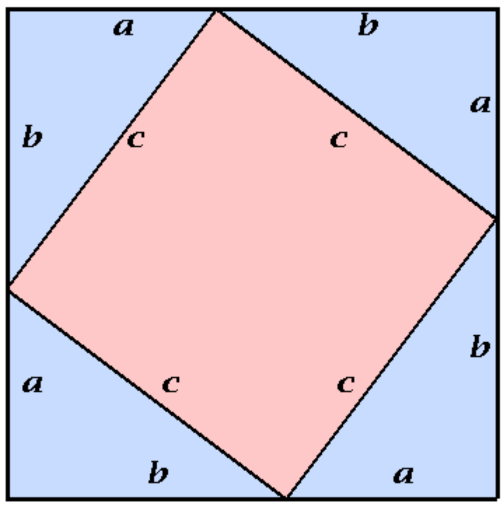
MATH CIRCLE: SUMS OF POWERS

SHOO SETO

1. Introduction

Adding up (sum) quantities which are exponentiated (powers) occur in many contexts. For instance, the Pythagorean theorem asserts that if a and b are sides to a right triangle with hypotenuse c, then

$$a^2 + b^2 = c^2.$$

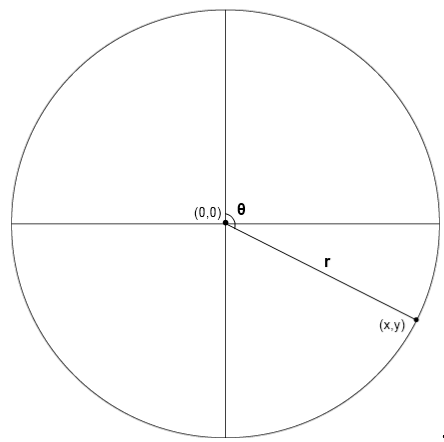


Exercise 1.1. Compute the area of the 4 blue triangles in two ways.

Sums of powers can also be used to write down equations of some familiar shapes, for instance, the circle is defined to be all the points of a fixed distance from the center. In coordinates, if we place the center at $(0,0)$, then the distance between a point (x,y) and the origin is given by

$$\sqrt{(x-0)^2+(y-0)^2} = R$$

So for fixed R, we have



.

What about for powers besides 2? That is, what would the graph of

$$|x|^p + |y|^p = 1$$

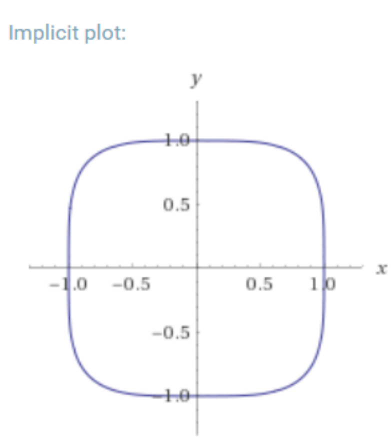
look like for different values of p?

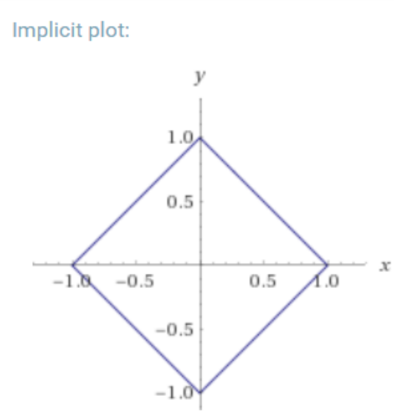
Exercise 1.2. Match the equations with the graph

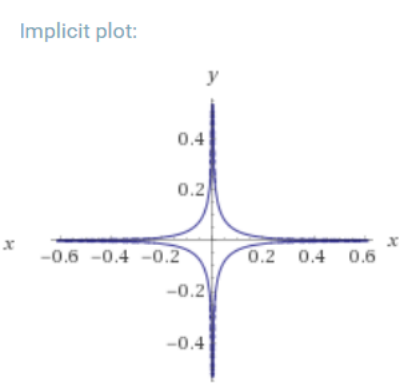
(1) $x^4 + y^4 = 1$

(2) $|x|^{\frac{1}{4}} + |y|^{\frac{1}{4}} = 1$

(3) $|x| + |y| = 1$.







