MATH 147A: INTRODUCTION TO DIFFERENTIAL GEOMETRY

SHO SETO

Contents

1. Curves in $\mathbb{R}^2$ and $\mathbb{R}^3$

1.1. **Curves.** We begin with two notions of a curve, a **level curve** and a **parametrized curve** and discuss their relationship.

Definition 1.1. Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$. A **level curve** in $\mathbb{R}^2$ is a set of points

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}$$

Example 1.1. The unit circle

$$x^2 + y^2 = 1$$

can be described as a level curve by being the zero set of $f(x, y) = x^2 + y^2 - 1$.

Level curves in higher dimension such as $\mathbb{R}^3$ require more defining equations.

Example 1.2. The x-axis in $\mathbb{R}^3$ is described as a level curve by

$$\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 \mid y = 0, z = 0\}$$

Another way to describe a curve is to view it as the path a moving point takes.

Definition 1.2. A **parametrized curve** in $\mathbb{R}^n$ is a map $\gamma : (a, b) \to \mathbb{R}^n$, for some a, b with $-\infty \leq a < b \leq \infty$. Written out explicitly, it would take the form

$$\gamma(t) = (x_1(t), x_2(t), \ldots, x_n(t)).$$

Example 1.3 (parabola)**.** The parabola $y = x^2$ can be described as a level curve by

$$\{(x, y) \in \mathbb{R}^2 \mid y - x^2 = 0\}.$$

It can be described as a parametrized curve by

$$\gamma(t) = (t, t^2).$$

This can be obtained by setting $x(t) = t$, then $y = x^2 = t^2$. Another possible parametrization is

$$\gamma(t) = (t^3, t^6).$$

The above two parametrizations are defined for any subinterval of $(-\infty, \infty)$. Now consider the parametrization by setting $y(t) = t$. Then we would obtain

$$\gamma(t) = (\sqrt{t}, t),$$

however, this parametrization is defined only for subintervals of $[0, \infty)$. In particular, this could only give us the positive x half of the parabola.

Example 1.4 (circle)**.** Now we will parametrize the unit circle $x^2 + y^2 = 1$. If we let $x(t) = t$, then $y(t) = \sqrt{1 - t^2}$. However, this parametrization would only give us the positive y half of the circle. To obtain the whole circle, we must consider examine the equation

$$x^2(t) + y^2(t) = 1.$$

By the Pythagorean theorem, we know that $\sin(t)^2 + \cos^2(t) = 1$ for any t, hence

$$\gamma(t) = (\cos(t), \sin(t))$$

for $t \in (-\infty, \infty)$ (actually any interval of length larger than 2π) will parametrize the whole circle.

Example 1.5. Now we find a parametrization of $y^2 - x^2 = 1$. This is a unit hyperbola and by setting one of the coordinates as t would lead to a similar issue as above. We will use the relationship of hyperbolic functions, namely $\cosh^2(t) - \sinh^2(t) = 1$ so that

$$\gamma(t) = (\sinh(t), \cosh(t)), \quad t \in \mathbb{R}.$$

If one prefers, we can also use the identity $\sec^2(t) - \tan^2(t) = 1$, however the interval of definition will require some care as $t = \frac{\pi}{2}$ is not defined.

Next we describe how to go from a parametrized curve to a level curve.

Example 1.6 (astroid)**.** Take the parametrized curve

$$\gamma(t) = (x(t), y(t)) = (\cos^3 t, \sin^3 t), \qquad t \in \mathbb{R}$$

Using the relation $\cos^2 t + \sin^2 t = 1$, we see that the coordinates of $\gamma(t)$ will satisfy

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1.$$

We now want to employ the methods of calculus to study these curves, hence we require the curves to be differentiable. For parameterized curves, this simply means that the coordinate functions $x_i(t)$ are differentiable. From now on **all parameterized curves (unless otherwises stated) are assumed to be smooth (infinitely differentiable)**.

Definition 1.3. If γ is a parametrized curve, its first derivative $\dot{\gamma}(t)$ is called the **tangent vector** of γ at the point $\gamma(t)$. We use the dot notation for the derivative.

Proposition 1.1. If the tangent vector of a parametrized curve is constant, the image of the curve is (part of) a straight line.

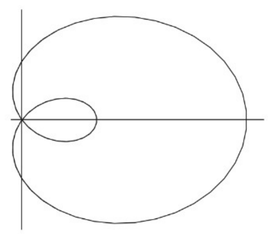
Proof. If $\dot{\gamma}(t) = \mathbf{a}$ for all t, where $\mathbf{a}$ is a constant vector, we have

$$\gamma(t) = \int \frac{d\gamma}{dt} dt = t\mathbf{a} + \mathbf{b}$$

for some constant vector $\mathbf{b}$. □

Example 1.7. The **limaçon** is the parametrized by

$$\gamma(t) = ((1 + 2\cos t)\cos t, (1 + 2\cos t)\sin t), \quad t \in \mathbb{R}.$$



Note that γ has a self-intersection at the origin. We can see this in the parametrization since $\gamma(t) = (0, 0)$ when $t = \frac{2\pi}{3}$ and $t = \frac{4\pi}{3}$. Now the tangent vector at the "two points" is

$$\dot{\gamma}(\tfrac{2\pi}{3}) = (\tfrac{\sqrt{3}}{2}, -\tfrac{3}{2}), \quad \dot{\gamma}(\tfrac{4\pi}{3}) = (-\tfrac{\sqrt{3}}{2}, -\tfrac{3}{2}).$$

1.2. **Arc-length.** By the Pythagorean theorem, we know the length of a vector $\mathbf{v} = (v_1, \ldots, v_n)$ is given by

$$\|\mathbf{v}\| = \sqrt{(v_1)^2 + \cdot + (v_n)^2}$$

To compute the length of a parametrized curve $\gamma(t)$ from t_0 to r, we will approximate the curve by straight lines. First partition the interval $[t_0, t]$ in to n pieces, i.e. let $t_k = t_0 + \frac{k(t-t_0)}{n}$. Then take the length of the straight line between $\gamma(t_k)$ and $\gamma(t_{k+1})$, sum the pieces up and letting $n \to \infty$, we have

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{\|\gamma(t_{k+1}) - \gamma(t_k)\|}{t_{k+1} - t_k} \frac{(t - t_0)}{n} = \int_{t_0}^{t} \|\dot{\gamma}(s)\| ds.$$

The term on the right is the **arc-length** of γ from t_0 to t.

Example 1.8 (Circumference)**.** For fixed $R > 0$, consider the parametrized curve $(R\cos t, R\sin t)$ for $t \in [0, 2\pi]$. This is a circle with radius R. We use the arc-length formula above and get

$$\int_0^{2\pi} \|\dot{\gamma}(s)\| ds = R \int_0^{2\pi} ds = 2\pi R$$

which is the familiar formula for the circumference.

If we consider the arc-length as a function of t, i.e.

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(s)\| ds,$$

then we can differentiate with respect to t to obtain

$$\frac{ds}{dt} = \frac{d}{dt} \int_{t_0}^{t} \|\dot{\gamma}(s)\| ds = \|\dot{\gamma}(t)\|.$$

Definition 1.4. If $\gamma : (a, b) \to \mathbb{R}^n$ is a parametrized curve, its **speed** at the point $\gamma(t)$ is $\|\dot{\gamma}\|$, and γ is said to be a **unit-speed** curve if $\dot{\gamma}(t)$ is a unit vector for all $t \in (a, b)$.

Proposition 1.2. Let $\mathbf{n}(t)$ be a unit vector that is a smooth function of t. Then $\dot{\mathbf{n}}(t)$ is zero or perpendicular to $\dot{\mathbf{n}}(t)$ for all t. For unit-speed curves γ, this means that $\dot{\gamma}$ is zero or perpendicular to $\dot{\gamma}$.

Proof. By direct computation,

$$0 = \frac{d}{dt}(\mathbf{n} \cdot \mathbf{n}) = 2\dot{\mathbf{n}} \cdot \mathbf{n}.$$

□

1.3. **Reparametrization.**

Definition 1.5. A parametrized curve $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \to \mathbb{R}^n$ is a **reparametrization** of a parametrized curve $\gamma : (a, b) \to \mathbb{R}^n$ if there is a smooth bijective map $\phi : (\tilde{a}, \tilde{b}) \to (a, b)$ such that the inverse ϕ^{-1} is smooth and

$$\tilde{\gamma}(s) = \gamma(\phi(s)) \quad \text{for all } s \in (\tilde{a}, \tilde{b}).$$

Intuitively, we are only changing how a point is moving along the same curve.

Example 1.9. Consider the parametrized curve $\gamma(t) = (\cos t, \sin t)$. Another parametrization could be given by $\tilde{\gamma}(t) = (\sin t, \cos t)$. To see that this is a reparametrization, we need to find a reparametrization map ϕ. One possible ϕ is $\phi(t) = \frac{\pi}{2} - t$, then $\tilde{\gamma}(t) = \gamma(\phi(t))$.

One useful reparametrization is to change a given curve to a unit-speed curve (a **unit-speed reparametrization**). We now investigate when this is possible.

Definition 1.6. A point $\gamma(t)$ of a parametrized curve is called a **regular point** if $\dot{\gamma}(t) \neq 0$; otherwise $\gamma(t)$ is a **singular point**. A curve is regular if all of its points are regular.

Proposition 1.3. Any reparametrization of a regular curve is regular

Proof. Suppose γ and $\tilde{\gamma}$ are reparametrization of the same curve, related by $\tilde{\gamma}t = \gamma(\phi(s))$. Let $\psi = \phi^{-1}$ be the inverse map. Since $\phi(\psi(t)) = t$, taking the derivative we have

$$\frac{d\phi}{ds}\frac{d\psi}{dt} = 1.$$

From this we can see that $\frac{d\phi}{ds} \neq 0$. Now $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(s))$ so

$$\frac{d\tilde{\gamma}}{ds} = \frac{d\gamma}{dt}\frac{d\phi}{d\tilde{t}}.$$

So $\frac{d\tilde{\gamma}}{ds} \neq 0$, if $\frac{d\gamma}{dt} \neq 0$. □

Proposition 1.4. If $\gamma(t)$ is a regular curve, its arc-length $s(t)$, starting at any point of γ, is a smooth function of t.

Proof. We know that

$$\frac{ds}{dt} = \|\dot{\gamma}(t)\|$$

regardless of γ being regular or not. By regularity, $\dot{\gamma}(t) \neq 0$ so in fact, it is smooth. □

Proposition 1.5. A parametrized curve has a unit-speed reparametrization if and only if it is regular.

Proof. Suppose $\gamma : (a, b) \to \mathbb{R}^n$ has a unit-speed reparametrization $\tilde{\gamma}(s)$. Then

$$1 = \left\|\frac{d\tilde{\gamma}}{ds}\right\| = \left\|\frac{d\gamma}{dt}\right\| \frac{dt}{ds}$$

and so $\frac{d\gamma}{dt} \neq 0$.

Conversely, suppose that γ is regular. Then $\frac{ds}{dt} = \|\dot{\gamma}\| > 0$. Since s is a smooth function of t with positive derivative, by the inverse function theorem, the inverse s^{-1} exists, smooth and maps intervals to intervals. Let $\phi = s^{-1}$ be the reparametrization map, so $\tilde{\gamma}(s) = \gamma(t)$ and

$$\frac{d\tilde{\gamma}}{ds}\frac{ds}{dt} = \frac{d\gamma}{dt}$$

and so

$$\left\|\frac{d\tilde{\gamma}}{ds}\right\|\frac{ds}{dt} = \left\|\frac{d\gamma}{dt}\right\| = \frac{ds}{dt}$$

hence $\tilde{\gamma}$ is unit speed. □

1.4. **Closed curves.**

Definition 1.7. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a smooth curve and let $T \in \mathbb{R}$. We say that γ is T-**periodic** if

$$\gamma(t+T) = \gamma(t) \qquad \text{for all} t \in \mathbb{R}.$$

If γ is not constant and is T-periodic for some $T \neq 0$, then γ is said to be **closed**.

Definition 1.8. The **period** of a closed curve γ is the smallest positive number T such that γ is T-periodic.

Example 1.10. The circle $\gamma(t) = (\cos(t), \sin(t))$ is 2π-periodic.

Proposition 1.6. If γ is a regular closed curve, a unit-speed reparametrization of γ is always closed.

