Laplacian is rotationally invariant. In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, the Laplacian is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

A rotationally symmetric solution, i.e. $u(r, \theta) = u(r)$ is given by

$$u(r) = c_1 \log r + c_2.$$

In higher dimensions, we can show rotational invariance through some linear algebra. We define a **rotation** in $\mathbb{R}^n$, to be a transformation given by an orthogonal matrix, i.e.

$$\vec{x}' = B\vec{x}$$

where $B \in O(n) = \{B \text{ is an } n \times n \text{ matrix such that } B^T B = BB^T = I\}$. Then the Laplacian is the trace of the Hessian matrix of u hence

$$\mathrm{Tr}(\mathrm{Hess}\, u(x)) = \mathrm{Tr}(B^T\,\mathrm{Hess}\, u(x')B) = \mathrm{Tr}(\mathrm{Hess}\, u(x')B^T B) = \mathrm{Tr}(\mathrm{Hess}\, u(x')).$$

In spherical coordinates,

$$\begin{aligned} x &= s\cos\phi \\ y &= s\sin\phi \\ z &= r\cos\theta \\ s &= r\sin\theta, \end{aligned}$$

we can view this as a change of coordinates from (x, y, z) to cylindrical coordinates (s, ϕ, z), then to spherical (r, θ, ϕ). Since z is fixed in the first transformation, we can apply the 2 dimensional change of coordinates so that

$$u_{xx} + u_{yy} = u_{ss} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}.$$

In the second change of coordinates, ϕ is fixed so that

$$u_{zz} + u_{ss} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Combining, we have

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}.$$

By chain rule,

$$\begin{aligned} u_s &= u_r r_s + u_\theta \theta_s + u_\phi \phi_s \\ &= u_r \frac{s}{r} + u_\theta \frac{\cos\theta}{r} \end{aligned}$$

so that

$$\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(u_{\theta\theta} + (\cot\theta)u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}\right).$$

The rotationally symmetric solutions in 3 dimensions is given by

$$u = -c_1 r^{-1} + c_2.$$

Example 2.1. If the domain that we are dealing with is rotationally symmetric, one strategy is to convert the Laplace or Poisson equation into polar coordinates. For example, consider the Dirichlet problem

$$\begin{cases} u_{xx} + u_{yy} = 1 & \text{in } r < a \\ u = 0 & \text{on } r = a. \end{cases}$$

Since the domain is rotationally symmetric with rotationally symmetric boundary condition, the solution should depend only on r. In polar coordinates,

$$u_{rr} + \frac{1}{r}u_r = 1.$$

We can view this as a first order equation of u_r, hence multiplying by the integrating factor, we have

$$(ru_r)_r = r.$$

Integrating twice, we have

$$u(r) = \frac{1}{4}r^2 + c_1 \ln(r) + c_2.$$

Inserting the boundary condition, we have

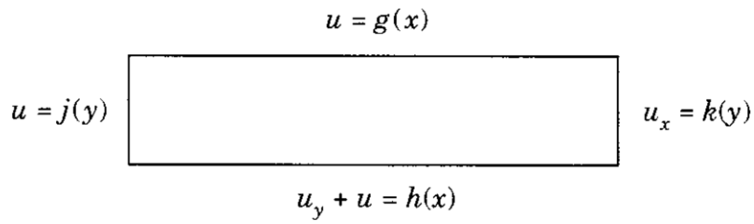
$$0 = \frac{a^2}{4} + c_1 \ln(a) + c_2.$$

hence we get a relation between the coefficients c_1 and c_2.

2.2. **Rectangles.** We will give explicit solutions over rectangles and cubes. Let D be a rectangle, that is

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < a, 0 < y < b\}$$

Example 2.2. We will find harmonic functions on D with the given boundary conditions



For simplicity, assume $h = k = j = 0$. We will find a separated solution, so assume $u(x,y) = X(x)Y(y)$. Then the Laplace equation can be rewritten as

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0.$$

Hence, there is a constant such that $X'' + \lambda X = 0$ and $Y'' - \lambda Y = 0$. Since $X(0) = X'(a) = 0$, we can show that $\lambda = \beta^2 > 0$. In fact, the explicit eigenfunctions and eigenvalues are

$$\begin{cases} X_n(x) = \sin\left(\frac{(n+\frac{1}{2})\pi x}{a}\right) \\ \beta_n^2 = \lambda_n = \left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{a^2}. \end{cases}$$

Next we solve for

$$Y'' - \lambda Y = 0 \quad \text{with } Y'(0) + Y(0) = 0.$$

Since this is a constant coefficient second order ODE with $\lambda > 0$, we have

$$Y(y) = A\cosh(\beta_n y) + B\sinh(\beta_n y).$$

Inserting the boundary conditions, we have

$$0 = Y'(0) + Y(0) = B\beta_n + A.$$

For convenience, let $B = -1$, so that $A = \beta_n$. Then

$$Y(y) = \beta_n \cosh(\beta_n y) - \sinh(\beta_n y).$$

Therefore, we can formally add the solutions $X_n Y_n$ to obtain

$$u(x,y) = \sum_{n=0}^{\infty} A_n \sin(\beta_n x)(\beta_n \cosh(\beta_n y) - \sinh(\beta_n y)).$$

This satisfies the 3 boundary conditions. It remains to insert the last boundary condition:

$$g(x) = \sum_{n=0}^{\infty} A_n(\beta_n \cosh(\beta_n b) - \sinh(\beta_n b))\sin(\beta_n x).$$

This is a Fourier sine series hence we can use the Fourier coefficient formula to get

$$A_n = \frac{2}{a}(\beta_n \cosh(\beta_n b) - \sinh(\beta_n b))^{-1} \int_0^a g(x)\sin(\beta_n x)dx.$$

2.3. **Poisson's Formula.** Next we discuss the Dirichlet problem for a circle,

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } x^2 + y^2 < a^2 \\ u = h(\theta) & \text{for } x^2 + y^2 = a^2 \end{cases}$$

We consider separated solutions $u(r, \theta) = R(r)\Theta(\theta)$ so that

$$0 = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''.$$

Dividing by $R\Theta$ and multiplying by r^2, we obtain

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ r^2R'' + rR' - \lambda R = 0. \end{cases}$$

For the angular component, we give periodic boundary conditions:

$$\Theta(\theta + 2\pi) = \Theta(\theta)$$

Hence we get explicit solutions

$$\Theta(\theta) = A\cos(n\theta) + B\sin(n\theta)$$

with $\lambda = n^2$, $(n = 0, 1, 2, \ldots)$.