Fun fact: shapes for $p = 4$ are called **squircles** and for $p > 2$ are more generally called **superellipse**.

Exercise 1.3. What is happening when p is getting larger and larger? What about when p is getting closer and closer to 0? What would you guess would happen when $p < 0$?

Some of mathematic's most difficult questions involve equations with sums of powers of integers. The name for such an equation is called a **Diophantine Equation**. Here are two of many problems that were (relatively) recently solved.

- Fermat's Last Theorem : $x^k + y^k = z^k$ is impossible for positive integers x, y, z for $k > 2$. Fermat (around 1637) claimed to have a "truely marvelous proof" of this fact but did not have enough room in the margins of a book he was reading at that time. Andrew Wiles, using math developed by many others throughout the years, gave a 129 page proof a bit later (September 1994)

- Sums of three cubes : integers solutions to $x^3 + y^3 + z^3 = n$. For $n = 1, 2$ there are infinitely many because

$$(9b^4)^3 + (3b - 9b^4)^3 + (1 - 9b^3)^3 = 1$$
$$(1 + 6c^3)^3 + (1 - 6c^3)^3 + (-6c^2)^3 = 2.$$

However, for $n = 3$, only two examples are known. See exercise. For higher numbers, there are some numbers that are impossible (4 and 5 mod 9) and until recently (March 11, 2019), it was unknown for 33. The combination for 33 was discovered by Andrew Booker and it is

$$(8866128975287528)^3 + (-8778405442862239)^3 + (-2736111468807040)^3 = 33.$$

As a side note, apparently it took a supercompute (64-core) at the University of Bristol three weeks of computing to get this. Also $n = 42$ is still open.

Exercise 1.4. Try to find the two (up to permutation of (x, y, z)) known examples of $x^3 + y^3 + z^3 = 3$. One of them is easy, the other involves the numbers ± 4 and ± 5 (not all of them). Using $4^3 = 64$ and $5^3 = 125$, try to guess the correct combination.

2. Main Activities

Today we will do something a bit more elementary.

2.1. Faulhaber's Formula.

Faulhaber's formula involves closed forms of the following sum:

$$1^p + 2^p + 3^p + \cdots + n^p.$$

Note that the formula depends on two quantities, p and n. There is a famous story of Gauss, as a child, was told to add up the numbers 1 to 100, so $n = 100$ and $p = 1$ in the above formula. He noticed that he could double the sum and regroup in pairs:

$$\begin{aligned}&1 \quad + 2 \quad + 3 \quad + 4 \; + \cdots + 97 + 98 + 99 + 100 + \\ &100 + 99 + 98 + 97 + \cdots + 4 \quad + 3 \;\; + 2 \quad + 1\end{aligned}$$

So there are 100 pairs that add up to 101 each. This is twice the original sum so,

$$1 + 2 + 3 + \cdots + 99 + 100 = \frac{100 * 101}{2} = 5050.$$

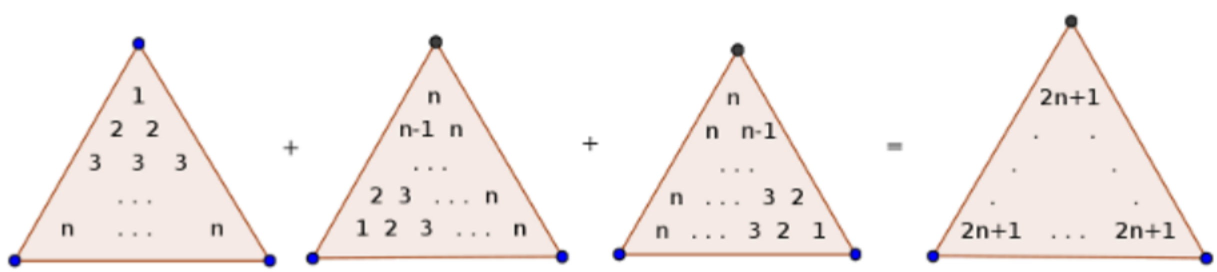
Exercise 2.1. There was nothing special about $n = 100$. Try to work out a formula for any n. That is,

$$1 + 2 + 3 + \cdots + (n-2) + (n-1) + n = ?$$

How about the next natural step which would be $p = 2$? This would be the sum of squares of consecutive integers.