Proof. Suppose γ has period T. Since the period is T, it is reasonable to define the "length" to be

$$L(\gamma) := \int_0^T \|\dot{\gamma}(t)\| dt.$$

A unit-speed reparametrizationd $\tilde{\gamma}$ is given by the arc-length

$$s = \int_0^t \|\dot{\gamma}(r)\| dr$$

so that $\tilde{\gamma}(s) = \gamma(t)$. Note that

$$\begin{aligned} s(t+T) &= \int_0^{t+T} \|\dot{\gamma}\| \\ &= \int_0^T \|\dot{\gamma}\| + \int_T^{t+T} \|\dot{\gamma}\| \\ &= L(\gamma) + s(t). \end{aligned}$$

Hence

$$\tilde{\gamma}(s) = \tilde{\gamma})(s+L)$$

so that $\tilde{\gamma}$ is a closed curve. So we can always assume that a closed curve is unit-speed and that its period is equal to its length. □

Definition 1.9. A curve γ is said to have a **self-intersection** at a point p of the curve if there exist parameter values $a \neq b$ such that

(1) $\gamma(a) = \gamma(b) = p$, and
(2) if γ is closed with period T, then $a - b$ is not an integer multiple of T.

1.5. **Level curves vs. parametrized curves.**

Theorem 1.1. Let $f(x,y)$ be a smooth function of two variables. Assume that, at every point of the level curve

$$\mathcal{C} = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$$

$\nabla f \neq (0,0)$. If $p = (x_0, y_0)$ is a point of $\mathcal{C}$, there is a regular parametrized curve $\gamma(t)$ defined on an open interval containing 0 such that γ passes through p when $t = 0$ and $\gamma(t)$ is contained in $\mathcal{C}$ for all t.

Theorem 1.2. Let γ be a regular parametrized plane curve, and let $\gamma(t_0) = (x_0, y_0)$. Then there is a smooth real-valued function $f(x,y)$ defined on open intervals containing x_0 and y_0 s.t. $\nabla f \neq (0,0)$ and $\gamma(t)$ is contained in the level curve $f(x,y) = 0$ for all values of t in some open interval containing t_0.

2. Curvature

2.1. **Curvature.**

To find a measure of how "curved" a curve is. Suppose that γ is a unit-speed curve in $\mathbb{R}^2$. As the parameter t of γ changes to $t + \Delta t$, the curve moves away from its tangent line at $\gamma(t)$ be a distance $(\gamma(t+\Delta t) - \gamma(t)) \cdot \mathbf{n}$ where $\mathbf{n}$ is a unit vector perpendicular to the tangent vector $\dot{\gamma}(t)$ at $\gamma(t)$ (Recall $v \cdot w = |v||w| \cos\theta$). By Taylor expansion:

$$\gamma(t+\Delta t) = \gamma(t) + \dot{\gamma}(t)\Delta t + \frac{1}{2}\ddot{\gamma}(t)(\Delta t)^2 + O((\Delta t)^3)$$

Since $\dot{\gamma} \cdot \mathbf{n} = 0$, an approximation when Δt is small for the deviation of the curve from the tangent line is given by

$$\frac{1}{2}\ddot{\gamma}(t) \cdot \mathbf{n}(\Delta t)^2.$$

This motivates the following definition

Definition 2.1. If γ is a unit-speed curve with parameter t, its **curvature** $\kappa(t)$ at the point $\gamma(t)$ is defined to be $\|\ddot{\gamma}(t)\|$.

Example 2.1. Consider the circle with radius R. It can be given by $\gamma(t) = (R\cos t, R\sin t)$. To give it a unit speed parametrization, we know that its arc-length is given by $s(t) = Rt$ so

$$\tilde{\gamma}(s) = (R\cos(s/R), R\sin(s/R))$$

is unit speed. Computing, we have

$$\ddot{\tilde{\gamma}}(s) = (-\tfrac{1}{R}\cos(\tfrac{s}{R}), -\tfrac{1}{R}\sin(\tfrac{s}{R})).$$

Therefore, the curvature is given by

$$\kappa(s) = \|\ddot{\tilde{\gamma}}(s)\| = \frac{1}{R}$$

We have a formula for regular curves (not necessarily unit-speed)

Proposition 2.1. Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$. Then its curvature is given by

$$\kappa = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}$$

Proof. Let s be a unit-speed parameter of $\gamma(t)$. Then

$$\dot{\gamma} = \frac{d\gamma}{dt} = \frac{d\gamma}{ds}\frac{ds}{dt},$$

Therefore

$$\kappa = \left\|\frac{d^2\gamma}{ds^2}\right\| = \left\|\frac{d}{ds}\left(\frac{\dot{\gamma}}{\dot{s}}\right)\right\| = \left\|\frac{\frac{d}{dt}\left(\frac{\dot{\gamma}}{\dot{s}}\right)}{\dot{s}}\right\| = \left\|\frac{\ddot{\gamma}\dot{s} - \ddot{s}\dot{\gamma}}{(\dot{s})^3}\right\| = \left\|\frac{\ddot{\gamma}(\dot{s})^2 - \dot{s}\ddot{s}\dot{\gamma}}{(\dot{s})^4}\right\|$$

Now

$$(\dot{s})^2 = \dot{\gamma} \cdot \dot{\gamma}$$

and taking the derivative we obtain

$$\dot{s}\ddot{s} = \dot{\gamma} \cdot \ddot{\gamma}.$$

Inserting these into the curvature equation

$$\kappa = \frac{\|\ddot{\gamma}(\dot{\gamma} \cdot \dot{\gamma}) - (\dot{\gamma} \cdot \ddot{\gamma})\dot{\gamma}\|}{\|\dot{\gamma}\|^2}$$

Using the triple product identity $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$ the numerator becomes

$$\ddot{\gamma}(\dot{\gamma} \cdot \dot{\gamma}) - (\dot{\gamma} \cdot \ddot{\gamma})\dot{\gamma} = \dot{\gamma} \times (\ddot{\gamma} \times \dot{\gamma})$$

Taking the norm and the fact that $\dot{\gamma}$ and $\dot{\gamma} \times \ddot{\gamma}$ are perpendicular,

$$\|\dot{\gamma} \times (\ddot{\gamma} \times \dot{\gamma})\| = \|\dot{\gamma}\|\|\ddot{\gamma} \times \dot{\gamma}\|.$$

□

2.2. **Plane Curves.** In $\mathbb{R}^2$, we can refine the definition of curvature into "signed curvature". Suppose that $\gamma(s)$ is a unit-speed curve in $\mathbb{R}^2$. Its unit tangent vector is given by

$$\mathbf{t} = \dot{\gamma}.$$

In the plane, there are two possible choices for a unit vector perpendicular to $\mathbf{t}$. We define $\mathbf{n}_s$ the **signed unit normal** of γ to be the unit vector obtained by rotating $\mathbf{t}$ counterclockwise by $\frac{\pi}{2}$. Since $\ddot{\gamma}$ is also perpendicular to $\mathbf{t}$, there must be a scalar κ_s such that

$$\ddot{\gamma} = \kappa_s \mathbf{n}_s;$$

κ_s is called the **signed curvature** of γ. If γ is a regular curve, we define the above quantities through its unit-speed parametrization. For unit tangent vectors, the direction of the tangent vector $\dot{\gamma}(s)$ is measured by the angle $\varphi(s)$ such that

$$\dot{\gamma}(s) = (\cos(\varphi(s)), \sin(\varphi(s)))$$

While the choice of the angle function $\varphi(s)$ is not unique, we can always find a smooth angle function.

Proposition 2.2. Let $\gamma : (a,b) \to \mathbb{R}^2$ be a unit-speed curve, let $s_0 \in (a,b)$ and let φ_0 be such that

$$\dot{\gamma}(s_0) = (\cos\varphi_0, \sin\varphi_0).$$

Then there is a unique smooth function $\varphi : (a,b) \to \mathbb{R}$ such that $\varphi(s_0) = \varphi_0$ and $\dot{\gamma}(s) = (\cos(\varphi(s)), \sin(\varphi(s)))$.

Proof. Let

$$\dot{\gamma}(s) = (f(s), g(s))$$

and since $\dot{\gamma}$ is a unit vector, $f^2 + g^2 = 1$. Define

$$\varphi(s) = \varphi_0 + \int_{s_0}^{s} (f\dot{g} - g\dot{f})dt.$$

This is a good candidate for the angle function since the integral term is actually $\left.\frac{g}{f}\right|_s - \left.\frac{g}{f}\right|_{s_0}$, which is the difference in the "slope" of the tangent vector. Since we are assuming γ is smooth, all the factors involved in φ are smooth hence φ is smooth. Let

$$F = f\cos\varphi + g\sin\varphi, \quad G = f\sin\varphi - g\cos\varphi.$$

then

$$\dot{F} = (\dot{f} + g\dot{\varphi})\cos\varphi + (\dot{g} - f\dot{\varphi})\sin\varphi.$$

Now the first term

$$\dot{f} + g\dot{\varphi} = \dot{f}(1 - g^2) + fg\dot{g} = f(f\dot{f} + g\dot{g}) = 0$$

and similarly $\dot{g} - f\dot{\varphi} = 0$. Hence F is constant and similarly G is constant. Plugging in initial conditions, $F(s_0) = 1$ and $G(s_0) = 0$. Hence

$$f\cos\varphi + g\sin\varphi = 1, \quad f\sin\varphi - g\cos\varphi = 0$$

which implies $f = \cos\varphi$ and $g = \sin\varphi$.

To show uniqueness, assume that there is another smooth function ψ such that $\psi(s_0) = \varphi_0$ and $\dot{\gamma}(s) = (\cos\psi(s), \sin\psi(s))$ for $s \in (a,b)$. Then $\varphi(s) - \psi(s) = 2\pi n(s)$ where $n : (a,b) \to \mathbb{Z}$. Since ψ and φ are smooth, $n(s)$ is smooth hence must be a constant. Since $n(0) = 0$, $\varphi = \psi$. $\square$

Definition 2.2. The smooth function φ in the above is called the **turning angle** of γ determined by the condition $\varphi(s_0) = \varphi_0$.

The signed curvature can be interpreted as

Proposition 2.3. Let $\gamma(s)$ be a unit-speed plane curve, and let $\varphi(s)$ be a turning angle for γ. Then

$$\kappa_s = \frac{d\varphi}{ds}.$$

Thus the signed curvature is the rate at which the tangent vector of the curve rotates.

Proof. The unit tangent vector is given by $\mathbf{t} = (\cos\varphi, \sin\varphi)$, so

$$\dot{\mathbf{t}} = \dot{\varphi}(-\sin\varphi, \cos\varphi) = \kappa_s \mathbf{n}_s$$

Since $\mathbf{n}_s = (-\sin\varphi, \cos\varphi)$, we get our result. $\square$

Definition 2.3. The **total signed curvature** of a unit-speed closed curve γ of length l is

$$\int_0^l \kappa_s(s)ds$$

Here we encounter our first "Gauss-Bonnet" type result

Corollary 2.1. The total signed curvature of a closed plane curve is an integer multiple of 2π.

Proof. Let γ be a unit-speed closed plane curve and let l be its length. The total signed curvature of γ is

$$\int_0^l \kappa_s(s)ds = \int_0^l \frac{d\varphi}{ds}ds = \varphi(l) - \varphi(0),$$

where φ is a turning angle for γ. Now γ is unit-speed and closed hence l-periodic, hence $\dot{\gamma}(l) = \dot{\gamma}(0)$, that is

$$(\cos\varphi(l), \sin\varphi(l)) = (\cos\varphi(0), \sin\varphi(0))$$

therefore $\varphi(l) - \varphi(0) = 2\pi n$. $\square$

For plane curves, its signed curvature determines the curve up to isometry.