In the radial direction, the ODE is of Euler type hence we assume that the solution is of the form $R(r) = r^\alpha$ so that

$$\alpha(\alpha - 1)r^\alpha + \alpha r^\alpha - n^2 r^\alpha = 0,$$

hence $\alpha = \pm n$, so $R(r) = c_1 r^n + c_2 r^{-n}$. Combining the angular and radial solutions, we have

$$u(r, \theta) = (c_1 r^n + c_2 r^{-n})(A\cos(n\theta) + B\sin(n\theta))$$

for $n = 1, 2, \ldots$. For $n = 0$, we have

$$u = c_1 + c_2 \log(r)$$

Since u is harmonic, and hence continuous on D, we do not consider the solutions r^{-n} and $\log r$ which are discontinuous at $r = 0$. Hence, summing the solutions, we have

$$u(r, \theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta)).$$

Inserting the Dirichlet boundary condition, we have

$$h(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos(n\theta) + B_n \sin(n\theta)).$$

Since this is the Fourier series for $h(\theta)$, we have the formulas for the coefficients

$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos(n\phi)d\phi$$

$$B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin(n\phi)d\phi$$

Much like the Dirichlet kernel, we can find a closed form. Plugging in the Fourier coefficient formulas into the solution, we have

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} h(\phi)d\phi + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\int_0^{2\pi} h(\phi)(\cos(n\phi)\cos(n\theta)+\sin(n\phi)\sin(n\theta))d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\left(1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\phi))\right)d\phi
\end{aligned}$$

By expressing cosine in terms of the complex exponential, we have

$$\begin{aligned}
&1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)} \\
&= 1 + \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}} \\
&= \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}.
\end{aligned}$$

Inserting this, we obtain **Poisson's formula**

$$u(r,\theta) = \frac{(a^2 - r^2)}{2\pi}\int_0^{2\pi}\frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi)+r^2}d\phi.$$

This formula can be used to express any harmonic function inside a circle in terms of its boundary values. In vector notation, viewing points as vectors originating from 0,

$$u(\vec{x}) = \frac{a^2 - |\vec{x}|^2}{2\pi a}\int_{|\vec{y}|=a}\frac{u(\vec{y})}{|\vec{x}-\vec{y}|^2}ds$$

Theorem 2.2. Let $h(\phi) = u(\vec{y})$ be any continuous function on the circle $C = \partial D$. Then the Poisson formula provides the only harmonic function in D for which

$$\lim_{x\to x_0} u(x) = h(x_0)$$

for all $x_0 \in C$.

Proof. We define the Poisson kernel

$$P(r,\theta) = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2}$$

so that

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta-\phi)h(\phi)d\phi$$

for $r < a$. The Poisson kernel has the following properties

(1) $P(r,\theta) > 0$ for $r < a$.

(2) $\int_0^{2\pi} P(r,\theta)d\theta = 2\pi$

(3) P is harmonic inside the circle.

Now $u(r,\theta)$ is harmonic since P is harmonic and we can differentiate under the integral sign in this situation ($P(r,\theta)$ needs to be continuous in the compact interval). Now we need to show the limit, so fix an angle θ_0, and consider r near a, then

$$u(r,\theta_0) - h(\theta_0) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta_0-\phi)(h(\phi)-h(\theta_0))d\phi.$$

First we will show that the kernel is concentrated at $\theta = 0$, that is, for any $\varepsilon > 0$, and for each $\delta \leq \frac{\theta}{2} \leq \pi - \delta$, we can find an r sufficiently close to a so that

$$|P(r,\theta)| = \frac{a^2-r^2}{a^2-2ar\cos\theta+r^2} = \frac{a^2-r^2}{(a-r)^2+4ar\sin^2(\theta/2)} \leq \frac{a^2-r^2}{(a-r)^2+4ar\sin^2(\delta)} < \varepsilon.$$

Using this r, we have that

$$\begin{aligned}|u(r,\theta_0)-h(\theta_0)| &\leq \frac{\varepsilon}{2\pi}\int_{\theta_0-\delta}^{\theta_0+\delta} P(r,\theta_0-\phi)d\phi + \frac{\varepsilon}{2\pi}\int_{|\phi-\theta_0|>\delta} |h(\phi)-h(\theta_0)|d\phi \\ &\leq (1+2H)\varepsilon,\end{aligned}$$

where $\delta > 0$ was chosen so that $|h(\phi)-h(\theta_0)| < \varepsilon$ whenever $|\phi-\theta_0| < \delta$ and $|h| \leq H$ for some constant H, since h is continuous. □

We can apply the Poisson formula to obtain the mean value property and the maximum principle:

Proposition 2.1 (Mean Value Property)**.** Let u be harmonic in a disk D, continuous up to the boundary. Then the value of u at the center of D equals the average of u on its circumference.

