Lecture Note for Math 220A
Complex Analysis of One Variable

Song-Ying Li

University of California, Irvine

Contents

List of Figures

Acknowledgement: I would like to thank J. N. Treuer for reading through the first draft and making some revisions.

I. Holomorphic and harmonic functions

1 Complex numbers and geometry

1.1 Complex number field

In the field of the real numbers $\mathbb{R}$, we know that $x^2 + 1$ has no roots. It would be ideal if every polynomial of degree n had n roots. To remedy this situation, we introduce an imaginary number $i = \sqrt{-1}$ ($i^2 = 1$). Then the polynomial $z^2 + 1$ has two roots: $\pm i$.

Viewing a point (x, y) in xy-plane, we introduce a corresponding complex number

$$z =: x + iy, \quad \mathbb{C} = \{z =: x + iy : x, y \in \mathbb{R}\}. \tag{1.1}$$

Here x is the real part of z which is denoted by $\operatorname{Re} z$ and y is the imaginary part of z which is denoted by $\operatorname{Im} z$.

• Two operations on $\mathbb{C}$.

For $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}$, we define two operations:

- Addition:

$$z_1 + z_2 =: (x_1 + x_2) + i(y_1 + y_2) \tag{1.2}$$

- Multiplication:

$$z_1 \cdot z_2 =: (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \tag{1.3}$$

It is easy to verify the following proposition.

Proposition 1.1 $(\mathbb{C}; +, \cdot)$ *forms a number field.*

It is called the complex number field.

Question: Is $(\mathbb{C}; +, \cdot)$ an order field?

For each $z = x + iy \in \mathbb{C}$, the complex conjugate $\overline{z}$ of z and the norm $|z|$ of z are defined as follows:

$$\overline{z} = x - iy \quad \text{and} \quad |z|^2 = \overline{z} \cdot z = x^2 + y^2. \tag{1.4}$$

Then

i. The additive inverse of z is

$$-z = -x + i(-y)$$

ii. When $z \neq 0$, the multiplicative inverse of z is

$$\frac{1}{z} =: \frac{\overline{z}}{|z|^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

iii. $|z|^2 = z \cdot \overline{z} = x^2 + y^2$;

iv. $x =: \operatorname{Re} z = \frac{1}{2}(z + \overline{z}), \quad y =: \operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$.

EXAMPLE 1 *What is the complex number* $\frac{1}{2-3i}$*?*

Solution.

$$\frac{1}{2-3i} = \frac{2+3i}{2^2+3^2} = \frac{2}{13} + i\frac{3}{13}.$$

More generally, one can compute $1/z$ as follows:

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{x}{|z|^2} - i\frac{y}{|z|^2}. \tag{1.5}$$

1.2 Geometry of the complex numbers

1.2.1 Euler's Formula

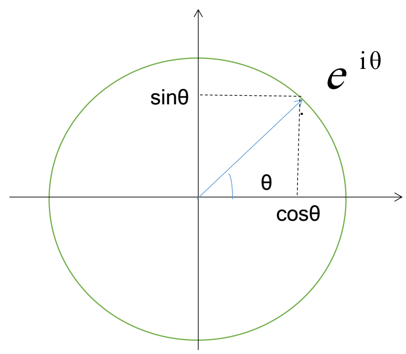
Let $|z|$ be the distance from $z = x + iy$ to $0 = 0 + i0$, let θ be the angle between the line segment $[0, z]$ and the x-axis. Then

$$x = |z|\cos\theta, \quad y = |z|\sin\theta, \quad z = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$$

where

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.6}$$

is Euler's formula.



One can prove Euler's formula by noting that

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

• **Fact:** Given two complex numbers $z = |z|e^{i\theta}$ and $w = |w|e^{i\varphi}$, one has

$$zw = |z||w|e^{i\theta}e^{i\varphi} = |z||w|e^{i(\theta+\varphi)} \quad \text{and} \quad |z\,w| = |z||w|. \tag{1.7}$$

Proposition 1.2 (Triangle Inequality) *If $z, w \in \mathbb{C}$, then*

$$|z + w| \leq |z| + |w|.$$

Proof.

