MATH CLUB: IDENTIFICATION SPACE

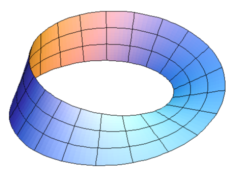
1. Möbius Strip

The **Möbius Strip** is a surface with only one side. What does this mean? For two-sided surfaces, to get to the other side, you would need to go off an edge, but for the Möbius strip, you can cover "both" sides by going in a straight line.

Making one in real life is easy! Start with a strip of paper (longer the better) and wrap it into a cylinder. Now instead of taping the two ends together, give one of the ends a half twist and then tape them together.

Exercise 1.1. Follow the above steps and make a Möbius strip

If you did it correctly, it should end up looking like this:



Now lets see if we can cover both "sides" of the strip by one straight line.

Exercise 1.2. Pick any point near the center of your strip and start drawing a straight line until you end up where you started.

If you cut a cylinder in half, you will end up with two cylinders with half the width. What do you think will happen if you cut the Möbius strip in half?

Exercise 1.3. Carefully take a pair of scissors and cut along the line that you just drew. What did you expect? What do you end up with?

The natural question you might ask now is "Why did I end up with just one circle?" Cutting along the interior is similar to carving out the interior and leaving just the border. To give you a visual aid:

Another thing you might notice is that the object you have has some twists.

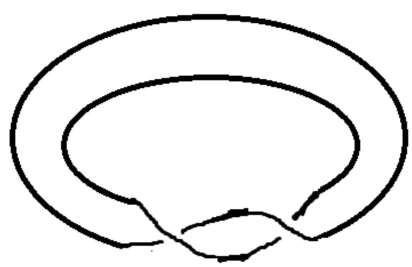
Exercise 1.4. Try to think of a way to count the number of twists your strip has. Remember that the Möbius strip has 1 twist.

Now lets do some similar analysis on this new object that we have. You can work with what you have or you can create a new one with more width.

Exercise 1.5. Draw a straight line across the center. Were you able to get both sides with one line?

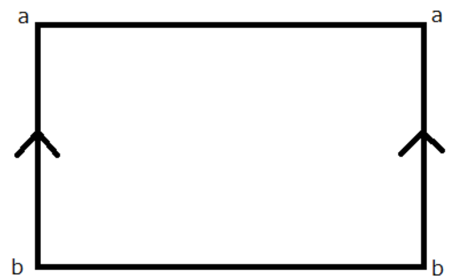
Exercise 1.6. Cut along the line that you drew. What do you think will happen?

Exercise 1.7. Try to think of a way to explain why you ended up with the object that you have.



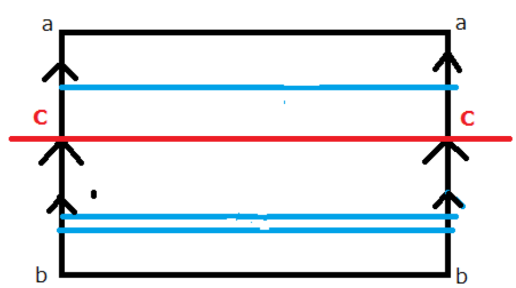
2. Identification Space

Sometimes it can be quite difficult to figure out how many twists your strip has. A tool that mathematicians (in particular *topologists*) use is whats called an identification space. Start by drawing a rectangle. Now we want to say that some edges are the same and that they also have a direction. For example, the identification space of a cylinder is



Exercise 2.1. What should the identification space for a Möbius strip look like?

How can we use the identification space to tell us information about the object? Let's begin with the identification space for a **cylinder**. Cutting a cylinder in half in the identification space is just drawing a horizontal line across. Label the new vertices created by the line, call it c. Start at a point between the vertices a and c, and begin drawing a horizontal line. You should end up at the other end, also between a and c. Since we are "identifying" the two edges, and hence the vertices, as the same point, we are right back to where we started from, thus creating a loop. We can do this for the region between c and b. So we get two loops, untwisted since it goes from one direction arrow to the same direction arrow and they are disjoint. But you already knew that.



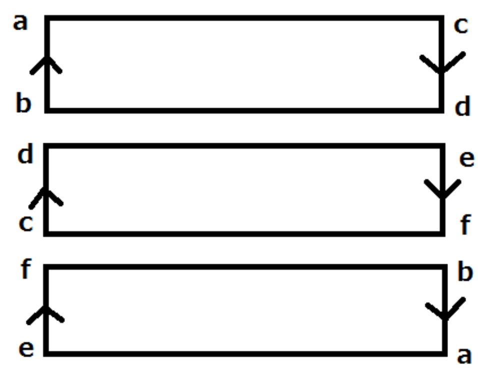
Exercise 2.2. Use the identification space of a Möbius strip to deduce that when you cut the strip in half, you end up with only one loop with two twists.

Now lets try it for the Möbius strip with two twists.

Exercise 2.3. Think of the identification space for a Möbius strip with two twists. Hint: You already drew one.

Exercise 2.4. Explain what happens when you cut the twice twisted Möbius strip down the middle using identification space.

We can keep going. The Möbius strip with 3 twists looks something like this



We have been cutting the strip in halves. What will happen when you cut it in **thirds**? This time lets use our identification space approach to see if we can figure out what will happen before we actually cut.

Exercise 2.5. The identification space for such a cut would look something like this

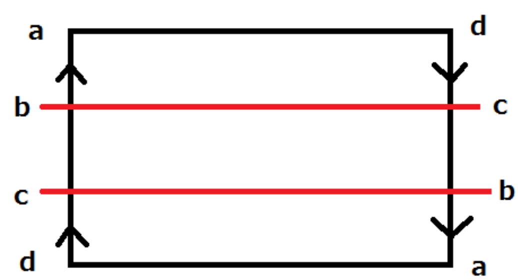


Figure out what happens when you cut it in thirds. Then actually cut away!

3. Activities

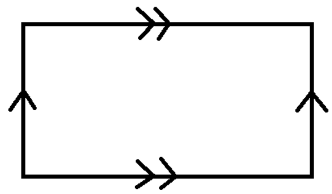
Exercise 3.1. What happens when you cut it in fifths?

Exercise 3.2. Pick your favorite number $n > 2$ and find out what would happen when you cut it in half.

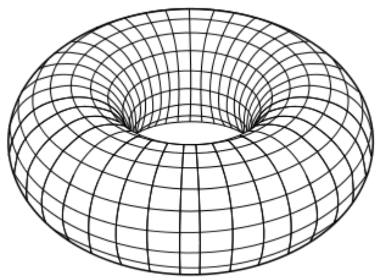
Exercise 3.3. Make a guess as to what happens when you cut a strip with n twists. Do you get one piece or two piece? Is there a formula for the number of twists you end up with? Are they linked together?

4. More Surfaces

We can do some more things with the identification space. Identify both top and bottom edges and both left and right edges:



If you imagine this folding up, you get a torus



Exercise 4.1. Can you think of the identification space for a torus with 2 holes? (these are called *genus 2* tori)

It gets crazier. First make the identification space of a cylinder, but identify the remaining edges as a mobius strip. This is called the *Klein Bottle*