Proof. Let $x = 0$ in the Poisson formula, then

$$u(0) = \frac{1}{2\pi a}\int_{|y|=a} u(y)ds$$

□

Proposition 2.2 ((Strong) Maximum Principle)**.** Let u be harmonic in D. If the maximum is attained in the interior, then u is a constant. Let $x_M \in \overline{D}$ be the maximum point. Then

$$u(x) \leq u(x_M) = M, \quad \text{for all } x \in D.$$

Proof. By applying mean value property at this point, we have

$$M = u(x_M) = \text{ average on circle } \leq M$$

so that $u = M$ on the circle. Repeating the argument, we can fill the domain D with circles so that u is a constant on D, if max is attained in the interior. □

Proposition 2.3 (Differentiability)**.** Let u be a harmonic function in any open set D of the plane. Then $u(\vec{x}) = u(x,y)$ possesses all partial derivatives of all orders in D.

Roughly, by the Poisson integral formula, we see that the integrand is differentiable and does not require the differentiability of the original function.

2.4. **Circles, Wedges, and Annuli.** We define the following domains:

$$\begin{aligned} \text{Wedge: } &\{(r,\theta) \mid 0<\theta<\theta_0, 0<r<a\} \\ \text{Annulus: } &\{(r,\theta) \mid 0<a<r<b\} \\ \text{Exterior of a circle: } &\{(r,\theta) \mid a<r<\infty\}. \end{aligned}$$

2.4.1. *Wedge.* We consider Laplace equation with homogeneous Dirichlet condition on the straight sides and an inhomogeneous Neumann condition on the circular side:

$$\begin{cases} \Delta u = 0 \\ u(r,0) = u(r,\beta) = 0 \\ \frac{\partial u}{\partial r}(a,\theta) = h(\theta). \end{cases}$$

By separation of variables, we get the two ODEs

$$\begin{cases} r^2 R'' + rR' - \lambda R = 0 \\ \Theta'' + \lambda\Theta = 0. \end{cases}$$

Inserting the homogeneous conditions gives us $\Theta(0) = \Theta(\beta) = 0$, hence we have a solution

$$\begin{cases} \Theta_n(\theta) = \sin\left(\frac{n\pi\theta}{\beta}\right) \\ \lambda_n = \left(\frac{n\pi}{\beta}\right)^2. \end{cases}$$

In the radial direction, we have

$$r^2 R'' + rR' - \lambda R = 0$$

This is of Euler type and has the solution $R(r) = r^\alpha$, where $\alpha^2 = \lambda$. Throwing out the singular solutions, we have the solution

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{n\pi/\beta} \sin\left(\frac{n\pi\theta}{\beta}\right).$$

With the inhomogenous boundary condition, we view it as a Fourier sine series with coefficients

$$A_n = a^{1-n\pi/\beta} \frac{2}{n\pi} \int_0^\beta h(\theta) \sin\left(\frac{n\pi\theta}{\beta}\right) d\theta.$$

2.4.2. *Annulus.* Consider the Dirichlet problem for the annulus in 2 dimensions,

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } 0 < a^2 < x^2 + y^2 < b^2 \\ u = g(\theta) & \text{for } r = a \\ u = h(\theta) & \text{for } r = b. \end{cases}$$

Since $r \neq 0$, we use the solutions of the circle except that we can now include the singular (in the circle case) terms so

$$u(r,\theta) = \frac{1}{2}(C_0 + D_0 \log r) + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n}) \cos(n\theta) + (A_n r^n + B_n r^{-n}) \sin(n\theta).$$

Plugging in the boundary conditions, and noting that $\{1, \cos(n\theta), \sin(n\theta)\}$ are orthogonal on the interval $(0, 2\pi)$, we have the relations

$$\begin{cases} A_n a^n + B_n a^{-n} = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\sin(n\theta)d\theta \\ A_n b^n + B_n b^{-n} = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\sin(n\theta)d\theta \end{cases}$$

and similarly for the constant and cosine terms. In matrix form, this is equivalent to solving for

$$\begin{pmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{pmatrix}\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}$$

where a, b are given and α_n, β_n are the corresponding Fourier coefficients. The determinant of the matrix is nonzero as long as $a \neq b$ and has a unique solution.

2.4.3. *Exterior of a circle.* We consider the Dirichlet problem for the exterior of a circle in 2 dimensions

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } x^2 + y^2 > a^2 \\ u = h(\theta) & \text{for } r = a \\ u \text{ is bounded} \end{cases}$$

We can solve this in the same way as a circle again however, the condition u being bounded implies that there are no terms r^n since $r^n \to \infty$ as $r \to \infty$. Hence we only keep the terms r^{-n}. Therefore,

$$u(r, \theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n\cos(n\theta) + B_n\sin(n\theta)).$$

Incorporating the boundary condition, we have

$$A_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos(n\theta)d\theta$$

and

$$B_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin(n\theta)d\theta.$$

Furthermore, there is a corresponding Poisson formula in this case given by

$$u(r, \theta) = \frac{(r^2 - a^2)}{2\pi}\int_0^{2\pi}\frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}d\phi.$$

3. Green's identities and Green's functions

In this section, we present a way to solve Laplace's equation on more general domains. Recall the divergence theorem, let D be a bounded region and F a vector field defined on D. Then

$$\iiint_D \operatorname{div} F dV = \iint_{\partial D} F \cdot n dS$$

where n is the unit outer normal vector of ∂D.