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z}+\overline{w}) \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + |w|^2 + (z\overline{w} + \overline{z\overline{w}}) \\ &= |z|^2 + |w|^2 + 2\mathrm{Re}\,(z\bar{w}) \\ &\leq |z|^2 + |w|^2 + 2|z\bar{w}| \\ &= |z|^2 + |w|^2 + 2|z\bar{w}| \\ &= |z|^2 + |w|^2 + 2|z||w| \\ &= (|z| + |w|)^2 \end{aligned}$$

This implies that

$$|z+w| \leq |z| + |w|.$$

Moreover, by mathematical induction, one can prove that if $z_1, \cdots, z_n \in \mathbb{C}$, then

$$|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|. \tag{1.8}$$

□

Proposition 1.3 (Cauchy-Schwartz Inequality) *If* $z_1, \cdots, z_n$ *and* $w_1, \cdots, w_n \in \mathbb{C}$*, then*

$$\left|\sum_{j=1}^{n} z_j \bar{w}_j\right|^2 \leq (\sum_{j=1}^{n} |z_j|^2)(\sum_{j=1}^{n} |w_j|^2). \tag{1.9}$$

Proof. For any $\lambda \in \mathbb{C}$, one has

$$\begin{aligned} 0 &\leq \sum_{j=1}^{n} |z_j - \lambda w_j|^2 \\ &= \sum_{j=1}^{n} (z_j - \lambda w_j)(\bar{z_j} - \bar{\lambda}\bar{w_j}) \\ &= \sum_{j=1}^{n} (z_j \bar{z_j} - \bar{\lambda} z_j \bar{w_j} - \lambda w_j \bar{z_j} + \lambda w_j \bar{\lambda} \bar{w_j}) \\ &= \sum_{j=1}^{n} |z_j|^2 - \left(\bar{\lambda}\sum_{j=1}^{n} z_j \bar{w_j} + \lambda \sum_{j=1}^{n} w_j \bar{z_j}\right) + |\lambda|^2 \sum_{j=1}^{n} |w_j|^2. \end{aligned}$$

If we choose $\lambda = \sum_{j=1}^{n} z_j \bar{w_j} / (\sum_{j=1}^{n} |w_j|^2)$ then

$$\begin{aligned} 0 &\leq \sum_{j=1}^{n} |z_j|^2 - 2\frac{\left|\sum_{j=1}^{n} z_j \bar{w_j}\right|^2}{\sum_{j=1}^{n} |w_j|^2} + \frac{\left|\sum_{j=1}^{n} z_j \bar{w_j}\right|^2}{\left(\sum_{j=1}^{n} |w_j|^2\right)^2} \sum_{j=1}^{n} |w_j|^2 \\ &= \sum_{j=1}^{n} |z_j|^2 - \frac{\left|\sum_{j=1}^{n} z_j \bar{w_j}\right|^2}{\sum_{j=1}^{n} |w_j|^2}. \end{aligned}$$

This implies that

$$\left|\sum_{j=1}^{n} z_j \bar{w}_j\right|^2 \leq (\sum_{j=1}^{n} |z_j|^2)(\sum_{j=1}^{n} |w_j|^2).$$

□

1.3 Holomorphic linear factional maps

1.3.1 Self-maps of unit circle and the unit disc.

Let

$$D(0,1) = \{z \in \mathbb{C} : |z| < 1\} \tag{1.10}$$

be the unit disk centered at the origin and

$$T = \partial D(0,1) = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in [0, 2\pi)\} \tag{1.11}$$

be the unit circle in the complex plane.

For any $a \in D(0,1)$, we define

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in D(0,1). \tag{1.10}$$

THEOREM 1.4 *For any $a \in D(0,1)$ we have*

i. $\phi_a(0) = a$ *and* $\phi_a(a) = 0$*;*

ii. $\phi_a^{-1} = \phi_a$*;*

iii. $\phi_a : T \to T$*, so it is one-to-one and onto;*

iv. $\phi_a : D(0,1) \to D(0,1)$*, so it is one-to-one and onto.*

Proof. For Part 1, it is easy to see that $\phi_a(0) = a$ and $\phi_a(a) = 0$.
For Part 2, since

$$\phi_a(\phi_a(z)) = \frac{a - \phi_a(z)}{1 - \bar{a}\phi_a(z)} = \frac{a(1 - \bar{a}z) - (a - z))}{1 - \bar{a}z - \bar{a}(a - z)} = \frac{z(1 - |a|^2)}{1 - |a|^2} = z.$$

This proves $\phi_a^{-1} = \phi_a$.
For Part 3, for any $z \in T$, one has

$$|\phi_a(z)| = |z||\phi_a(z)| = |\bar{z}||\phi_a(z)| = |\bar{z}\phi_a(z)| = \frac{|a\bar{z} - |z|^2|}{|1 - \bar{a}z|} = \frac{|1 - a\bar{z}|}{|1 - \bar{a}z|} = 1.$$

This implies $\phi_a : T \to T$. Combining this with Part 2, one has $\phi_a : T \to T$ is one-to-one and onto.