Theorem 2.1. Let $k : (a,b) \to \mathbb{R}$ be any smooth function. Then, there is a unit-speed curve $\gamma : (a,b) \to \mathbb{R}^2$ whose signed curvature is k. Furthermore, if $\tilde{\gamma} : (a,b) \to \mathbb{R}^2$ is any other unit-speed curve whose signed curvature is k, there is a direct isometry of M of $\mathbb{R}^2$ such that

$$\tilde{\gamma}(s) = M(\gamma(s)), \quad \text{for all } s \in (a,b)$$

Proof. For the first part, fix $s_0 \in (a,b)$ and define for any $s \in (a,b)$,

$$\varphi(s) = \int_{s_0}^s k(u)du$$

and

$$\gamma(s) = \left(\int_{s_0}^s \cos\varphi(t)dt, \int_{s_0}^s \sin\varphi(t)dt\right).$$

Then the tangent vector of γ is

$$\dot{\gamma}(s) = (\cos\varphi(s), \sin\varphi(s)),$$

and by Proposition 2.3, its signed curvature is

$$\frac{d\varphi}{ds} = \frac{d}{ds}\int_{s_0}^s k(u)du = k(s).$$

Now let $\tilde{\varphi}$ be a smooth turning angle for $\tilde{\gamma}$. Then

$$\dot{\tilde{\gamma}}(s) = (\cos\tilde{\varphi}(s), \sin\tilde{\varphi}(s)),$$

therefore

$$\tilde{\gamma}(s) = \left(\int_{s_0}^{s} \cos \tilde{\varphi}(t) dt, \int_{s_0}^{s} \sin \tilde{\varphi}(t) dt \right) + \tilde{\gamma}(s_0).$$

Furthermore, $\tilde{\gamma}$ has signed curvature $k(s) = \frac{d\tilde{\varphi}}{ds}$ so

$$\tilde{\varphi}(s) = \int_{s_0}^{s} k(u) du + \tilde{\varphi}(s_0).$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be translation by $\tilde{\gamma}(s_0)$, and let $\theta = \tilde{\varphi}(s_0)$. Then

$$\begin{aligned} \tilde{\gamma}(s) &= T \left(\int_{s_0}^{s} \cos(\varphi(t) + \theta) dt, \int_{s_0}^{s} \sin(\varphi(t) + \theta) dt \right) \\ &= T \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} (\gamma(s))^T \end{aligned}$$

□

Proposition 2.4. The signed curvature κ_s of a regular plane curve is a smooth function of s.

Proof. Since regular curves always have a unit-speed parametrization, let γ be a unit-speed curve. By Proposition 2.2, there is a unique smooth turning angle φ. By Proposition 2.3, $\kappa_s = \frac{d\varphi}{ds}$. Hence κ_s is smooth. □

Example 2.2. Any regular plane curve γ whose curvature is a positive constant is part of a circle. Let κ be the curvature of γ and let κ_s be its signed curvature. Then $\kappa_s = \pm\kappa$. Since κ_s is continuous, it must be constant in this case. From Theorem 2.1, if we can find a parametrized circle whose signed curvature is κ_s, then every curve of constant curvature must come from some isometry of this circle.

A unit-speed parametrized circle of radius R centered at the origin is given by

$$\gamma(s) = \left(R \cos\left(\frac{s}{R}\right), R \sin\left(\frac{s}{R}\right) \right).$$

Its tangent vector is

$$\mathbf{t} = \dot{\gamma} = \left(-\sin\left(\frac{s}{R}\right), \cos\left(\frac{s}{R}\right) \right) = \left(\cos\left(\frac{s}{R} + \frac{\pi}{2}\right), \sin\left(\frac{s}{R} + \frac{\pi}{2}\right) \right).$$

So the turning angle is $\varphi = \dfrac{s}{R} + \dfrac{\pi}{2}$ hence $\kappa_s = \dfrac{d\varphi}{ds} = \dfrac{1}{R}$. If $\kappa_s > 0$, let $R = \kappa_s^{-1}$. If we parametrize the circle so that it turns clockwise, i.e.,

$$\tilde{\gamma}(s) = \left(R \cos\left(\frac{2}{R}\right), -R \sin\left(\frac{s}{R}\right) \right),$$

then its signed curvature is $-\frac{1}{R}$, which gives the case for $\kappa_s < 0$.

In summary knowing the signed curvature essentially tells us what the curve is in $\mathbb{R}^2$.

2.3. **Space curves.** Now we consider curves in $\mathbb{R}^3$. These will be called **space curves**. In 3 dimensions, curvature is no longer sufficient. Let $\gamma(s)$ be a unit-speed curve in $\mathbb{R}^3$, and let $\mathbf{t} = \dot{\gamma}$ be its unit tangent vector.

Definition 2.4. If curvature $\kappa(s) \neq 0$, define the **principal normal** of γ at the point $\gamma(s)$ to be the vector

$$\mathbf{n}(s) = \frac{1}{\kappa(s)} \dot{\mathbf{t}}(s).$$

Define the **binormal** vector of γ at the point $\gamma(s)$ to be

$$\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$$

It is clear that $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ forms a orthonormal basis and since $\mathbf{b}$ is a unit vector, it is perpendicular to $\dot{\mathbf{b}}$. Furthermore,

$$\dot{\mathbf{b}} = \frac{d}{ds}(\mathbf{t} \times \mathbf{n}) = \mathbf{t} \times \dot{\mathbf{n}}.$$

So $\dot{\mathbf{b}}$ is perpendicular to both $\mathbf{t}$ and $\mathbf{b}$ hence is parallel to $\mathbf{n}$ hence

$$\dot{\mathbf{b}} = -\tau \mathbf{n}$$

for some scalar τ, which we call **torsion** of γ. The torsion of a regular curve is simply the torsion of its unit-speed parametrization.

Proposition 2.5. Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ with nowhere-vanishing curvature. Then

$$\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2}$$

Proof. Direct computation. □

Proposition 2.6. Let γ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then, the image of γ is contained in a plane if and only if τ is zero at every point of the curve.

Proof. We can assume γ is unit-speed. First assume that γ is contained in a plane, say $v \cdot N = d$ for some constant unit normal vector N. Recall that this characterizes the plane, if we fix some coordinates then $N = (a, b, c)$ and $v = (x, y, z)$ so $N \cdot v = ax + by + cz = d$. Then we have $\mathbf{t} \cdot N = 0$ and $\dot{\mathbf{t}} \cdot N = 0$. Using the fact that $\dot{\mathbf{t}} = \kappa \mathbf{n}$, we can conclude that $\mathbf{n} \cdot N = 0$ if $\kappa \neq 0$. So $\mathbf{t}$ and $\mathbf{n}$ are both perpendicular to N hence $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is parallel to N. Since we assumed N is a unit vector, by continuity of the function $\mathbf{b}(s)$, we must have either $\mathbf{b} = N$ or $\mathbf{b} = -N$ and in both cases $\dot{\mathbf{b}} = 0$.

Conversely, suppose that $\tau = 0$ everywhere, hence $\dot{\mathbf{b}} = 0$. So $\mathbf{b}$ is a constant vector. Then

$$\frac{d}{ds}(\gamma \cdot \mathbf{b}) = \mathbf{t} \cdot \mathbf{b} = 0$$

so $\gamma \cdot \mathbf{b} = d$ for some constant d. □

We computed $\dot{\mathbf{t}}$ and $\dot{\mathbf{b}}$. Now we compute $\dot{\mathbf{n}}$:

$$\dot{\mathbf{n}} = -\tau \mathbf{n} \times \mathbf{t} + \kappa \mathbf{b} \times \mathbf{n} = -\kappa \mathbf{t} + \tau \mathbf{b}$$

hence we obtain the **Frenet-Serret equations**:

$$\begin{pmatrix} \dot{\mathbf{t}} \\ \dot{\mathbf{n}} \\ \dot{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}$$

Proposition 2.7. Let γ be a unit-speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Then γ is a parametrization of (part of) a circle.

Proof. Since $\tau = 0$, $\mathbf{b}$ is a constant vector and γ is contained in a plane P perpendicular to $\mathbf{b}$. Now

$$\frac{d}{ds}\left(\gamma + \frac{1}{\kappa}\mathbf{n}\right) = 0$$

so $\gamma + \frac{1}{\kappa}\mathbf{n}$ is a constant vector, say $\mathbf{a}$. Then

$$\|\gamma - \mathbf{a}\| = \frac{1}{\kappa}$$

So γ lies on the sphere with center $\mathbf{a}$ and radius $\frac{1}{\kappa}$. The intersection of the plane and the sphere is a circle. □

Proposition 2.8. Let P be an 3×3 orthogonal matrix and let $a \in \mathbb{R}^3$ so that $M(v) = Pv + a$ is a direct isometry of $\mathbb{R}^3$. If γ is unit-speed in $\mathbb{R}^3$, the curve $\Gamma = M(\gamma)$ is also unit-speed. Furthermore, if $\mathbf{t}, \mathbf{n}, \mathbf{b}$ and $\mathbf{T}, \mathbf{N}, \mathbf{B}$ are the tangent, principal normal, and binormal for γ and Γ, respectively, then $\mathbf{T} = P\mathbf{t}$, $\mathbf{N} = P\mathbf{n}$ and $\mathbf{B} = P\mathbf{b}$.

Proof. We have

$$\|\dot{\Gamma}\|^2 = \|P\dot{\gamma}\|^2 = \dot{\gamma}^T P^T P \dot{\gamma} = \dot{\gamma}^T \dot{\gamma} = \|\gamma\|^2 = 1.$$

Also, since P is an isometry, $\|\ddot{\Gamma}\| = \|\ddot{\gamma}\| = \kappa$. $\mathbf{T} = P\mathbf{t}$ and $\mathbf{N} = P\mathbf{n}$ are straightforward. The binormal vectors can be deduced from

$$(At) \times (An) = \det(A)(A^T)^{-1}(t \times n)$$

where we use $A = P \in SO(3)$ so that $\det(P) = 1$ and $P^{-1} = P^T$. $\square$

Remark A **direct isometry** is an isometry of the form $F(v) = Pv + a$ where $P \in SO(n)$.

Theorem 2.2. Let $\gamma(s)$ and $\tilde{\gamma}(s)$ be two unit-speed curves in $\mathbb{R}^3$ with the same curvature $\kappa(s) > 0$ and the same torsion $\tau(s)$ for all s. Then, there is a direct isometry M of $\mathbb{R}^3$ such that

$$\tilde{\gamma}(s) = M(\gamma(s)).$$

Furthermore, if k and t are smooth functions with $k > 0$ everywhere, there is a unit-speed curve in $\mathbb{R}^3$ whose curvature is k and whose torsion is t.

Proof. Let $\mathbf{t}, \mathbf{n}$, and $\mathbf{b}$ be the tangent, principal normal, and binormal of γ and $\tilde{\mathbf{t}}, \tilde{\mathbf{n}}$, and $\tilde{\mathbf{b}}$ be those for $\tilde{\gamma}$. Let s_0 be fixed and θ the angle between them. Let ρ be the rotation that takes $\mathbf{t}(s_0)$ to $\tilde{\mathbf{t}}(s_0)$, i.e. $\rho(\mathbf{t}(s_0)) = \tilde{\mathbf{t}}(s_0)$. Let ρ' be the rotation fixing $\tilde{\mathbf{t}}(s_0)$ and $\rho'(\rho(\mathbf{n}(s_0))) = \tilde{\mathbf{n}}(s_0)$. By orthonormality, $\rho'\rho(\mathbf{b}(s_0)) = \tilde{\mathbf{b}}(s_0)$. Let T be the translation action by $\tilde{\gamma}(s_0) - \gamma(s_0)$. Then $M := T\rho'\rho$ is a direct isometry and by Proposition 2.8, $\Gamma = M(\gamma)$ is a unit speed curve, with $\mathbf{T}(s), \mathbf{N}(s)$, and $\mathbf{B}(s)$ the tangent, principal normal, and binormal along $\Gamma(s)$. By construction, $\mathbf{T}(s_0) = \tilde{\mathbf{t}}(s_0)$, $\mathbf{N}(s_0) = \tilde{\mathbf{n}}(s_0)$, and $\mathbf{B}(s_0) = \tilde{\mathbf{b}}(s_0)$. We want to show that they are in fact the same for all s. We consider the following

$$A(s) = \tilde{\mathbf{t}}(s) \cdot \mathbf{T}(s) + \tilde{\mathbf{n}}(s) \cdot \mathbf{N}(s) + \tilde{\mathbf{b}}(s) \cdot \mathbf{B}(s).$$

We have $A(s_0) = 3$ and since all the vectors involved are unit vectors, $A(s) = 3$ if and only if $\tilde{\mathbf{t}} = \mathbf{T}$, $\tilde{\mathbf{n}} = \mathbf{N}$, and $\tilde{\mathbf{b}} = \mathbf{B}$. Thus we want to show that A is a constant. By direct computation $\cdot A = 0$, where we use the fact that both curves satisfy the Frenet-Serret equations with the same curvature and torsion. Hence we proved the first part.