We first prove **Green's first identity**, for u, v real valued functions,

$$\iint_{\partial D} v\frac{\partial u}{\partial n}dS = \iiint_D \nabla v \cdot \nabla u dx + \iiint_D v\Delta u dx.$$

This is a simple consequence of the divergence theorem and the product rule

$$\operatorname{div}(v\nabla u) = \nabla v \cdot \nabla u + v\Delta u.$$

Setting $v = 1$, we have

$$\iint_{\partial D} \frac{\partial u}{\partial n} dS = \iiint_D \Delta u dx.$$

We can apply this to obtain a necessary solution for the solution to exist for the Neumann problem in any domain D: If

$$\begin{cases} \Delta u = f & \text{in } D \\ \frac{\partial u}{\partial n} = h & \text{on } \partial D, \end{cases}$$

Then integrating over D we have

$$\iint_{\partial D} h dS = \iiint_D f dV.$$

Hence, for a solution to exist, the above must be satisfied between the boundary data and the inhomogeneous term.

3.0.1. *Mean value property.* We also show a mean value property for harmonic functions in 3 dimensions. Let B be a ball of radius a. Suppose $\Delta u = 0$ on B. On a sphere, we have the normal direction being the same as the radial direction, hence

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r}.$$

Therefore, plugging into Green's first identity, we obtain

$$\iint_{\partial B} \frac{\partial u}{\partial r} dS = 0.$$

In spherical coordinates, the surface integral is given by

$$\int_0^{2\pi} \int_0^{\pi} u_r(a, \theta, \phi) a^2 \sin(\phi) d\phi d\theta = 0.$$

Dividing out the constant by $4\pi a^2$ and moving the r derivative outside the integral, we get

$$\frac{\partial}{\partial r} \left(\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(r, \theta, \phi) \sin(\phi) d\phi d\theta \right) \Bigg|_{r=a} = 0.$$

Note that this holds for all $r = a$. Hence we see that the quantity

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(r, \theta, \phi) \sin(\phi) d\phi d\theta = \frac{1}{A(\partial B)} \iint_{\partial B} u dS$$

is independent of r. Taking the limit $r \to 0$, we get

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(0) \sin(\phi) d\phi d\theta = u(0),$$

hence

$$\frac{1}{A(\partial B)} \iint_{\partial B} u dS = u(0).$$

This proves the mean value property in three dimensions. From the mean value property, we automatically have the maximum principle

Proposition 3.1. If D is any solid region, a nonconstant harmonic function in D cannot take its maximum value inside D, but only on ∂D.

3.0.2. *Uniqueness of Dirichlet problem.* Given two harmonic functions u_1 and u_2 with the same boundary data, their difference $u := u_1 - u_2$ is harmonic and is zero at the boundary. Using Green's first identity with $v = u$, we obtain

$$0 = \iint_{\partial D} u \frac{\partial u}{\partial n} dS = \iiint_D |\nabla u|^2 dV.$$

Hence $|\nabla u| = 0$ and u is a constant in D. Since the boundary is 0, we have that $u = 0$. This establishes the uniqueness.

We define the term on the right as the following

Definition 3.1. Define the **energy** of u as

$$E[u] = \frac{1}{2} \iiint_D |\nabla u|^2 dV.$$

We have the following

Proposition 3.2 (Dirichlet principle)**.** Let u be the unique harmonic function in D with boundary data $u = h(x)$ on ∂D. Let w be any other function with the same boundary data. Then the harmonic function u is the minimizer of the energy, i.e.

$$E[w] \geq E[u].$$

Proof. Let $v := u - w$. Then

$$\begin{aligned} E[w] &= \frac{1}{2} \iiint_D |\nabla (u - v)|^2 dV \\ &= E[u] + E[v] \end{aligned}$$

where the mixed term vanish by applying Green's first identity. □

3.1. **Green's second identity.** By subtracting Green's first identity, we have **Green's second identity**

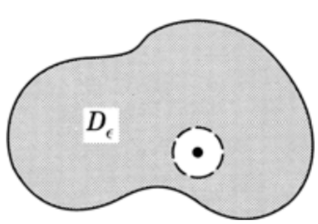
$$\iiint_D u\Delta v - v\Delta u) dV = \iint_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS.$$

3.1.1. *Representation formula.* Much like the Poisson formula, we want to obtain a integral representation for any harmonic function. We have the following

Theorem 3.1. In dimension 3, if $\Delta u = 0$ in D, then

$$u(x_0) = \frac{1}{4\pi} \iint_{\partial D} \left(-u(x) \frac{\partial}{\partial n} \left(\frac{1}{\|x - x_0\|} \right) + \frac{1}{\|x - x_0\|} \frac{\partial u}{\partial n} \right) dS.$$

Proof. Choose polar coordinates centered at x_0. Apply Green's second identity with $v = -\frac{1}{4\pi r}$. Let $D_\varepsilon = D - B(x_0, \varepsilon)$.



Then $\Delta u = 0$ and $\Delta v = 0$ in D_ε. Apply Green's second identity to this pair to obtain

$$-\iint_{\partial D_\varepsilon} \left(u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{\partial u}{\partial n}\frac{1}{r}\right) dS = 0.$$

Now the boundary ∂D_ε consists of 2 parts, with the normal vector for the inner sphere pointing inward so that $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$, thus

$$-\iint_{\partial D} \left(u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{\partial u}{\partial n}\frac{1}{r}\right) dS = -\iint_{r=\varepsilon} \left(u\frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{\partial u}{\partial r}\frac{1}{r}\right) dS.$$

This holds for any small $\varepsilon > 0$. We now take the limit as $\varepsilon \to 0$. On $\{r = \varepsilon\}$, we have

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2} = -\frac{1}{\varepsilon^2},$$

Hence

$$\begin{aligned}-\iint_{r=\varepsilon} \left(u\frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{\partial u}{\partial r}\frac{1}{r}\right) dS &= \frac{1}{\varepsilon^2}\iint_{r=\varepsilon} u dS + \frac{1}{\varepsilon}\iint_{r=\varepsilon}\frac{\partial u}{\partial r} dS \\ &= 4\pi\bar{u} + 4\pi\varepsilon\frac{\partial \bar{u}}{\partial r},\end{aligned}$$

where the bar denotes the average of the function on the sphere $r = \varepsilon$. Let $\varepsilon \to 0$ so that the last term equals $4\pi u(0)$. This gives the integral formula. □

3.2. **Green's function.** We investigate further the nature of the integral formulas.