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + (n-1)^2 + n^2 = ?$$

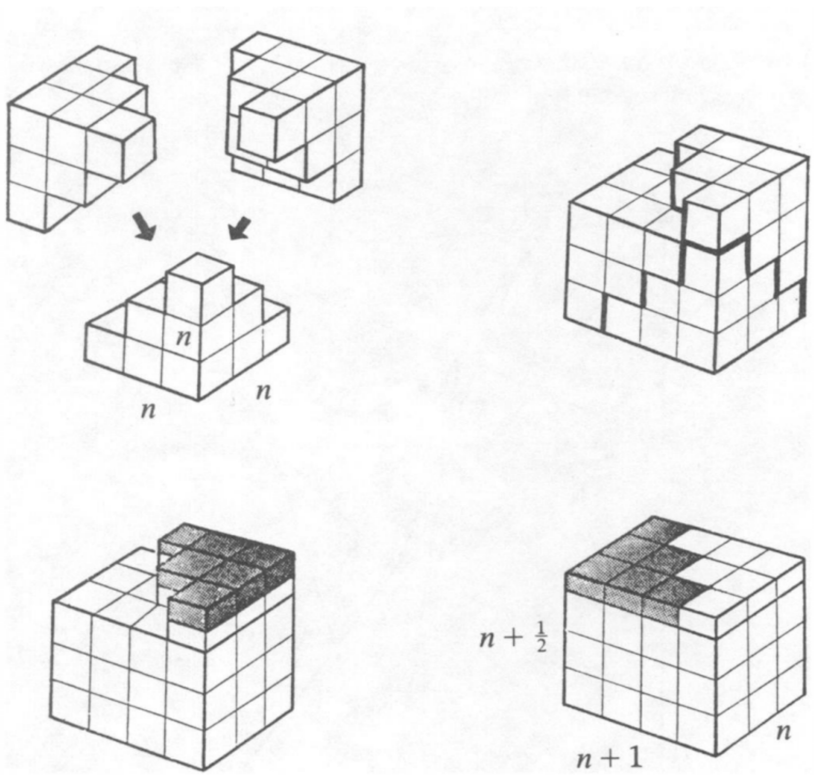
We can try finding a nice formula like the $p = 1$ case. Consider the following:



Here the left hand side is 3 times the sum of the squares up to n. Consider plugging in small values of n, like $n = 3$ or $n = 4$ to convince yourself that the picture is true.

Exercise 2.2. Derive a closed formula for the sum of squares and check that it works for some small values of n. Hint: How many $2n + 1$'s are there in the right triangle? Is there a nice formula to compute it quickly?

Here is another "proof without words" of the formula:

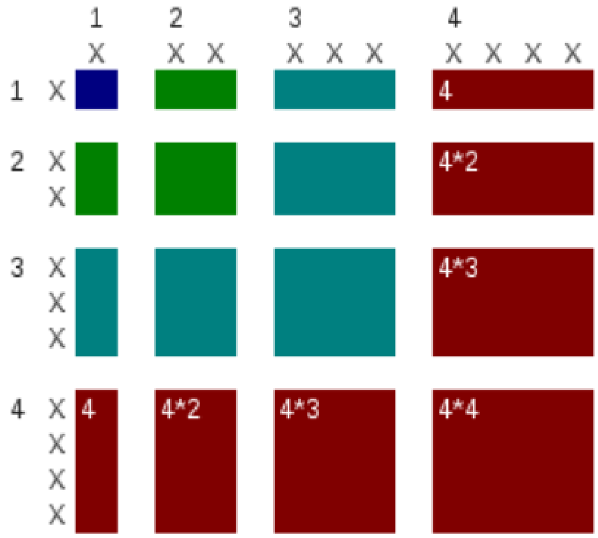


—MAN-KEUNG SIU
University of Hong Kong

Let's keep going, the next natural step is $p = 3$, i.e. the sum of cubes

$$1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 = ?$$

For this, consider this geometric picture:



You can visualize this as a cube getting chopped up into layers and getting spread out on a plane.