For the second part, the Frenet-Serret equation has a unique solution give an initial condition $\mathbf{T}(s_0) = e_1$, $\mathbf{N}(s_0) = e_2$ and $\mathbf{B}(s_0) = e_3$, and in fact they stay orthonormal for all values of s. Now define

$$\gamma(s) = \int_{s_0}^{s} \mathbf{T}(u)du.$$

Then $\dot{\gamma} = \mathbf{T}$ and $\dot{\mathbf{T}} = k\mathbf{N}$. Further more, $\mathbf{B} = \lambda \mathbf{T} \times \mathbf{N}$ for some $\lambda = \pm 1$. By continuity, $\lambda = 1$ so $\mathbf{B}$ is the binormal, hence t is the torsion. $\square$

3. Global properties of curves

3.1. Simple closed curves.

Definition 3.1. A **simple closed curve** in $\mathbb{R}^2$ is a closed curve in $\mathbb{R}^2$ that has no self-intersection

A simple closed curve splits $\mathbb{R}^2$ into two connected regions, its interior and exterior. A simple closed curve is **positively oriented** if the signed unit normal points towards the interior and **negatively oriented** if not.

3.2. The isoperimetric inequality.

The isoperimetric inequality is the following

Theorem 3.1. Let γ be a simple closed curve with length $l(\gamma)$ and area of the interior $A(\gamma)$. Then

$$A(\gamma) \leq \frac{l(\gamma)^2}{4\pi}$$

with equality if and only if γ parametrizes a circle.

In preparation to prove this, we need to use the following formula for the area of the interior of a simple closed curve.

Proposition 3.1. If $\gamma(t) = (x(t), y(t))$ is a positively-oriented simple closed curve in $\mathbb{R}^2$ with period T, then

$$A(\gamma) = \frac{1}{2}\int_0^T (x\dot{y} - y\dot{x})dt.$$

The proof can be found in multi-variable calculus textbooks, for instance the section on Green's theorem of Stewart Calculus.

Another identity we will use is called the (weak) Wirtinger's Inequality

Proposition 3.2. Let $F : [0,\pi] \to \mathbb{R}$ be a smooth function such that $F(0) = F(\pi) = 0$. Then,

$$\int_0^\pi \left(\frac{dF}{dt}\right)^2 dt \geq \int_0^\pi F(t)^2 dt.$$

and equality holds if and only if $F(t) = a\sin t$ for all $t \in [0,\pi]$, where a is a constant.

Proof. First note that

$$\int_0^\pi (F' - F\cot(t))^2 dt \geq 0.$$

Expanding this we get

$$0 \leq \int_0^\pi (F')^2 dt - \int_0^\pi 2FF'\cot(t)dt + \int_0^\pi F^2\cot^2(t)dt.$$

Since $(F^2)' = 2FF'$, integrating the second term by parts we obtain

$$-\int_0^\pi 2FF'\cot(t)dt = -(F^2\cot(t))\Big|_0^\pi - \int_0^\pi F^2\csc^2(t)dt.$$

The boundary terms will vanish by taking the limit or considering their Taylor expansions. Using the fact that $\csc^2(t) - \cot^2(t) = 1$, we obtain

$$\int_0^\pi (F')^2 dt - \int_0^\pi F^2 dt \geq 0.$$

Solving the ODE

$$F' = F\cot(t)$$

yields $F(t) = a\sin(t)$. □

Now we prove the isoperimetric inequality

Proof of Theorem 3.1. Let γ be a simple closed curve with length $l(\gamma)$. Let s be its arc length parameter. For convenience, we reparametrize so that the period is π, namely

$$t = \frac{\pi s}{l(\gamma)}.$$

Furthermore, we assume that $\gamma(0) = 0$. We calculate the area and length using polar coordinates

$$x(t) = r(t)\cos\theta(t), \quad y(t) = r(t)\sin\theta(t),$$

note that the angle $\theta(t)$ is a function of our parametrization t. Then

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2, \quad x\dot{y} - y\dot{x} = r^2\dot{\theta}.$$

By chain rule,

$$\dot{x}^2 + \dot{y}^2 = \left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right)\left(\frac{ds}{dt}\right)^2 = \frac{l(\gamma)^2}{\pi^2}.$$

By Green's theorem, to obtain the isoperimetric inequality, we want to show that

$$\frac{l(\gamma)^2}{4\pi} - A(\gamma) = \frac{1}{4}\int_0^\pi (\dot{r}^2 + r^2\dot{\theta}^2 dt - \frac{1}{2}\int_0^\pi r^2\dot{\theta}dt$$

is nonnegative and equality if and only if γ is a circle. Rearranging terms and multiplying by 4, we want

$$\int_0^\pi r^2(\dot\theta - 1)^2 dt + \int_0^\pi (\dot r^2 - r^2) dt$$

Is nonnegative. The first term is integrating a squared term and the second follows from Wirtinger's inequality. Wirtinger's inequality is equal when $r(t) = a\sin(t)$ and the first term vanishes when $\theta = t + c$ for some constant. $\square$

4. Surfaces

4.1. **Introduction.** Heuristically, a surface is a subset of $\mathbb{R}^3$ that "looks" like a piece of $\mathbb{R}^2$. A motivating example is the surface of the Earth. The Earth is globally a sphere however, locally looks like a flat plane. To be precise,

Definition 4.1. A subset $S \subset \mathbb{R}^3$ is a **surface** if, for every point $p \in S$, there is an open set $U \subset \mathbb{R}^2$ and an open set $W \subset \mathbb{R}^3$ containing p such that $S \cap W$ is homeomorphic to U. A homeomorphism $\sigma : U \to S \cap W$ from the above definition is called a **surface patch** or **parametrization** of $S \cap W$ of S. A collection of such surface patches whose images cover the whole of S is called an **atlas** of S.

Example 4.1. The most basic example of a surface is a plane in $\mathbb{R}^3$. Let a be a point in the plane and $\mathbf{p}$ and $\mathbf{q}$ be two non parallel vectors perpendicular to the defining normal vector of the plane. Then a surface patch is given by

$$\sigma(u, v) = a + u\mathbf{p} + v\mathbf{q}.$$

Example 4.2. A circular cylinder is the set of points in $\mathbb{R}^3$ that are at a fixed distance from a fixed straight line. A level surface description is given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}.$$

A surface patch can be given by

$$\sigma(u, v) = (\cos(u), \sin(u), v),$$

however we must be careful on how we define the open set that this is defined over. Note that $(0, 2\pi] \times \mathbb{R}$ would cover the cylinder but is not open. Any "larger" set would make σ not injective. If we let

$$U = (0, 2\pi) \times \mathbb{R},$$

then we do not cover a line on the cylinder. Hence we need to consider another surface patch by considering another open set

$$\tilde U = (-\pi, \pi) \times \mathbb{R}.$$

Then an atlas is given by $\{\sigma|_U, \sigma|_{\tilde U}\}$.

Example 4.3. A unit 2-sphere S^2 is given as a level set by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

One way to parametrize this is by using latitude and longitude,i.e.

$$\sigma(\theta, \varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta).$$

with

$$U = \{(\theta, \varphi) \mid -\frac{\pi}{2} < \theta < \frac{\pi}{2}, 0 < \varphi < 2\pi\}.$$

This covers most of the sphere except a semicircle on the negative x-axis. To obtain an atlas, we consider another surface patch

$$\tilde\sigma(\theta, \varphi) = (-\cos\theta\cos\varphi, -\sin\theta, -\cos\theta\sin\varphi)$$

with the same U.

Example 4.4. A non-example of a surface is given by the circular cone defined by the level set

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}.$$

We can see that this is not a surface in our definition since any surface patch containing the vertex will not be homeomorphic to a ball. If we remove the vertex, then we have a union of surfaces, each homeomorphic to a cylinder.

Definition 4.2. Given two surface patches, σ and $\tilde{\sigma}$, we can consider the composition $\Phi := \sigma^{-1} \circ \tilde{\sigma}$, called **transition map** defined on the preimage of the intersection of the surface patches. Then we have

$$\tilde{\sigma}(\tilde{u}, \tilde{v}) = \sigma(\Phi(\tilde{u}, \tilde{v}))$$

where defined.

4.2. **Smooth Surfaces.**

Definition 4.3. A surface patch $\sigma : U \to \mathbb{R}^3$ is called **regular** if it is smooth and the vectors $\sigma_u := \frac{\partial \sigma}{\partial u}$ and $\sigma_v := \frac{\partial \sigma}{\partial v}$ are linearly independent at all points $(u, v) \in U$. A **smooth surface** S is one where every point $p \in S$, there is a regular surface patch with $p \in \sigma(U)$

Proposition 4.1. The transition maps of a smooth surface are smooth.

Proof. Let $\sigma : U \to \mathbb{R}^3$ and $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ be two (regular) surface patches. Suppose that $p \in S$ lies in both surfaces patches, i.e.

$$\sigma(u_0, v_0) = \tilde{\sigma}(\tilde{u}_0, \tilde{v}_0) = p.$$

Written out in coordinates, we have

$$\sigma(u, v) = (f(u, v), g(u, v), h(u, v)).$$

Using subscripts as partial derivatives, linear independence of σ_u and σ_v implies the Jacobian $J(\sigma)$

$$\begin{pmatrix} f_u & f_v \\ g_u & g_v \\ h_u & h_v \end{pmatrix}$$

is a rank 2 matrix everywhere. Without loss of generality, assume

$$\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$$

is full rank, i.e. invertible, at p. Consider the projection

$$\pi(x, y, z) = (x, y)$$

and define $F := \pi \circ \sigma$, so that

$$F(u, v) = (f(u, v), g(u, v)).$$

We apply the inverse function theorem to obtain open sets $W \subset \mathbb{R}^2$ containing (u_0, v_0) and $V \subset \mathbb{R}^2$ containing $F(u_0, v_0)$ where $F : W \to V$ is bijective with smooth inverse $F^{-1} : V \to W$. Similarly define $\tilde{F} := \pi \circ \tilde{\sigma}$. Then

$$F^{-1} \circ \tilde{F} = \sigma^{-1} \circ \tilde{\sigma}$$

on $\tilde{W} = \tilde{\sigma}^{-1}\sigma(W)$. Since F^{-1} and $\tilde{F}$ are smooth, the transition map is smooth. $\square$

The next result states how we can **reparametrize** surface patches.

Proposition 4.2. Let U and $\tilde{U}$ be open subsets of $\mathbb{R}^2$ and let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch. Let $\Phi : \tilde{U} \to U$ be a bijective smooth map with smooth inverse map $\Phi^{-1} : U \to \tilde{U}$. Then, $\tilde{\sigma} = \sigma \circ \Phi : \tilde{U} \to \mathbb{R}^3$ is a regular surface patch.

Proof. First, the patch $\tilde{\sigma}$ is smooth since Φ is smooth. It remains to show that $\tilde{\sigma}$ is regular. Let $\Phi(x, y) = (u(x, y), v(x, y))$. Then

$$\frac{\partial \tilde{\sigma}}{\partial x} = \frac{\partial \sigma}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \sigma}{\partial v}\frac{\partial v}{\partial x}$$

and

$$\frac{\partial \tilde{\sigma}}{\partial y} = \frac{\partial \sigma}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \sigma}{\partial v}\frac{\partial v}{\partial y}$$

and by direct computation

$$\frac{\partial \tilde{\sigma}}{\partial x} \times \frac{\partial \tilde{\sigma}}{\partial y} = \det(J(\Phi))\frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v}.$$

Since Φ is smooth bijective with smooth inverse, $\det(J(\Phi)) \neq 0$ so linear independence of one implies the other. □

In summary we assume all surfaces are smooth and all surface patches are regular unless otherwise stated. Furthermore, we assume the surfaces are connected.

4.3. **Smooth maps.** What does it mean for a map between two surfaces, $f : S_1 \to S_2$ to be **smooth**? We have a well-defined notion of differentiability on maps between Euclidean space since we have a natural set of coordinates. On surfaces then, it would depend on the parametrization. Let σ_1 and σ_2 be surface patches for S_1 and S_2 respectively. Then we can compose with the map f to get a map between Euclidean spaces,

$$\sigma_2^{-1} \circ f \circ \sigma_1 : U_1 \subset \mathbb{R}^2 \to U_2 \subset \mathbb{R}^2.$$

We then claim that f is **smooth** if the composition map is a smooth map between Euclidean spaces. This is well-defined since the transition maps between different surface patches are smooth. A particular class of smooth maps we will be interested on those who are bijective with smooth inverses. These will be called **diffeomorphisms** and if there exists a diffeomorphism $f : S_1 \to S_2$, then S_1 and S_2 are said to be **diffeomorphic**.