Definition 3.2. The **Green's function** $G(x, x_0)$ for the operator $-\Delta$ and the domain D at the point $x_0 \in D$ is a function defined for $x \in D$ such that

(1) $G(x, x_0)$ possesses continuous second derivatives and $\Delta G = 0$ in D except at the point x_0.

(2) $G = 0$ on ∂D.

(3) The function $G(x, x_0) + \frac{1}{4\pi\|x-x_0\|}$ is finite at x_0 and has continuous second derivatives everywhere and is harmonic at x_0.

Theorem 3.2. If $G(x, x_0)$ is the Green's function, then the solution for the Dirichlet problem is given by the formula

$$u(x_0) = \iint_{\partial D} u(x)\frac{\partial G(x, x_0)}{\partial n} dS.$$

Proof. By the representation formula, we have

$$u(x_0) = \iint_{\partial D} \left(u\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} v\right) dS,$$

where $v(x) = -\frac{1}{4\pi\|x-x_0\|}$. Define $H(x) := G(x, x_0) - v(x)$. Then H is harmonic in D by definition of the Green's function. Applying Green's second identity to u and H, we have

$$0 = \iint_{\partial D} \left(u\frac{\partial H}{\partial n} - \frac{\partial u}{\partial n} H\right) dS.$$

Since $G = H + v$, adding the representation formula and the identity, we obtain

$$u(x_0) = \iint_{\partial D} \left(u\frac{\partial G}{\partial n} - \frac{\partial u}{\partial n} G\right) dS.$$

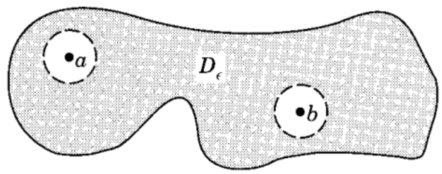
Since G vanishes on ∂D, the second term vanishes we we obtain the integral formula. □

3.2.1. *Symmetry of Green's function.*

Proposition 3.3. For any $a, b \in D$, where D is some bounded region,

$$G(a,b) = G(b,a).$$

Proof. Let $u(x) = G(x,a)$ and $v(x) = G(x,b)$. Let $D_\varepsilon = D - B(a,\varepsilon) - B(b,\varepsilon)$.



By Green's second identity, we have

$$\iiint_{D_\varepsilon} (u\Delta v - v\Delta u)dx = \iint_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$$
$$+ \iint_{|x-a|=\varepsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS + \iint_{|x-b|=\varepsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS.$$

Since the Green's function vanishes on ∂D and is harmonic in the deleted neighborhood, we have

$$\iint_{|x-a|=\varepsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS + \iint_{|x-b|=\varepsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS = 0.$$

This holds for any ε. Now we focus on the a term. Let $r = |x-a|$, and on the sphere, we have $\frac{\partial}{\partial n} == \frac{\partial}{\partial r}$ so that

$$\iint_{r=\varepsilon} \left[\left(-\frac{1}{4\pi r} + H\right)\frac{\partial v}{\partial n} - v\frac{\partial}{\partial n}\left(-\frac{1}{4\pi r} + H\right)\right] r^2 \sin\phi d\phi d\theta$$
$$= \iint_{r=\varepsilon} \left(-\frac{1}{4\pi r}\frac{\partial v}{\partial n} + v\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right)\right) r^2 \sin\phi d\phi d\theta + \iint_{r=\varepsilon} \left(H\frac{\partial v}{\partial n} - v\frac{\partial H}{\partial n}\right) r^2 \sin\phi d\phi d\theta$$
$$= -\frac{1}{4\pi}\iint_{r=\varepsilon} v \sin\phi d\phi d\theta.$$

By mean value property, the last is

$$\lim_{\varepsilon\to 0}\int_0^{2\pi}\int_0^{\pi} \frac{v}{4\pi\varepsilon^2}\varepsilon^2 \sin\phi d\phi d\theta = v(a).$$

and similarly, we have

$$\iint_{|x-b|=\varepsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS \to -u(b)$$

hence

$$0 = v(a) - u(b) = G(a,b) - G(b,a).$$

□

3.3. **Half-space and sphere.**

3.3.1. *Solutions on half-space.* We will first determine the Green's function for a half-space in three dimensions. Let $D := \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 \mid z > 0\}$. Define a **reflected point** as $\mathbf{x}^* = (x, y, -z)$. Then the Green's function for D is given by

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi\|\mathbf{x} - \mathbf{x}_0\|} + \frac{1}{4\pi\|\mathbf{x} - \mathbf{x}_0^*\|}.$$

We need to check the three properties of the Green's function.

(1) G is finite and differentiable except at $\mathbf{x}_0$. Using polar coordinates centered at $\mathbf{x}_0$, we see that $\Delta G = 0$.

(2) By symmetry, we see that $G = 0$ on ∂D.

(3) Since $G + \dfrac{1}{4\pi\|\mathbf{x} - \mathbf{x}_0\|} = \dfrac{1}{4\pi\|\mathbf{x} - \mathbf{x}_0^*\|}$ has no singularities, hence is harmonic at $\mathbf{x}_0$.

