MATH 147A FINAL

Choose 8 of the 10. Each problem is worth 10 points for a total of 80 points. No extra credit for doing more problems. Please circle your choice of 8.

(1) Given a surface patch σ, show that
$$\|\sigma_u \times \sigma_v\|^2 = EG - F^2.$$
Conclude that $EG - F^2 \geq 0$. When is $EG = F^2$?

(2) Show that the Gaussian curvature of the surface $z = f(x, y)$, where f is a smooth function is
$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

(3) Prove that a smooth map $f : S_1 \to S_2$ is a local isometry **if and only if** f is conformal and equiareal.

(4) Let γ be a unit speed curve on an oriented surface S. Show that
$$II(\dot{\gamma}, \dot{\gamma}) = \kappa_n$$
where κ_n is the normal curvature of γ.

(5) Compute the Gauss curvature of the elliptic paraboloid
$$\sigma(u, v) = (u, v, u^2 + v^2),$$
then determine the area of the part with $z \leq 1$.

(6) Show that the normal curvature of **any** curve on a sphere of radius R is $\pm\frac{1}{R}$.

(7) Compute the Gauss curvature of *Enneper's surface* given by the surface patch
$$\sigma(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right).$$

(8) Show that the Archimedes map is equiareal. The Archimedes map is the map from the unit sphere to the unit cylinder such that it maps the point on the sphere inside the cylinder to the closest point on the cylinder and parallel to the z-axis.

(9) Show that the stereographic projection is a conformal map. You will need to compute the stereographic projection and the appropriate surface patches yourself.

(10) (This problem will be graded out of 15 points, however your total exam score will not exceed 80. Also, this problem is not particularly difficult, just a bit long) The **Gauss-Bonnet Theorem** states for S a smooth orientable *closed* surface,
$$\int_S K dA = 2\pi\chi(S),$$
where K is the Gaussian curvature and $\chi(S)$ is the *Euler characteristic*. In particular, a sphere and a torus are closed surfaces. We computed the Euler characteristic for a sphere in the homework as $\chi(S^2) = 2$. Verify the Gauss-Bonnet formula for the sphere. Next using the parametrization for a torus given by
$$\sigma(u, v) = (b + r\cos u)\cos v, (b + r\cos u)\sin v, r\sin u)$$
with $b > r$ and $0 < u, v < 2\pi$, use the Gauss-Bonnet theorem to compute the Euler characteristic of the torus $\chi(T)$.

Date: June 9, 2016.