Finally, for part 4, notice that

$$\begin{aligned}|\phi_a(z)|^2 < 1 \quad &\Longleftrightarrow \quad |z-a|^2 < |1-\overline{a}z|^2 \\ &\Longleftrightarrow \quad |z|^2 - 2\mathrm{Re}\,\overline{a}z + |a|^2 < 1 - 2\mathrm{Re}\,\overline{a}z + |a|^2|z|^2 \\ &\Longleftrightarrow \quad |z|^2 + |a|^2 < 1 + |a|^2|z|^2 \\ &\Longleftrightarrow \quad |a|^2(1-|z|^2) < 1 - |z|^2 \\ &\Longleftrightarrow \quad |z|^2 < 1.\end{aligned}$$

This proves the first part of Part 4, combining this with Part 2, one has completed the proof of Part 4 and the proof of the theorem. $\square$

1.3.2 Maps from line to circle and upper half plane to disc.

THEOREM 1.5 (Cayley Transformation) *The transformation*

$$C(z) = \frac{z-i}{z+i} \tag{1.11}$$

is a one-to-one and onto map of the real line $\mathbb{R}$ *to the unit circle* $T \setminus \{1\}$ *and a bijection from the upper half plane* $\mathbb{R}^2_+ = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ *to the unit disc* $D(0,1)$.

Proof. If $w = C(z)$ then

$$w(z+i) = z - i \iff z(1-w) = iw + i = i(w+1) \iff z = i\frac{1+w}{1-w}.$$

Thus

$$C^{-1}(z) = i\frac{1+z}{1-z}. \tag{1.12}$$

Moreover,

$$|C(z)| = 1 \iff |z-i|^2 = |z+i|^2 \iff -\mathrm{Im}\, z = \mathrm{Im}\, z \iff \mathrm{Im}\, z = 0.$$

This implies that $C : \mathbb{R} \to T \setminus \{1\}$ is one-to-one and onto. Similarly, one has

$$|C(z)| < 1 \iff |z-i|^2 < |z+i|^2 \iff -\mathrm{Im}\, z < \mathrm{Im}\, z \iff \mathrm{Im}\, z > 0.$$

Thus $C : \mathbb{R}^2_+ \to D(0,1)$ is one to one and onto. $\square$

2 Smooth functions on domains in $\mathbb{C}$

2.1 Notation and definitions

Let

$$z = x + iy, \quad \overline{z} = x - iy \in \mathbb{C}.$$

• We use the following notation

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big). \tag{2.1}$$

Then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}, \quad \frac{\partial}{\partial y} = i\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\Big),$$

$$\frac{\partial x}{\partial x} = 1, \quad \frac{\partial y}{\partial y} = 1, \quad \frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = 0$$

and

$$\frac{\partial z}{\partial z} = \frac{\partial \overline{z}}{\partial \overline{z}} = 1, \quad \frac{\partial z}{\partial \overline{z}} = \frac{\partial \overline{z}}{\partial z} = 0. \tag{2.2}$$

• Differential: $dz = dx + idy$ and $d\overline{z} = dx - idy$. Moreover,

$$dz\Big(\frac{\partial}{\partial z}\Big) = 1, \quad dz\Big(\frac{\partial}{\partial \overline{z}}\Big) = 0. \tag{2.3}$$

• **Domains in the complex plane:**

An open and connected set D in $\mathbb{R}^2$ is called a domain in $\mathbb{R}^2$ or $\mathbb{C}$.
$\overline{D}$ denotes the closure of D.

EXAMPLE 2 *Examples of domains:*

(i) $\mathbb{R}^2_+ = \{z \in \mathbb{C} : \text{Im}\,(z) > 0\}$, the upper half plane.

(ii) $D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$, the unit disc centered at 0.

The closure of $D(0,1)$ is $\overline{D}(0,1) = \{z \in \mathbb{C} : |z| \le 1\}$.

• C^k **functions**

Let D be a domain in $\mathbb{C}$. Let u be a real-valued function on D ($u : D \to \mathbb{R}$). Let $C^0(D)$ ($C^0(\overline{D})$) denote the set of all continuous functions on D (on $\overline{D}$). For any non-negative integer k, one has the following definitions.