Now we use the Green's function to solve

$$\begin{cases} \Delta u = 0 & \text{for } z > 0 \\ u(x, y, 0) = h(x, y). \end{cases}$$

Since the unit normal is $(0, 0, -1)$, we have $\dfrac{\partial G}{\partial n} = -\dfrac{\partial G}{\partial z}|_{z=0}$. Now

$$\begin{aligned} \frac{\partial G}{\partial n} = -\frac{\partial G}{\partial z}\bigg|_{z=0} &= \frac{1}{4\pi}\left(\frac{z + z_0}{\|\mathbf{x} - \mathbf{x}_0^*\|^3} - \frac{z - z_0}{\|\mathbf{x} - \mathbf{x}_0\|^3}\right)\bigg|_{z=0} \\ &= \frac{1}{2\pi}\frac{z_0}{\|\mathbf{x} - \mathbf{x}_0\|^3}. \end{aligned}$$

Therefore, we have the solution to the Laplace equation on the half space as

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi}\iint_{\mathbb{R}^2} \frac{h(x, y)dxdy}{((x - x_0)^2 + (y - y_0)^2 + z_0^2)^{\frac{3}{2}}}$$

3.3.2. *Solutions on Sphere.* We compute the Green's function on the ball $D = \{\|\mathbf{x}\| < a\}$. Define the **reflected point** of a ball by

$$\mathbf{x}_0^* = \frac{a^2\mathbf{x}_0}{\|\mathbf{x}_0\|^2}.$$

Let $\|\mathbf{x} - \mathbf{x}_0\| = r$ and $\|\mathbf{x} - \mathbf{x}_0^*\| = r^*$. We claim that the Green's function for the ball is

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi r} + \frac{a}{4\pi\|\mathbf{x}_0\|r^*}$$

for $\mathbf{x}_0 \neq 0$, and

$$G(\mathbf{x}, 0) = -\frac{1}{4\pi\|\mathbf{x}\|} + \frac{1}{4\pi a}.$$

For $\mathbf{x}_0 \neq 0$, we check the three properties. The first and third properties are straightforward since $\mathbf{x}_0^*$ lies outside the circle, and we can use polar coordinates with different centers. For the second property, we notice that for $\|\mathbf{x}\| = a$, we have

$$\left\|\frac{r_0}{a}\mathbf{x} - \frac{a}{r_0}\mathbf{x}_0\right\| = \|\mathbf{x} - \mathbf{x}_0\|,$$

where $r_0 = \|\mathbf{x}_0\|$. Factoring out $\frac{r_0}{a}$, we get

$$\frac{r_0}{a} r^* = r$$

for any $\|\mathbf{x}\| = a$, hence $G = 0$ on the sphere $\|\mathbf{x}\| = a$.

Now we use the Green's function to give a formula for the solution of the Dirichlet problem in a ball in dimension 3,

$$\begin{cases} \Delta u = 0 & \text{in } \|\mathbf{x}\| < a \\ u = h & \text{on } \|\mathbf{x}\| = a. \end{cases}$$

First consider $\mathbf{x}_0 \neq 0$. We need to calculate $\frac{\partial G}{\partial n}$ on the boundary $\|\mathbf{x}\| = a$.

Now $r^2 = \|\mathbf{x} - \mathbf{x}_0\|^2$, hence $2r\nabla r = 2(\mathbf{x} - \mathbf{x}_0)$ so that $\nabla r = \frac{\mathbf{x}-\mathbf{x}_0}{r}$ and similarly for r^*. Also computing,

$$\nabla G = \frac{\mathbf{x} - \mathbf{x}_0}{4\pi r^3} - \frac{a}{r_0} \frac{\mathbf{x} - \mathbf{x}_0^*}{4\pi (r^*)^3}.$$

Using the fact that $\mathbf{x}_0^* = \frac{a^2}{r_0^2}\mathbf{x}_0$ and for $\|\mathbf{x}\| = a$, $r^* = \frac{a}{r_0} r$, we have

$$\nabla G = \frac{1}{4\pi r^3} \left(\mathbf{x} - \mathbf{x}_0 - \frac{r_0^2}{a^2}\mathbf{x} + \mathbf{x}_0 \right)$$

on the surface, hence

$$\frac{\partial G}{\partial n} = \nabla G \cdot \frac{\mathbf{x}}{a} = \frac{a^2 - r_0^2}{4\pi a r^3}.$$

Inserting this in to the integral formula, we obtain

$$u(\mathbf{x}_0) = \frac{a^2 - \|\mathbf{x}_0\|^2}{4\pi a} \iint_{\|\mathbf{x}\|=a} \frac{h(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^3} dS.$$

In 2 dimensions, the Green function is given by

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \log \|\mathbf{x} - \mathbf{x}_0\| - \frac{1}{2\pi} \log \left(\frac{\|\mathbf{x}_0\|}{a} \|\mathbf{x} - \mathbf{x}_0^*\| \right),$$

where the third property is replaced by $\frac{1}{2\pi} \log \|\mathbf{x} - \mathbf{x}_0\|$.

4. General Eigenvalue Problem

Throughout this section, we will use the L^2 inner product and norm,

$$\begin{aligned} (f, g) &= \iiint_D f(x)\overline{g(x)} dx \\ \|f\|^2 &= (f, f). \end{aligned}$$

We now consider the Dirichlet eigenvalue problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases}$$

where D is an arbitrary domain (open) in 3 dimensions, with smooth boundary. We denote the eigenvalues by

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$

repeating up to multiplicity.

Consider the minimum problem:

$$m = \min\left\{ \frac{\|\nabla w\|^2}{\|w\|^2} \mid w = 0 \text{ on } \partial D, w \not\equiv 0 \right\}.$$

The quotient we minimize is called the **Rayleigh quotient**.