Proposition 4.3. Let $f : S_1 \to S_2$ be a diffeomorphism. If σ_1 is a surface patch on S_1, then $f \circ \sigma_1$ is a surface patch on S_2.

Proof. Immediately follows from Proposition 4.2. □

We also consider a condition slightly weaker.

Definition 4.4. A smooth map $f : S_1 \to S_2$ between smooth surfaces is called a **local diffeomorphism** if, for any point $p \in S_1$, there is an open set U such that $f(U)$ is open in S_2 and $f|_U : U \to f(U)$ is a diffeomorphism.

Example 4.5. Consider a map between the $y - z$ plane and the unit cylinder given by

$$f(0, y, z) = (\cos(y), \sin(y), z).$$

While f is not a diffeomorphism, we will show that it is a local diffeomorphism. Let $\pi(u, v) = (0, u, v)$ parametrize the $y - z$ plane. Let $\{\sigma|_U, \sigma|_{\tilde{U}}\}$ be an atlas for the unit cylinder given earlier. If $2n\pi < a < 2(n + 1)\pi$, then

$$f(\pi(a, z)) = \sigma(a - 2n\pi, z)$$

If a is an even multiple of 2π, we use $\sigma|_{\tilde{U}}$.

4.4. **Tangents and derivatives.** In order to do calculus on the surfaces, we define the notion of a tangent vector to a surface.

Definition 4.5. A **tangent vector** to a surface S at a point $p \in S$ is the tangent vector at p of a curve in S passing through p. The **tangent space** T_pS of S at p is the set of all tangent vectors to S at p.

Proposition 4.4. Let $\sigma : U \to \mathbb{R}^3$ be a patch of a surface S containing a point $p \in S$, and let (u, v) be coordinates in U. The tangent space to S at p is the vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v.

Proof. Let γ be a smooth curve in S. Then

$$\gamma(t) = \sigma(u(t), v(t)).$$

Then

$$\frac{\partial \gamma}{\partial t} = \sigma_u u_t + \sigma_v v_t$$

So $\frac{\partial \gamma}{\partial t}$ is a linear combination of σ_u and σ_v. Now we must show that if we have a vector in the span of σ_u and σ_v, say $\lambda\sigma_u + \mu\sigma_v$, then there is a curve that gives that vector as its tangent vector. That is achieved by the curve

$$\gamma(t) = \sigma(u_0 + \lambda t, v_0 + \mu t).$$

□

The plane defined by the tangent space will be called the **tangent plane**. Now we want to define a notion of a derivative for smooth map $f : S_1 \to S_2$ between smooth surfaces.

Definition 4.6. Let $w \in T_p S_1$, then there is a curve γ passing through p at t_0 such that $w = \dot{\gamma}(t_0)$. Let $\tilde{\gamma} = f \circ \gamma$. Then there is a corresponding $\tilde{w} = \dot{\tilde{\gamma}}(t_0)$. The **derivative** $D_p f$ of f at the point $p \in S_1$ is the map $D_p f : T_p S_1 \to T_{f(p)} S_2$ such that $D_p f(w) = \tilde{w}$.

We need to first make sure that this is well-defined, i.e. only depends on f, p, and w. Let $\sigma : U \to \mathbb{R}^3$ be a surface patch of S containing p, say $\sigma(u_0, v_0)$, and let

$$f(\sigma(u, v)) = \tilde{\sigma}(\alpha(u, v), \beta(u, v)).$$

Let $w = \lambda\sigma_u + \mu\sigma_v$ be the tangent vector at p of a curve $\gamma(t) = \sigma(u(t), v(t))$. Then the corresponding curve $\tilde{\gamma}$ is given by

$$\tilde{\gamma}(t) = \tilde{\sigma}(\alpha(u(t), v(t)), \beta(u(t), v(t))).$$

Then by directly computing we have

$$\begin{aligned} D_p f(w) &= (\dot{u}\alpha_u + \dot{v}\alpha_v)\tilde{\sigma}_\alpha + (\dot{u}\beta_u + \dot{v}\beta_v)\tilde{\sigma}_\beta|_{t_0} \\ &= (\lambda\alpha_u + \mu\alpha_v)\tilde{\sigma}_\alpha + (\lambda\beta_u + \mu\beta_v)\tilde{\sigma}_\beta|_{t_0}. \end{aligned}$$

Since $\dot{u}(t_0) = \lambda$ and $\dot{v}(t_0) = \mu$, the right hand side is independent of the curve γ. Note that we can rewrite in matrix form as

$$\tilde{w} = \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = D_p f(w)$$

using the basis $\{\tilde{\sigma}_\alpha, \tilde{\sigma}_\beta\}$ and $\{\sigma_u, \sigma_v\}$. What we have shown is that

Proposition 4.5. If $f : S_1 \to S_2$ is a smooth map between surfaces and $p \in S$, the derivative $D_p f : T_p S_1 \to T_{f(p)} S_2$ is a linear map.

Furthermore,

Proposition 4.6.

(1) If S is a surface and $p \in S$, the derivative at p of the identity map $S \to S$ is the identity map $T_p S \to T_p S$.
(2) If S_1, S_2, and S_3 are surfaces and $f_1 : S_1 \to S_2$ and $f_2 : S_2 \to S_3$ are smooth maps, then for all $p \in S_1$,
$$D_p(f_2 \circ f_1) = D_{f_1(p)}(f_2) D_p f_1.$$
(3) If $f : S_1 \to S_2$ is a diffeomorphism, then for all $p \in S_1$ the linear map $D_p f : T_p S_1 \to T_{f(p)} S_2$ is invertible.

Proof.

(1) Suppose f is the identity map since $\gamma(t) = f(\gamma(t))$, $\tilde{w} = \gamma'(t) = Iw$, hence $D_p f = I$.
(2) Let γ_1 be a curve in S_1 and let $\gamma_2 = f_1(\gamma_1)$. Then $D_p f_1(\gamma_1') = \gamma_2'$. If $\gamma_3 = f_2(\gamma_2)$, then $\gamma_3' = D_{f_1(p)} f_2(\gamma_2') = D_{f_1(p)} f_2 D_p f_1(\gamma_1')$.

(3) Since $f^{-1} \circ f$ is the identity map, $D_p(f^{-1} \circ f) = I$. From part (2), we know that $D_p(f^{-1} \circ f) = D_{f(p)} f^{-1} D_p f$ hence $D_p f$ is invertible.

□

We now give a simple criterion for a smooth map to be a local diffeomorphism

Proposition 4.7. Let S and $\tilde{S}$ be a smooth map. Then f is a local diffeomorphism if and only if, for all $p \in S$, the linear map $D_p f : T_p S \to T_{f(p)} \tilde{S}$ is invertible.

Proof. Suppose f is a local diffeomorphism, then it is a diffeomorphism from some open set U to $f(U)$. Then the previous proposition tells us the derivative is invertible.

Conversely, suppose $D_p f$ is invertible. Let $\sigma : U \to \mathbb{R}^3$ be a surface patch containing p and $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ a surface patch containing $f(p)$. By shrinking U if necessary, we assume $f(\sigma(U)) \subset \tilde{\sigma}(\tilde{U})$, that is, there are smooth functions $\alpha, beta$ such that

$$f(\sigma(u, v)) = \tilde{\sigma}(\alpha(u, v), \beta(u, v)).$$

Then the map $F(u, v) = (\alpha(u, v), \beta(u, v))$ has the Jacobian matrix

$$\begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix}$$

which is the matrix of $D_p f$ under a suitable basis, hence invertible. Now we apply the inverse function theorem to F to find open sets $V \subset U$ and $\tilde{V} \subset \tilde{U}$ such that $F : V \to \tilde{V}$ is a diffeomorphism. Then f is a diffeomorphism between $\sigma(V)$ and $\tilde{\sigma}(\tilde{V})$, hence f is a local diffeomorphism. □

4.5. **Normals and orientability.** Given a surface patch $\sigma : U \to \mathbb{R}^3$, define the **standard unit normal** by

$$N_\sigma = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

This is not independent of the surface patch, if $\tilde{\sigma}$ is another surface patch, then

$$N_\sigma = \pm N_{\pm\tilde{\sigma}}$$

Definition 4.7. A surface S is **orientable** if there exists an atlas $\mathcal{A}$ for S with property that, if Φ is the transition map between any two surface patches in $\mathcal{A}$, then $\det(D(\Phi)) > 0$ where Φ is defined.

Definition 4.8. An **oriented surface** is a surface S together with a smooth choice of unit normal N at each point, i.e. a smooth map $N : S \to \mathbb{R}^3$ such that for all $p \in S$, $N(p)$ is a unit vector perpendicular to $T_p S$. It can be shown that oriented surfaces are orientable.

Example 4.6 (Möbius Strip)**.** While we will mostly deal with oriented surfaces, here is an example of a non-orientable surface. It is defined by the surface patch

$$\sigma(t, \theta) = \left(\left(1 - t \sin \frac{\theta}{2}\right) \cos \theta, \left(1 - t \sin \frac{\theta}{2}\right) \sin \theta, t \cos \frac{\theta}{2} \right),$$

with the domain of definition to be

$$U = \{(t, \theta) \in \mathbb{R}^2 \mid -\frac{1}{2} < t < \frac{1}{2}, 0 < \theta < 2\pi\}.$$

and

$$\tilde{U} = \{(t, \theta) \in \mathbb{R}^2 \mid -\frac{1}{2} < t < \frac{1}{2}, -\pi < \theta < \pi\}.$$

Computing the normal, we have

$$\sigma_t = \left(-\sin \frac{\theta}{2} \cos \theta, -\sin \frac{\theta}{2} \sin \theta, \cos \frac{\theta}{2} \right),$$

and

$$\sigma_\theta = (-\sin \theta, \cos \theta, 0),$$

and

$$\sigma_t \times \sigma_\theta = \left(-\cos\theta\cos\frac{\theta}{2}, -\sin\theta\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right).$$

If the Mobius strip was orientable, then there would be a well-defined unit normal N defined at every point of S. However, at the point $\sigma(0,0) = \sigma(0,2\pi)$

$$N = \lim_{\theta\to 0+} N_\sigma = (-1,0,0)$$

but also

$$N = \lim_{\theta\to 2\pi^+} N_\sigma = (1,0,0).$$

5. Examples of surfaces

5.1. Level surfaces.

Definition 5.1. A **level surface** is given as a zero set of a smooth function, i.e.

$$\{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0\}$$

Theorem 5.1. Let S be a subset of $\mathbb{R}^3$ with the following property: for each point $p \in S$, there is an open subset $W \subset \mathbb{R}^3$ containing the point p and a smooth function $f : W \to \mathbb{R}$ such that

(1) $S \cap W = \{(x,y,z) \in W \mid f(x,y,z) = 0\}$;
(2) The gradient ∇f does not vanish at p.

Then S is a smooth surface.

Proof. Let $p = (x_0, y_0, z_0)$ and assume $\frac{\partial f}{\partial z}|_p \neq 0$. Consider the map $F : W \to \mathbb{R}^3$ defined by

$$F(x,y,z) = (x,y,f(x,y,z)).$$

The Jacobian of F is

$$DF = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{pmatrix}$$

which is full rank, hence invertible at p. By the inverse function theorem, there exists an open set V containing $F(x_0,y_0,z_0) = (x_0,y_0,0)$ and a smooth map $G : V \to W$ such that $\tilde{W} = G(V)$ is open and $F : \tilde{W} \to V$ and $G : V \to \tilde{W}$ are inverse bijections. Since V is open, we can find open subsets $U_1 \subset \mathbb{R}^2$ containing (x_0,y_0) and $U_2 \subset \mathbb{R}$ containing 0 such that $U_1 \times U_2 \subset V$, hence we assume that $V = U_1 \times U_2$. Since $F(G(x,y,w)) = (x,y,w)$ and $F(x,y,z) = (x,y,f(x,y,z))$, by injectivity of F, if $(x,y,f(x,y,z)) = (x,y,w)$, then $G(x,y,w) = (x,y,z)$, i.e.

$$G(x,y,w) = (x,y,g(x,y,w))$$

for some smooth map $G : U_1 \times U_2 \to \mathbb{R}$ and

$$f(x,y,g(x,y,w)) = w$$

for all $(x,y) \in U_1$ and $w \in U_2$. Now define $\sigma : U_1 \to \mathbb{R}^3$ by

$$\sigma(x,y) = (x,y,g(x,y,0)).$$

Then σ is a homeomorphism from U_1 to $S \cap \tilde{W}$. It is smooth and regular since

$$\sigma_x \times \sigma_y = (-g_x, -g_y, 1) \neq (0,0,0).$$

By doing this construction at each $p \in S$, we construct an atlas for S. □

Example 5.1. Consider the unit sphere S^2. It can be given as the zero set of the function $f(x,y,z) = x^2 + y^2 + z^2 - 1$. Then $\nabla f = (2x, 2y, 2z)$ hence $\|\nabla f\| = 2$ on S^2. Hence the previous theorem tells us that S^2 is a surface.