Exercise 2.3. Find a closed form for $p = 3$.

What about for $p \geq 4$? Visualizing a 4-dimensional "square" or "cube" or perhaps it should be called a "quart" since 4th degree polynomials are called quartic, are a little harder to visualize. However, we can still figure out the formula if we know the previous ones! Consider the sum

$$\sum_{k=1}^{n} (1+k)^5 - k^5.$$

Here we are using *sigma notation* which means add up the numbers with k ranging from $k = 1$ to $k = n$. That is,

$$\begin{aligned}\sum_{k=1}^{n} (1+k)^5 - k^5 &= ((2)^5 - 1^5) + ((3)^5 - 2^5) + \cdots + ((n^5) - (n-1)^5) + ((1+n)^5 - n^5) \\ &= (n+1)^5 - 1^5 \\ &= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n.\end{aligned}$$

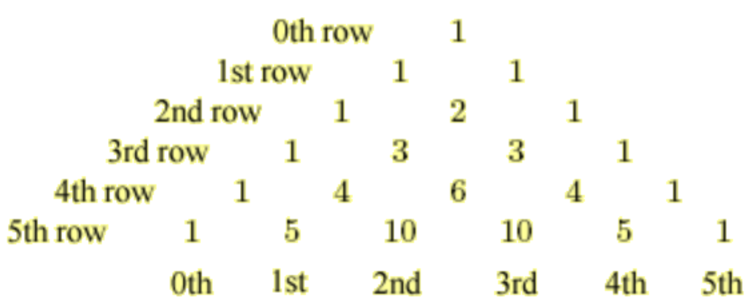
Exercise 2.4. Sum up the series in a different way to derive the formula for $\sum_{k=1}^{n} k^4$.

Now the pattern is somewhat apparent as to how one would compute for all values p. The caveat here is that we need to know the formula for the lower p's first. Also if we do it this way, we would also need a way to quickly computer $(n+1)^p$. While we won't be able to present the formula for general p here, we can at least discuss the expansion of $(n+1)^p$.

2.2. **Binomial Theorem.** The **binomial expansion** is the following

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ is called the **binomial coefficient** and the exclamation mark is the **factorial** given by $n! = 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$. The binomial coefficient computes the number of ways to *choose* k things from a set of n things. Now consider the following **Pascal's triangle**



The pattern is the number is the sum of the two numbers on top of the number. What the number also represents is *How many times while going down to the n-th row did you go left?*. For instance, to get to the 5th row, 2nd number, while traversing down, we had to pick going left exactly twice. Hence the number is $\binom{5}{2}$. Going back to the binomial expansion, let $a = x$, $b = 1$ and $n = 5$. Then

$$(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1.$$

2.3. **Geometric Series.** (If time permits)
Compute out the following:

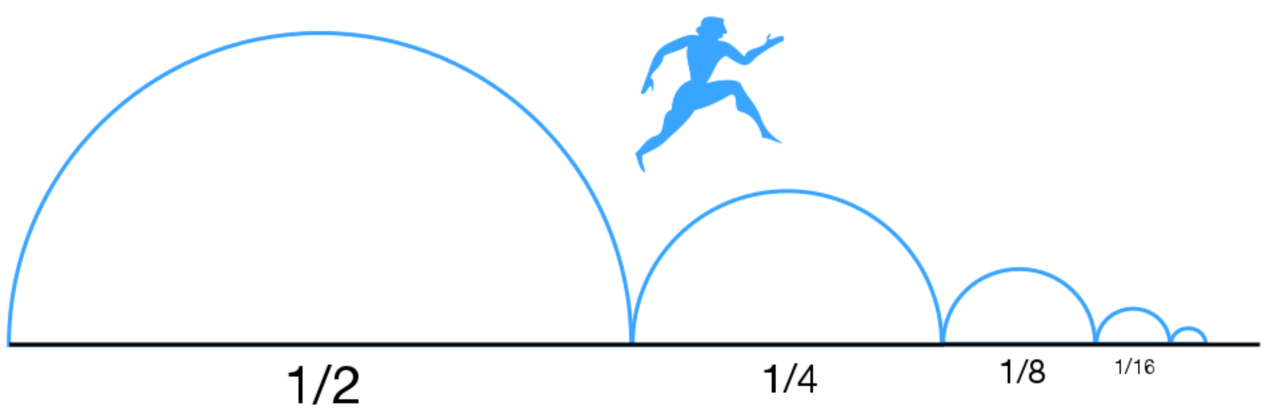
$$(1-x)(1+x)$$
$$(1-x)(1+x+x^2)$$
$$(1-x)(1+x+x^2+x^3)$$
$$(1-x)(1+x+x^2+x^4+x^5)$$

notice the pattern? What would $(1-x)(1+x+x^2+x^3+\cdots+x^{n-1}+x^n)$ be?

Exercise 2.5. Find a nicer way to express

$$\sum_{k=0}^{n} x^k = 1+x+x^2+x^3+\cdots+x^{n-1}+x^n.$$

Then think of what would happen if $|x|<1$, say $x=\frac{1}{2}$ and n gets large.



Bonus Exercise: Prove all the formulas using induction.

3. Further generalities

Consider the following "argument": Let

$$S = 2 + 4 + 8 + 16 + 32 + \cdots = \sum_{k=1}^{\infty} 2^k.$$

Then

$$\begin{aligned} 2S &= 4 + 8 + 16 + 32 + \cdots \\ &= -2 + 2 + 4 + 8 + 16 + 32 + \cdots \\ &= -2 + S. \end{aligned}$$

Therefore, $S = -2$. Is there anything wrong with this argument? Tangentially related, there

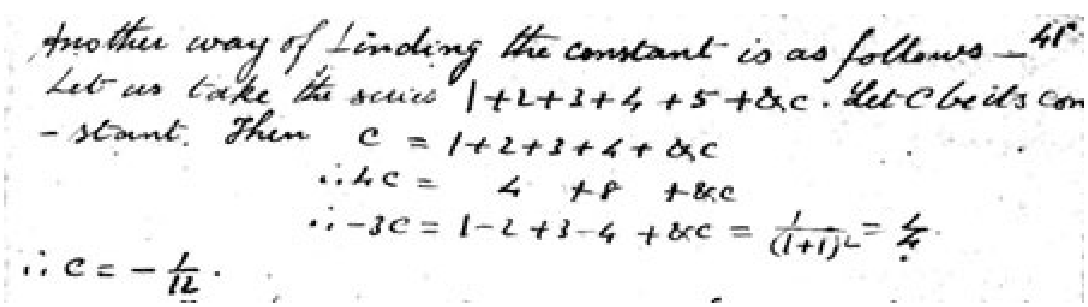
are some pretty exotic ways to interpret infinite sums.

Definition 3.1. The **Riemann zeta function** is defined as the sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Using tools from calculus, we can show $\zeta(2) = \frac{\pi^2}{6}$. This is sometimes called the Basel problem. Another interesting result is the value of $\zeta(-1) = -\frac{1}{12}$.

Exercise 3.2. Try to make sense of the following page from Srinivasa Ramanjun's notebook:


Another way of finding the constant is as follows — 48
Let us take the series 1+2+3+4+5+&c. Let C be its constant. Then C = 1+2+3+4+&c
∴ 4C = 4 + 8 + &c
∴ −3C = 1−2+3−4+&c = $\frac{1}{(1+1)^2}$ = $\frac{1}{4}$
∴ C = $-\frac{1}{12}$.

Hint: Try squaring the geometric series