Theorem 4.1. Assume that $u(x)$ is a solution to the minimum problem. Then the value of the minimum equals the smallest eigenvalue λ_1 of the Dirichlet problem and $u(x)$ is its eigenfunction, i.e.

$$\lambda_1 = \min\left\{ \frac{\|\nabla w\|^2}{\|w\|^2} \mid w = 0 \text{ on } \partial D, w \not\equiv 0 \right\}, \quad \text{ and } -\Delta u = \lambda_1 u, \text{ in } D.$$

Proof. Consider a test function w, which is any C^2 function w such that $w = 0$ on ∂D and $w \not\equiv 0$. By assumption, we have

$$m = \frac{\|\nabla u\|^2}{\|u\|^2} \le \frac{\|\nabla w\|^2}{\|w\|^2}$$

for any test function w, where m is the minimum. Let v be any test function and consider $w := u + \varepsilon v$, ε is any constant. Then the function

$$f(\varepsilon) = \frac{\|\nabla(u + \varepsilon v)\|^2}{\|u + \varepsilon v\|^2} = \frac{\|\nabla u\|^2 + 2\varepsilon(\nabla u, \nabla v) + \|\nabla v\|^2}{\|u\|^2 + 2\varepsilon(u, v) + \|\nabla v\|^2}$$

has a minimum at $\varepsilon = 0$, hence $f'(0) = 0$. Computing, we have

$$0 = f'(0) = \frac{2\|u\|^2(\nabla u, \nabla v) - 2\|\nabla u\|^2(u, v)}{\|u\|^4}.$$

Therefore, we have

$$(\nabla u, \nabla v) = m(u, v).$$

Integrating by parts, or by Green's first identity, we have

$$(\Delta u + mu, v) = 0$$

which holds for all test functions v, therefore m is an eigenvalue for $-\Delta$ with eigenfunction u. To show that m is the smallest eigenvalue, let $-\Delta v_j = \lambda_j v_j$ where λ_j is any other eigenvalue. Then

$$m \le \frac{\|\nabla v_j\|^2}{\|v\|^2} = \lambda_j.$$

□

Theorem 4.2. Suppose that $\lambda_1, \dots \lambda_{n-1}$ are already known, with the eigenfunctions $v_1, \dots, v_{n-1}$, respectively. Then

$$\lambda_n = \min\left\{ \frac{\|\nabla w\|^2}{\|w\|^2} \mid w \not\equiv 0, w = 0 \text{ on } \partial D, w \in C^2, 0 = (w, v_1) = (w, v_2) = \cdots = (w, v_{n-1}) \right\},$$

assuming that the minimum exists. Furthermore, the minimizing function is the nth eigenfunction v_n.

Proof. Let u denote the minimizing function, which we assume exists. Note that $u = 0$ on ∂D and u is orthogonal to the previous eigenfunctions. Let m be the minimum value. Let $w = u + \varepsilon v$ where v satisfies the constraints in the minimizing problem. By the same reasoning as above,

$$\iiint_D (\Delta u + mu) v dV = 0$$

and

$$\begin{aligned}\iiint_D (\Delta u + mu) v_j dV &= \iiint_D u(\Delta v_j + m v_j) dV \\ &= (m - \lambda_j) \iiint_D u v_j dV = 0,\end{aligned}$$

since u is orthogonal to v_j. We also used the fact that u and v_j are vanishing on the boundary. Let h be an arbitrary trial function, which is C^2, $h = 0$ on ∂D, and $h \not\equiv 0$. By Gram-Schmidt process, we let

$$v(x) = h(x) - \sum_{k=1}^{n-1} \frac{(h, v_k)}{(v_k, v_k)} v_k(x).$$

This v is not orthogonal to all the v_k. Since each function satisfies the constraint conditions, v also satisfies as well. Hence

$$\iiint_D (\Delta u + mu) h dV = 0.$$

since $h = v + c_k v_k$, where c_k are the coefficients in the Gram-Schmidt decomposition. Since this holds for any test function, we get that

$$\Delta u + mu = 0.$$

It remains to show that $m = \lambda_n$. Suppose $m < \lambda_{n-1}$, then it is attained by some u. However, by induction, $\lambda_{n-1} = \dfrac{\|\nabla v_{n-1}\|^2}{\|v_{n-1}\|^2}$ which was the minimizer of the Rayleigh quotient with less constraints, hence u would be smaller, which is a contradiction. So we know $\lambda_{n-1} \leq m$. By the same reason as the $n = 1$ case, we can show that $m \leq \lambda_n$. Since u must be linearly independent to v_{n-1}, we have that $m = \lambda_n$, note that its value can be equal to λ_{n-1}. □

We can then prove that the first eigenvalue is simple (multiplicity one) so that in fact, our eigenvalues can be ordered as

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$$

First we prove a maximum principle in this setting.

Lemma 4.1. The first eigenfunction can be chosen to be nonnegative, $u \geq 0$.

Proof. Let u_+ and u_- be the nonnegative and negative parts of u. By the minimal characterization, we know that

$$\lambda_1 = \min\{\frac{\|\nabla u_+\|^2}{\|u_+\|^2}, \frac{\|\nabla u_-\|^2}{\|u_-\|^2}\} \leq \frac{\|\nabla u\|^2}{\|u\|^2} = \lambda_1$$

This would give a contradiction unless one of u_+ or u_- is zero. We choose $u_- = 0$. □

Next we use a maximum principle to show that

Lemma 4.2. The first eigenfunction is positive, i.e. $u > 0$, in the interior of D.