Example 5.2. Consider the zero set given by the function $f(x,y,z) = x^2 + y^2 - z^2$. This set is a cone. Its gradient is $\nabla f = (2x, 2y, -2z)$. At $(0,0,0)$, we have $\nabla f = 0$ and it is the only singular point, so removing the vertex, we get that the cone is a smooth.

5.2. **Quadric surfaces.**

6. First Fundamental Form

Analogous to how the local geometry of space curves is completely determined by two geometric invariants, the curvature and the torsion, we now try to determine the local geometry for surfaces. These will be the first and second fundamental form.

6.1. **Lengths of curves on surfaces.**

Recall the arc-length of a curve γ is given by

$$\int \|\dot{\gamma}(t)\| dt.$$

To calculate such an quantity, we need to know what the norm $\|\cdot\|$ is. Recall that we can obtain a norm in an inner product space by defining $\|x\|^2 = \langle x, x\rangle$. In $\mathbb{R}^n$, the inner product is usually taken to be the dot product.

Definition 6.1. Let S be a surface and $p \in S$. The **first fundamental form** of S at p associates to tangent vectors $v, w \in T_pS$ the scalar

$$\langle \mathbf{v}, \mathbf{w}\rangle_{p,S} = \mathbf{v} \cdot \mathbf{w}.$$

Hence the first fundamental form of S at p is the dot product restricted to tangent vectors to S at p.

On a surface patch σ, we can express the first fundamental form under the basis $\{\sigma_u(p), \sigma_v(p)\}$. Let du and dv be the dual vectors of σ_u, σ_v respectively, i.e. linear maps $du : T_pS \to \mathbb{R}$ and $dv : T_pS \to \mathbb{R}$ such that

$$\begin{cases} du(\sigma_u) = 1 \\ du(\sigma_v) = 0 \\ dv(\sigma_v) = 1 \\ dv(\sigma_u) = 0. \end{cases}$$

Let $\mathbf{v} = \lambda\sigma_u + \mu\sigma_v$. Then

$$\begin{aligned} \langle \mathbf{v}, \mathbf{v}\rangle &= \langle \lambda\sigma_u + \mu\sigma_v, \lambda\sigma_u + \mu\sigma_v\rangle \\ &= \lambda^2\langle \sigma_u, \sigma_u\rangle + 2\lambda\mu\langle \sigma_u, \sigma_v\rangle + \mu^2\langle \sigma_v, \sigma_v\rangle. \end{aligned}$$

Letting $E = \|\sigma_u\|^2$, $F = \langle \sigma_u, \sigma_v\rangle$ and $G = \|\sigma_v\|^2$, we have

$$\langle \mathbf{v}, \mathbf{v}\rangle = E\lambda^2 + 2F\lambda\mu + G\mu^2 = Edu(\mathbf{v})^2 + 2Fdu(\mathbf{v})dv(\mathbf{v}) + Gdv(\mathbf{v})^2.$$

We write the first fundamental for of the surface patch $\sigma(u, v)$ as

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

Some texts may write the coefficients E, F, G as $E = g_{11}$, $F = g_{12}$ and $G = g_{22}$. The reason for this is that the first fundamental form can be written as a quadratic form

$$\langle v, w\rangle = v^tAw$$

where the matrix A is given by

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}.$$

Suppose that $\tilde{\sigma}(\alpha, \beta)$ is a reparametrization of $\sigma(u, v)$, with the first fundamental form given by

$$\tilde{E}d\alpha^2 + 2\tilde{F}d\alpha d\beta + \tilde{G}d\beta^2.$$

We compute their relation. From $\sigma(u, v) = \tilde{\sigma}(\alpha(u, v), \beta(u, v))$, by chain rule,

$$\begin{aligned} \sigma_u &= \alpha_u\tilde{\sigma}_\alpha + \beta_u\tilde{\sigma}_\beta \\ \sigma_v &= \alpha_v\tilde{\sigma}_\alpha + \beta_v\tilde{\sigma}_\beta \end{aligned}$$

and so the dual vectors are given by
$$\begin{pmatrix} d\alpha \\ d\beta \end{pmatrix} = \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$
Then we see that
$$\begin{aligned} ds^2 &= \begin{pmatrix} d\alpha & d\beta \end{pmatrix} \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix} \\ &= \begin{pmatrix} du & dv \end{pmatrix} \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix}^T \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} \\ &= \begin{pmatrix} du & dv \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}. \end{aligned}$$
Using this, we are now ready to give the length of a curve γ on a surface. Let γ lie on a surface patch σ. Then
$$\gamma(t) = \sigma(u(t), v(t))$$
for some smooth functions u, v. By chain rule, we have $\dot{\gamma} = \dot{u}\sigma_u + \dot{v}\sigma_v$ and so
$$\begin{aligned} \int \|\dot{\gamma}\| dt &= \int \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle} dt \\ &= \int \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} dt. \end{aligned}$$

Example 6.1 (Plane)**.** For the plane given by
$$\sigma(u, v) = \mathbf{a} + u\mathbf{p} + v\mathbf{q},$$
with $\mathbf{p}, \mathbf{q}$ unit vectors such that $\mathbf{p} \perp \mathbf{q}$. Then $\sigma_u = \mathbf{p}$ and $\sigma_v = \mathbf{q}$ therefore $E = G = 1$ and $F = 0$, so the first fundamental form is given by
$$ds^2 = du^2 + dv^2.$$

Example 6.2 (Surface of revolution)**.** Consider the surface given by
$$\sigma(u, v) = (f(u)\cos(v), f(u)\sin(v), g(u)).$$
Further assume that $f(u) > 0$ for all u and the curve given by $u \mapsto (f(u), 0, g(u))$ is unit speed, i.e. $\dot{f}^2 + \dot{g}^2 = 1$. The basis vectors are given by
$$\sigma_u = (\dot{f}\cos(v), \dot{f}\sin(v), \dot{g})$$
and
$$\sigma_v = (-f\sin(v), f\cos(v), 0).$$
So taking the appropriate dot products, we get
$$E = 1, \quad F = 0, \quad G = f^2.$$
Hence the first fundamental form is given by
$$ds^2 = du^2 + f(u)^2 dv^2.$$

Example 6.3 (Generalized cylinder)**.** We consider a generalized cylinder given by
$$\sigma(u, v) = \gamma(u) + v\mathbf{a}.$$
where γ is unit-speed, $\mathbf{a}$ is a unit vector, and γ is contained in a plane perpendicular to $\mathbf{a}$. We have
$$\sigma_u = \dot{\gamma}, \quad \sigma_v = \mathbf{a}.$$
Therefore
$$E = G = 1, \quad F = 0,$$
and so the first fundamental form is simply
$$ds^2 = du^2 + dv^2$$

6.2. **Isometries of surfaces.**

Definition 6.2. If S_1 and S_2 are surfaces, a smooth map $f : S_1 \to S_2$ is called a **local isometry** if it takes any curve in S_1 to a curve of the same length in S_2. If a local isometry $f : S_1 \to S_2$ exists, we say that S_1 and S_2 are **locally isometric**.

To express the condition for a local isometry in a more useful form, we define the following

Definition 6.3. Let $f : S_1 \to S_2$ be a smooth map and let $p \in S_1$. For $\mathbf{v}, \mathbf{w} \in T_pS_1$, define the **pullback of** f by

$$f^*\langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_pf(\mathbf{v}), D_pf(\mathbf{w}) \rangle_{f(p)}.$$

Then $f^*\langle\ ,\ \rangle_p$ is a symmetric bilinear form on T_pS_1.

Theorem 6.1. A smooth map $f : S_1 \to S_2$ is a local isometry if and only if the symmetric bilinear forms $\langle\ ,\ \rangle_p$ and $f^*\langle\ ,\ \rangle_p$ on T_pS_1 are equal for all $p \in S_1$.

Proof. If γ_1 is a curve on S_1, the length of the part of γ_1 with endpoints $\gamma_1(t_0)$ and $\gamma_1(t_1)$ is

$$\int_{t_0}^{t_1} \langle \dot\gamma_1, \dot\gamma_1 \rangle^{1/2} dt.$$

The length of the corresponding part of the curve $\gamma_2 = f \circ \gamma_1$ on S_2 is

$$\int_{t_0}^{t_1} \langle \dot\gamma_2, \dot\gamma_2 \rangle^{1/2} dt = \int_{t_0}^{t_1} \langle Df(\dot\gamma_1), Df(\dot\gamma_1) \rangle^{1/2} dt = \int_{t_0}^{t_1} f^*\langle \dot\gamma_1, \dot\gamma_1 \rangle^{1/2} dt.$$

Conversely, suppose the lengths are the same for any curve γ. Then

$$\langle \mathbf{v}, \mathbf{v} \rangle = f^*\langle \mathbf{v}, \mathbf{v} \rangle$$

for all $\mathbf{v}$. Since they are symmetric bilinear forms, they give the same form. □

Thus f is a local isometry if and only if D_pf is an isometry for all $p \in S_1$. It follows that every local isometry is a local diffeomorphism. Let $f : S_1 \to S_2$ be a local isometry and let $p \in S_1$. If D_pf is not invertible, then there is a nonzero tangent vector $\mathbf{v} \in T_p$ such that $D_pf(v) = 0$. However $0 \neq \|\mathbf{v}\|^2 = \|D_pf(\mathbf{v})\|^2 = 0$, which is a contradiction so D_pf is invertible, hence f is a local diffeomorphism.

Corollary 6.1. A local diffeomorphism $f : S_1 \to S_2$ is a local isometry if and only if, for any surface patches σ_1 of S_1 and $f \circ \sigma_1$ of S_2 have the same first fundamental form.

From this, we see that a cylinder and a plane are locally isometric since they have the same first fundamental form. Now we consider another class of surfaces which are isometric to the plane.

Definition 6.4. A **tangent developable** is the union of the tangent lines to a curve in $\mathbb{R}^3$.

Let γ be a unit-speed curve. Then we parametrize the tangent developable as

$$\sigma(u, v) = \gamma(u) + v\dot\gamma(u).$$

Now

$$\sigma_u \times \sigma_v = v\ddot\gamma \times \dot\gamma.$$

So for σ to be regular, we require $\ddot\gamma \neq 0$, i.e. $\kappa > 0$. Under the Frenet-Serret frame, we have

$$\sigma_u \times \sigma_v = -\kappa v \mathbf{b}.$$

Proposition 6.1. Any tangent developable is locally isometric to a plane.

Proof. Let γ be unit-speed and $\kappa > 0$. Now

$$E = \|\sigma_u\|^2 = 1 + v^2\kappa^2$$
$$F = \sigma_u \cdot \sigma_v = 1$$
$$G = \|\sigma_v\|^2 = 1.$$

We know there exists a plane curve with κ hence its tangent developable which is a subset of the plane has the same first fundamental form. □

6.3. **Conformal mappings of surfaces.** Using the first fundamental form, we can express the angle of intersection between two curves γ and $\tilde{\gamma}$ as

$$\cos\theta = \frac{\langle \dot{\gamma}, \dot{\tilde{\gamma}} \rangle}{\|\dot{\gamma}\| \|\dot{\tilde{\gamma}}\|}.$$

Definition 6.5. If S_1 and S_2 are surfaces, a **conformal map** $f : S_1 \to S_2$ is a local diffeomorphism such that, if γ_1 and $\tilde{\gamma}_1$ are any two curves on S_1 that intersect, say at $p \in S_1$, and if γ_2 and $\tilde{\gamma}_2$ are their images under f, the angle of intersection of γ_1 and $\tilde{\gamma}_1$ at p is equal to the angle of intersection of γ_2 and $\tilde{\gamma}_2$ at $f(p)$.