4.1. **Minimax principle.** We start with an approximation scheme where we let $w_1, \dots, w_n$ be n arbitrary test functions (C^2 functions that vanish on ∂D.) Define the matrix $A = (a_{ij})$ and $B = (b_{ij})$, with entries given by

$$a_{ij} = (\nabla w_i, \nabla w_j) = \iiint_D \nabla w_i \cdot \nabla w_j dV$$
$$b_{ij} = (w_i, w_j) = \iiint_D w_i w_j dV.$$

Consider the characteristic equation

$$\det(A - \lambda B) = 0.$$

Note that it is a polynomial equation in λ. Denote the roots by

$$\lambda_1^* \leq \cdots \leq \lambda_n^*.$$

First we show that

$$\lambda_n^* = \max_{c \neq 0} \frac{Ac \cdot c}{Bc \cdot c}.$$

Lemma 4.3. For A a real $n \times n$ symmetric matrix, the largest root λ_n of $\det(A - \lambda I) = 0$ is given by

$$\lambda_n = \max_{c \neq 0} \frac{Ac \cdot c}{c \cdot c}.$$

Proof. From linear algebra, a $n \times n$ real symmetric matrix has an orthonormal eigenbasis consisting of eigenvectors $\{v_i\}$. Then, for any $v \in \mathbb{R}^n$, we can write as $v = c_1 v_1 + \cdots + c_n v_n$. We have

$$\begin{aligned} \frac{Av \cdot v}{v \cdot v} &= \frac{A(c_1 v_1 + \cdots + c_n v_n) \cdot v}{v \cdot v} \\ &= \frac{c_1^2 \lambda_1 \|v_1\|^2 + \cdots + c_n^2 \lambda_n \|v_n\|^2}{\|v\|^2} \\ &\leq \lambda_n \frac{\|v\|^2}{\|v\|^2} = \lambda_n. \end{aligned}$$

The maximum is attained when $v = v_n$. □

Now for B positive definite and symmetric, we can decompose as $B = b^2$, with b positive definite and symmetric and so $B^{-1} = b^{-1} b^{-1}$. Note that

$$\det(A - \lambda B) = \det(b^{-1} A b^{-1} - \lambda) \det(B) = 0$$

hence we apply the lemma to $b^{-1} A b^{-1} - \lambda I$, hence

$$\lambda_n^* = \max_{c \neq 0} \frac{b^{-1} A b^{-1} c \cdot c}{c \cdot c}.$$

Let $c = bv$, then

$$\lambda_n^* = \max_{v \neq 0} \frac{Av \cdot v}{Bv \cdot v}.$$

Returning to the minimax principle, we see that the matrix B defined above is positive definite since

$$x^T Bx = \sum_{i,j=1}^{n} (x_i w_i, x_j w_j) = \int \sum_i (x_i w_i) \sum_j (x_j w_j) = \int (\sum_i x_i w_i)^2 > 0.$$

so that

$$\lambda_n^* = \max_{c \neq 0} \frac{\|\nabla(\sum c_j w_j)\|^2}{\|\sum c_i w_i\|^2}$$

This leads to the minimax principle

Theorem 4.3. If $w_1, \ldots, w_n$ is an arbitrary set of test functions and λ_n^* is defined as above, then the nth eigenvalue is given by

$$\lambda_n = \min \lambda_n^*$$

where the minimum is taken over all possible choices of n test functions $w_1, \ldots, w_n$.

Proof. Fix any choice of n test functions $w_1, \ldots, w_n$. We can then choose constants $c_1, \ldots c_n$ not all zero so that for eigenfunctions $v_1, \ldots v_{n-1}$ and $w := \sum c_j w_j$,

$$(w, v_k) = \sum_{j=1}^{n} c_j (w_j, v_k) = 0.$$

so that w is orthogonal to the first $n-1$ eigenfunctions. Then

$$\lambda_n \leq \frac{\|\nabla w\|^2}{\|w\|^2} \leq \lambda_n^*.$$

Since this holds for any choice of n test functions, we can take the minimum on both sides.

For the opposite inequality, we simply let $w_i = v_i$ to obtain the minimum. □

Using the minimax principle, we can give an inequality relation about eigenvalues in different domains:

Theorem 4.4. Let $D_1 \subset D_2$. Then $\lambda_n(D_1) \geq \lambda_n(D_2)$, where $\lambda_n(D)$ is the nth eigenvalue of the Dirichlet problem on D.

Proof. Let $w_1, \ldots, w_n$ be test functions and c_i be coefficients such that for $w = \sum c_j w_j$,

$$\lambda_n^*(D_1) = \frac{\|\nabla w\|^2}{\|w\|^2},$$

that is, the Rayleigh quotient is maximized among the fixed test functions. Since $D_1 \subset D_2$, we can extend these test functions to be test functions on D_2 so that

$$\lambda_n^*(D_1) = \lambda_n^*(D_2)$$

Taking the minimum over test functions of D_1, we have

$$\lambda_n(D_1) = \min_{\text{test functions of } D_1} \lambda_n^*(D_1) = \min_{\text{test functions of } D_1} \lambda_n^*(D_2) \geq \min_{\text{test functions of } D_2} \lambda_n^*(D_2) = \lambda_n(D_2).$$

□

Now we prove that the eigenvalues increase:

Corollary 4.1. For $D \subset \mathbb{R}^n$, the eigenvalues of the Dirichlet problem,

$$\begin{cases} -\Delta u = \lambda u & \text{on } D \\ u = 0 & \text{on } \partial D. \end{cases}$$

form an infinite sequence λ_n such that $\lambda_n \to \infty$ as $n \to \infty$.

Appendix A. Some Linear Algebra

Theorem A.1. If A is an $n \times n$ real symmetric matrix, then it is diagonalizable under an orthonormal basis of eigenvectors.

Corollary A.1. If A is positive definite and symmetric, then it can be decomposed into its "square root" matrices $A = a^2$ where a is positive definite and symmetric.

Proof. Since A is positive definite and symmetric, we can write as

$$A = PDP^T = (PD^{1/2}P^T)(PD^{1/2}P^T) = a^2$$

where P is a matrix of (normalized) eigenvectors, hence $P^TP = I$, D is a diagonal matrix of eigenvalues and $D^{1/2}$ is the square root of the entries, which is possible since A is positive definite hence each eigenvalue is positive. □