Definition 6.6. As a special case, if $\sigma : U \to \mathbb{R}^3$ is a surface, then σ may be viewed as a map from an open subset of the plane, parametrized by (u, v) and the image S of σ, and say that σ is a **conformal parametrization** or a **conformal surface patch** of S if this map between surfaces is conformal.

Theorem 6.2. A local diffeomorphism $f : S_1 \to S_2$ is conformal if and only if there is a function $\lambda : S_1 \to \mathbb{R}$ such that

$$f^*\langle \mathbf{v}, \mathbf{w} \rangle_p = \lambda(p)\langle \mathbf{v}, \mathbf{w} \rangle_p$$

for all $p \in S_1$ and $\mathbf{v}, \mathbf{w} \in T_pS_1$.

Proof. Suppose $f : S_1 \to S_2$ is a conformal map. Then

$$\frac{\langle \dot{\gamma}, \dot{\tilde{\gamma}} \rangle}{\|\dot{\gamma}\|^{\frac{1}{2}} \|\dot{\tilde{\gamma}}\|^{\frac{1}{2}}} = \frac{f^*\langle \dot{\gamma}, \dot{\tilde{\gamma}} \rangle}{f^*\|\dot{\gamma}\|^{\frac{1}{2}} f^*\|\dot{\tilde{\gamma}}\|^{\frac{1}{2}}}$$

for all pairs of intersecting curves γ and $\tilde{\gamma}$ on S_1. Note that for any $\mathbf{v}, \mathbf{w} \in T_pS_1$ there exists such curves. Choose an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of T_pS_1 with respect to the fundamental form $\langle\ ,\ \rangle$. Let

$$\lambda = f^*\|\mathbf{v}_1\|^2, \quad \mu = f^*\langle \mathbf{v}_1, \mathbf{v}_2 \rangle, \quad \nu = f^*\|\mathbf{v}_2\|^2.$$

Applying to $\mathbf{v} = \mathbf{v}_1$ and $\mathbf{w} = \cos\theta \mathbf{v}_1 + \sin\theta \mathbf{v}_2$, we get

$$\cos\theta = \frac{\lambda\cos\theta + \mu\sin\theta}{\sqrt{\lambda(\lambda\cos^2\theta + 2\mu\sin\theta\cos\theta + \nu\sin^2\theta)}}.$$

Leting $\theta = \frac{\pi}{2}$, we get $\mu = 0$. Then

$$\lambda = \lambda\cos^2\theta + \nu\sin^2\theta, \qquad \text{for all } \theta \in \mathbb{R},$$

hence $\theta = \frac{\pi}{2}$, we get $\lambda = \nu$. This implies $f^*\langle \mathbf{v}, \mathbf{w} \rangle = \lambda\langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v}, \mathbf{w} \in T_pS_1$. The converse is immediate. ☐

Interpreting this in terms of surface path,

Corollary 6.2. A local diffeomorphism $f : S_1 \to S_2$ is conformal if and only if, for any surface patch σ of S_1, the first fundamental forms of the patches σ of S_1 and $f \circ \sigma$ of S_2 are proportional.

Example 6.4 (Stereographic projection)**.** Consider the unit sphere S^2. The stereographic projection maps points on the sphere to a plane by considering the ray from the north pole $N = (0, 0, 1)$ to a point $q = (x, y, z) \in S^2$ and extending the ray to a point on the $z = 0$ plane. It is given as the map

$$\Pi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

Its inverse gives a surface parametrization of the sphere given by

$$\sigma_1(u, v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

We can parametrize the $z = 0$ plane by $\sigma_2(u, v) = (u, v, 0)$ and the first fundamental form is given by $du^2 + dv^2$. The two parametrizations are related by

$$\Pi(\sigma_1(u, v)) = \sigma_2(u, v)$$

Computing the first fundamental form of σ_1, we compute

$$(\sigma_1)_u = \left(\frac{2(v^2-u^2+1)}{(u^2+v^2+1)^2}, \frac{-4uv}{(u^2+v^2+1)^2}, \frac{4u}{(u^2+v^2+1)^2}\right)$$
$$(\sigma_1)_v = \left(\frac{-4uv}{(u^2+v^2+1)^2}, \frac{2(u^2-v^2+1)}{(u^2+v^2+1)^2}, \frac{4u}{(u^2+v^2+1)^2}\right).$$

Taking the corresponding dot products, we get

$$E = G = \frac{4}{(u^2+v^2+1)^2}, \quad F = 0$$

hence the first fundamental forms are a (non-constant) scalar multiple of each other.

6.4. **Equiareal maps and a theorem of Archimedes.** Recall from calculus:

Definition 6.7. The **area** $A_\sigma(R)$ of the part $\sigma(R)$ of a surface patch $\sigma : U \to \mathbb{R}^3$ corresponding to a region $R \subset U$ is

$$A_\sigma(R) = \int_R \|\sigma_u \times \sigma_v\| du dv.$$

In terms of the coefficients of the first fundamental form,

Proposition 6.2. If $\sigma_u \cdot \sigma_u = E$, $\sigma_u \cdot \sigma_v = F$ and $\sigma_v \cdot \sigma_v = G$, then

$$\|\sigma_u \times \sigma_v\| = \sqrt{EG - F^2}$$

Proof. From the magnitude of cross product formula, we have

$$\|\sigma_u \times \sigma_v\|^2 = \|\sigma_u\|^2 \|\sigma_v\|^2 \sin^2\theta = EG\sin^2\theta,$$

where θ is the angle between the vectors σ_u and σ_v. We then have

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{(\sigma_u \cdot \sigma_v)^2}{\|\sigma_u\|^2\|\sigma_v\|^2} = 1 - \frac{F^2}{EG}$$

so combining the two, we get

$$\|\sigma_u \times \sigma_v\|^2 = EG(1 - \frac{F^2}{EG}) = EG - F^2.$$

$\square$

The area is well-defined by the following.

Proposition 6.3. The area of a surface patch is unchanged by reparametrization

Proof. Let $\sigma : U \to \mathbb{R}^3$ be a surface patch and let $\tilde{\sigma} : \tilde{U} \to \mathbb{R}^3$ be a reparametrization of σ. Let $\Phi : \tilde{U} \to U$ be the reparametrization map, that is

$$\tilde{\sigma}(\tilde{u}, \tilde{v}) = \sigma \circ \Phi(\tilde{u}, \tilde{v}).$$

Note that

$$\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} = \det(D\Phi)\sigma_u \times \sigma_v,$$

where $D\Phi$ is the Jacobian of Φ. Hence by change of variables

$$\int_{\tilde{R}} \|\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}\| d\tilde{u}d\tilde{v} = \int_{\tilde{R}} |\det(D\Phi)| \|\sigma_u \times \sigma_v\| d\tilde{u}d\tilde{v} = \int_R \|\sigma_u \times \sigma_v\| du dv.$$

$\square$

Definition 6.8. Let S_1 and S_2 be two surfaces. A local diffeomorphism $f : S_1 \to S_2$ is said to be **equiareal** if it takes any region in S_1 to a region of the same area in S_2.

Theorem 6.3. A local diffeomorphism $f : S_1 \to S_2$ is equiareal if and only if, for any surface patch $\sigma(u, v)$ on S_1, the coefficients of the first fundamental forms E_1, F_1, G_1 and E_2, F_2, G_2 of the patches σ_1 and $\sigma_2 = f \circ \sigma_1$, respectively, satify

$$E_1G_1 - F_1^2 = E_2G_2 - F_2^2$$

Proof. Suppose f is equiareal. For contradiction, assume without loss of generality that at some point p, $\sqrt{E_1G_1 - F_1^2} > \sqrt{E_2G_2 - F_2^2}$. Then in some sufficiently small neighborhood, say R the same inequality holds. Then for $\tilde{R} = f(R)$,

$$A_{\sigma_1}(R) = \int_R \sqrt{E_1G_1 - F_1^2} dA > \int_{\tilde{R}} \sqrt{E_2G_2 - F_2^2} d\tilde{A} = A_{\sigma_2}(\tilde{R})$$

which is a contradiction. The converse is immediate. □

6.5. **Spherical Geometry.** Consider a unit sphere $x^2 + y^2 + z^2 = 1$ sitting inside a unit cylinder $x^2 + y^2 = 1$. Map a point $p \in S^2$ to a point q on the cylinder by taking the straight projection from the z-axis to the nearest point on the cylinder. Such a map is given by

$$f(x, y, z) = \left(\frac{x}{(x^2 + y^2)^{\frac{1}{2}}}, \frac{y}{(x^2 + y^2)^{\frac{1}{2}}}, z \right).$$

Theorem 6.4 (Archimedes theorem)**.** The map f is an equiareal diffeomorphism.

Proof. Consider the parametrization of the sphere given by

$$\sigma_1(\theta, \varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$$

Applying f, we have

$$\sigma_2(\theta, \varphi) = f(\sigma_1(\theta, \varphi)) = (\cos\varphi, \sin\varphi, \sin\theta).$$

Computing the first fundamental form, we have $E_1 = 1$, $F_1 = 0$, $G_1 = \cos^2\theta$ and $E_2 = \cos^2\theta$, $F_2 = 0$, $G_2 = 1$. □

We need the following fact:

Proposition 6.4 (Geodesics on a sphere)**.** Let p and q be distinct points of S^2. If $p \neq -q$, the short great circle arc joining p and q is the unique curve of shortest length joining p and q.

Proof. By using rotational symmetry, we can assume $p = (0, 0, 1)$, and let q lie on some great circle containing p. Then the length of the segment connecting p and q is $\frac{\pi}{2} - \alpha$ for some $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$. The first fundamental form in spherical coordinate parametrization is given by $d\theta^2 + \cos^2\theta d\varphi^2$. Hence the length of a curve $\gamma(t)$ such that $\gamma(t_0) = p$ and $\gamma(t_1) = q$ is given by

$$\int_{t_0}^{t_1} \sqrt{\dot{\theta}^2 + \cos^2\theta\dot{\varphi}^2} dt.$$

The integrand is bounded below by

$$\int_{t_0}^{t_1} |\dot{\theta}| dt = \int_{\alpha}^{\frac{\pi}{2}} d\theta = \frac{\pi}{2} - \alpha.$$

when this minimum is achieved, we must have

$$\sqrt{\dot{\theta}^2 + \cos^2\theta\dot{\varphi}^2} = |\dot{\theta}|,$$

hence $\cos\theta\dot{\varphi} = 0$, hence φ is a constant, which must be zero. □

Archimedes' Theorem can be applied to obtain the following

Theorem 6.5. The area of a spherical triangle on the unit sphere S^2 with internal angles α, β, γ

$$\alpha + \beta + \gamma - \pi$$

Proof. A spherical triangle is cut out by three great circles. Two great circles intersect twice and form a lune of area 2φ where φ is the interior angle. The entire surface of the surface is covered by the three pairs of lunes, however the triangle is counted in excess twice each, and there are two triangles, both of equal area by symmetry so

$$4\alpha + 4\beta + 4\gamma - 4A = 4\pi,$$

where A is the area of the spherical triangle. □

7. Curvature of a surface

The curvature of a curve measured the difference between the curve and a straight line. We form an analogous object which will measure the difference between the surface and a plane.

7.1. **The second fundamental form.** Let σ be a surface patch for an oriented surface with standard unit normal $\mathbf{N}$. As the parameters change from (u, v) to $(u + \Delta u, v + \Delta v)$, the surface moves away from the plane through $\sigma(u, v)$ parallel to the tangent plane by a distance

$$(\sigma(u + \Delta u, v + \Delta v) - \sigma(u, v)) \cdot \mathbf{N}.$$

By Taylor expansion, we have

$$\sigma(u+\Delta u, v+\Delta v)-\sigma(u, v) = \sigma_u \Delta u+\sigma_v \Delta v+\frac{1}{2}(\sigma_{uu}(\Delta u)^2+2\sigma_{uv}\Delta u\Delta v+\sigma_{vv}(\Delta v)^2)+\text{higher order terms.}$$

Since σ_u and σ_v are tangent to the surface, it is perpendicular to $\mathbf{N}$, so the deviation of σ from its tangent plane is

$$\frac{1}{2}(L(\Delta u)^2 + 2M\Delta u\Delta v + N(\Delta v)^2) + \text{ higher order terms}$$

where

$$L = \sigma_{uu} \cdot \mathbf{N}, \quad M = \sigma_{uv} \cdot \mathbf{N}, \quad N = \sigma_{vv} \cdot \mathbf{N}.$$

Motivated by this, we have

Definition 7.1. The **second fundamental form** of the surface patch σ is given by

$$Ldu^2 + 2Mdudv + Ndv^2$$

Example 7.1. Consider the plane

$$\sigma(u, v) = \mathbf{a} + u\mathbf{p} + v\mathbf{q}$$

Since $\sigma_u = \mathbf{p}$ and $\sigma_v = \mathbf{q}$ are constant, we have $\sigma_{uu} = \sigma_{uv} = \sigma_{vv} = 0$, hence the second fundamental form is zero.

Example 7.2. Consider a surface of revolution

$$\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u));$$

assuming $f(u) > 0$ for all u and that the curve $\gamma(t) = \sigma(t, 0)$ is unit speed. We can compute the relevant quantities:

$$\sigma_u = (\dot{f} \cos v, \dot{f} \sin v, \dot{g}), \quad \sigma_v = (-f \sin v, f \cos v, 0)$$

$$\sigma_{uu} = (\ddot{f} \cos v, \ddot{f} \sin v,\ddot{g}), \quad \sigma_{uv} = (-\dot{f} \sin v, \dot{f} \cos v, 0), \quad \sigma_{vv} = (-f \cos v, -f \sin v, 0),$$

$$\mathbf{N} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = (-\dot{g} \cos v, -\dot{g} \sin v, \dot{f})$$

and so the coefficients of the second fundamental form are

$$L = \dot{f}\ddot{g} - \ddot{f}\dot{g}, \quad M = 0, \quad N = f\dot{g}.$$

For the unit sphere S^2 in latitude-longitude coordinates, $u = \theta,\ v = \varphi,\ f(\theta) = \cos\theta,\ g(\theta) = \sin\theta$. Then the second fundamental form is

$$d\theta^2 + \cos^2\theta d\varphi^2.$$

7.2. **The Gauss and Weingarten maps.** We give an alternative approach to curvature by giving an equivalent definition of the second fundamental form. In this approach, we investigate how the normal vector varies across the surface.

Definition 7.2. The **Gauss map** of an oriented surface S is the map $G : S \to S^2$ such that $G(p) = \mathbf{N}(p)$, where $\mathbf{N}(p)$ is the unit normal at p.

We can take its derivative as a map between surfaces so that

$$D_pG : T_pS \to T_{G(p)}S^2$$

Since the tangent plane at $G(p)$ is the plane with normal vector $\mathbf{N}(p)$, we can identify the tangent spaces of S and S^2 hence

$$D_pG : T_pS \to T_pS.$$

Definition 7.3. Let $p \in S$. The **Weingarten map** $W_{p,S}$ of S at p is defined by

$$W_{p,S} = -D_pG.$$

The **second fundamental form** of S at p is the bilinear form on T_pS given by

$$II(\mathbf{v}, \mathbf{w}) := \langle W_{p,S}(\mathbf{v}), \mathbf{w}\rangle, \quad \mathbf{v}, \mathbf{w} \in T_pS.$$

We need to show that this definition is equivalent to the one given in the previous section.

Lemma 7.1. Let $\sigma(u, v)$ be a surface patch with standard unit normal $\mathbf{N}(u, v)$. Then

$$\mathbf{N}_u \cdot \sigma_u = -L, \quad \mathbf{N}_u \cdot \sigma_v = \mathbf{N}_v \cdot \sigma_u = -M, \quad \mathbf{N}_v \cdot \sigma_v = -N$$

Proof. Since σ_u and σ_v are tangent to the surface patch, we have

$$\mathbf{N} \cdot \sigma_u = \mathbf{N} \cdot \sigma_v = 0$$

Differentiating the above with respect to u and v give us the identities. □

Proposition 7.1. Let p be a point of a surface S, let $\sigma(u, v)$ be a surface patch of S with p in its image. Then, for any $\mathbf{v}, \mathbf{w} \in T_pS$,

$$II(\mathbf{v}, \mathbf{w}) = Ldu(\mathbf{v})du(\mathbf{w}) + M(du(\mathbf{v})dv(\mathbf{w}) + du(\mathbf{w})dv(\mathbf{v})) + Ndv(\mathbf{v})dv(\mathbf{w}).$$

Proof. By linearity, it suffices to show for $\mathbf{v} = \sigma_u$ and $\mathbf{w} = \sigma_v$. Let $\sigma(u_0, v_0) = p$. Then, with the derivatives evaluated at (u_0, v_0),

$$W(\sigma_u) = -D_pG(\sigma_u) = -\frac{d}{du}|_{u=u_0} G(\sigma(u, v_0)) = -\frac{d}{du}_{u=u_0} \mathbf{N}(u, v_0) = -\mathbf{N}_u.$$

Hence

$$II(\sigma_u, \sigma_u) = \langle W(\sigma_u), \sigma_u\rangle = -\mathbf{N}_u \cdot \sigma_u = L,$$

and similar computations hold for the other cases. □

Corollary 7.1. The second fundamental form is a symmetric bilinear form. Equivalently, the Weingarten map is self-adjoint.

7.3. **Normal and geodesic curvatures.** On a surface, we can analyze the curvature of curves on the surface. Consider curves on a cylinder versus curves on a sphere. It is possible for a curve to be straight (in the ambient space $\mathbb{R}^3$ on a cylinder, but it is not possible to have such a curve on a sphere.

Let γ be a unit speed curve on an oriented surface S. Since $\dot\gamma$ is perpendicular to $\mathbf{N}$, we have an orthogonal basis $\{\dot\gamma, \mathbf{N}, \mathbf{N} \times \dot\gamma\}$. Since $\ddot\gamma$ is perpendicular to $\dot\gamma$, we can write this as a linear combination of $\mathbf{N}$ and $\mathbf{N} \times \dot\gamma$:

$$\ddot\gamma = \kappa_n\mathbf{N} + \kappa_g\mathbf{N} \times \dot\gamma.$$

Definition 7.4. κ_n is called the **normal curvature** and κ_g is called the **geodesic curvature**.

The following can be directly verified.

Proposition 7.2. With the above notation, we have

$$\begin{aligned}\kappa_n =& \ddot{\gamma}\cdot\mathbf{N}, \quad \kappa_g = \ddot{\gamma}\cdot(\mathbf{N}\times\dot{\gamma}),\\ \kappa^2 =& \kappa_n^2+\kappa_g^2\\ \kappa_n =& \kappa\cos\psi, \quad \kappa_g = \pm\kappa\sin\psi\end{aligned}$$

where κ is the curvature of γ and ψ is the angle between $\mathbf{N}$ and the principal normal $\mathbf{n}$ of γ.

The normal curvature and the second fundamental form are related by the following

Proposition 7.3. If γ is a unit-speed curve on an oriented surface S, its normal curvature is given by

$$\kappa_n = II(\dot{\gamma},\dot{\gamma})$$

This result means that if two curves touch each other at a point p on a surface S, then they have the same normal curvature at the point p.

Proof. Since $\dot{\gamma}$ is a tangent vector to S, $\mathbf{N}\cdot\dot{\gamma}=0$. Hence, by product rule, $\mathbf{N}\cdot\ddot{\gamma} = -\dot{\mathbf{N}}\cdot\dot{\gamma}$ so

$$\kappa_N = \mathbf{N}\cdot\ddot{\gamma} = -\dot{\mathbf{N}}\cdot\gamma = \langle W(\dot{\gamma}),\dot{\gamma}\rangle = II(\dot{\gamma},\dot{\gamma}).$$

□

Proposition 7.4 (Meusnier's Theorem)**.** Let p be a point of a surface S and let $\mathbf{v}$ be a unit tangent vector to S at p. Let Π_θ be the plane containing the line through p parallel to $\mathbf{v}$ and making an angle θ with the tangent plane T_pS, and assume that Π_θ is not parallel to T_pS. Suppose that Π_θ intersects S in a curve with curvature κ_θ.

Proof. Assume γ_θ is a unit-speed parametrization of the curve of intersection of Π_θ and S. Then at p, $\dot{\gamma}_\theta = \pm\mathbf{v}$, so $\ddot{\gamma}_\theta$ is perpendicular to $\mathbf{v}$ and is parallel to Π_θ (since it is a plane curve of Π_θ. Hence $\psi = \frac{\pi}{2}-\theta$ in Proposition 7.2. □

7.4. **Parallel transport and covariant derivative.** We now wish to take derivatives of vectors in a tangent plane, however we must be careful that the derivative vector is also an element of the same tangent plane.

Definition 7.5. Let γ be a curve on a surface S and let $\mathbf{v}$ be a tangent vector field along γ. The **covariant derivative** of $\mathbf{v}$ along γ is the orthogonal projection $\nabla_\gamma\mathbf{v}$ of $\frac{d\mathbf{v}}{dt}$ onto the tangent plane $T_{\gamma(t)}S$ at a point $\gamma(t)$, i.e.,

$$\nabla_\gamma\mathbf{v} = \dot{\mathbf{v}} - (\dot{\mathbf{v}}\cdot\mathbf{N})\mathbf{N},$$

where $\mathbf{N}$ is a unit normal to σ.

8. Gaussian curvature

Definition 8.1. Let W be the Weingarten map of an oriented surface S at a point $p\in S$. The **Gaussian curvature** K of S at p is defined by

$$K = \det(W).$$

Proposition 8.1. Let σ be a surface patch of an oriented surface S. Then, with the above notation, the matrix of W with respect to the basis $\{\sigma_u,\sigma_v\}$ of T_pS is

$$\begin{pmatrix} E & F\\ F & G\end{pmatrix}^{-1}\begin{pmatrix} L & M\\ M & N\end{pmatrix}$$

Corollary 8.1.

$$K = \frac{LN-M^2}{EG-F^2}$$

Appendix A. Real Analysis

Suppose we are given the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = 1 + x + 2x^2 + 3xy + 4xy^2 + 5y^4.$$

By using only differentiation and evaluating at a point, say $(0,0)$, how can we obtain the coefficients from each term? This is the motivation in defining a Taylor series of a smooth function. In this example, we see that

$$f(0,0) = 0, \quad \frac{\partial f}{\partial x}(0,0) = 1, \quad \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0) = 2$$
$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 3, \quad \frac{1}{2}\frac{\partial^3 f}{\partial x \partial y^2}(0,0) = 4, \quad \frac{1}{4!}\frac{\partial^4 f}{\partial y^4}(0,0) = 5.$$

Writing this in general, we obtain

Definition A.1 (Taylor Expansion)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Its **Taylor Expansion** is given by

$$f(x,y) \sim \sum_{\alpha,\beta=0}^{\infty} \frac{1}{\alpha!\beta!}\frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial y^\beta} x^\alpha y^\beta.$$

Note that the function is not necessarily equal to its Taylor series, however is a useful approximation. Another theorem we use often is

Theorem A.1 (Inverse Function Theorem)**.** Let $f : U \to \mathbb{R}^n$ be a smooth map defined on an open subset U of $\mathbb{R}^n$. Assume that at the point $x_0 \in U$, the Jacobian matrix $J(f)$ is invertible. Then, there is an open subset V of $\mathbb{R}^n$ and a smooth map $g : V \to \mathbb{R}^n$ such that

1. $y_0 = f(x_0) \in V$
2. $g(y_0) = x_0$
3. $g(V) \subset U$
4. $g(V)$ is an open subset of $\mathbb{R}^n$
5. $f(g(y)) = y$ for all $y \in V$.

In particular, $f : g(V) \to V$ and $g = f^{-1} : V \to g(V)$.

Theorem A.2 (Change of variables)**.** Suppose that Φ is a continuously differentiable transformation such that $\det D\Phi \neq 0$ and maps the region $\tilde{R}$ to R, i.e., $\Phi(\tilde{R}) = R$. Suppose that f is continuous on R.

$$\int_{\tilde{R}} f(\tilde{u},\tilde{v}) |\det D\Phi| d\tilde{u} d\tilde{v} = \int_R f(u,v) du dv$$

Appendix B. Topology

For a general topological space, we have the following definition of a continuous function.

Definition B.1. Let X and Y be two topological spaces. A function $f : X \to Y$ is continuous if given an open set $V \subset Y$, the preimage

$$f^{-1}(V) = \{x \in X \mid f(x) = y \text{ for } y \in V\}$$

is open.

If we specialize to metric spaces, the usual $\epsilon - \delta$ definition is equivalent.