(i) We say that $u \in C^k(D)$ $(C^k(\overline{D}))$ if the partial derivatives of u satisfy

$$\frac{\partial^m u}{\partial x^\alpha \partial y^\beta}(x,y) \in C^0(D) \ \left(C^0(\overline{D})\right) \tag{2.4}$$

for all $\alpha, \beta \geq 0$ and $m =: \alpha + \beta \leq k$.
(ii) We say a complex valued function $f = u + iv \in C^k(D)$ if $u \in C^k(D)$ and $v \in C^k(D)$. Here u and v are real-valued functions.
Remark: Since $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/(2i)$, one can see that any function $u(x, y)$ on D can be viewed as a function of z and $\bar{z}$.

EXAMPLE 3 *Examples of transforming functions of x and y into functions of z and $\overline{z}$.*

(i) $u(x,y) = x - y = \frac{1}{2}(z + \overline{z}) + \frac{i}{2}(z - \overline{z})$
(ii) $u = x^2 + y^2 - xy = z\overline{z} + \frac{1}{4i}(z^2 - \overline{z}^2)$.

Proposition 2.1 *Let $f \in C^1(D)$. Then*

$$df =: \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}. \tag{2.5}$$

Proof. Since

$$\frac{\partial f}{\partial z}dz = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})(dx + idy) = \frac{1}{2}(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy) + \frac{i}{2}(\frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx)$$

Similarly,

$$\frac{\partial f}{\partial \overline{z}}d\overline{z} = \frac{1}{2}(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy) - \frac{i}{2}(\frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx)$$

Therefore,

$$\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

2.2 Polynomial of degree n

Definition 2.2 *A polynomial P_n of x and y is of degree n if and only if*

$$P_n(x,y) = \sum_{\alpha+\beta=0}^{n} a_{\alpha\beta}x^\alpha y^\beta.$$

and for some α, β with $\alpha + \beta = n$, $a_{\alpha\beta} \neq 0$.

Proposition 2.3 *Transforming a polynomial of x and y into a polynomial of z and $\overline{z}$.*

(a) Any polynomial of degree n in x and y can be written as:

$$P_n(x,y) = \sum_{\alpha+\beta=0}^{n} a_{\alpha\beta} x^\alpha y^\beta = \sum_{\alpha+\beta=0}^{n} A_{\alpha\beta} z^\alpha \overline{z}^\beta$$

(b) $P_n(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$ if and only if $a_{\alpha\beta} = A_{\alpha\beta} = 0$ for all $\alpha, \beta \in \mathbb{N}$.

Proof. Notice that

$$\begin{aligned} P_n(x,y) &= \sum_{k=0}^{n} \sum_{\alpha+\beta=k} a_{\alpha\beta} x^\alpha y^\beta \\ &= \sum_{k=0}^{n} \sum_{\alpha+\beta=k} a_{\alpha\beta} \left(\frac{1}{2}(z+\bar{z})\right)^\alpha \left(\frac{1}{2i}(z-\bar{z})\right)^\beta \\ &= \sum_{k=0}^{n} \sum_{\alpha+\beta=k} A_{\alpha\beta} z^\alpha \overline{z}^\beta \\ &= \sum_{\alpha+\beta=0}^{n} A_{\alpha\beta} z^\alpha \overline{z}^\beta. \end{aligned}$$

We sometimes denote the last summation as $P_n(z,\bar{z})$. Part (a) is proved.

Proof of Part (b): The backwards direction is easy. For the forward direction, notice that

$$\frac{\partial^k (x^p y^q)}{\partial x^\alpha \partial y^\beta} = \begin{cases} 0, & \text{if either } p < \alpha \text{ or } q < \beta \\ \frac{p!}{\alpha!}\frac{q!}{\beta!} x^{p-\alpha} y^{q-\beta}, & \text{if } p \geq \alpha \text{ and } q \geq \beta. \end{cases} \tag{2.6}$$

If for all $(x,y) \in \mathbb{R}^2$, $P_n(x,y) = 0$, then after applying $\frac{\partial^k}{\partial x^\alpha \partial y^\beta}$ to both sides and evaluating at $(0,0)$, one has

$$\alpha!\beta! a_{\alpha\beta} = 0.$$

Thus, $\forall \alpha, \beta$, $a_{\alpha\beta} = 0$. Similarly, notice that

$$\frac{\partial^k (z^p \overline{z}^q)}{\partial z^\alpha \partial \overline{z}^\beta} = \begin{cases} 0, & \text{if either } p < \alpha \text{ or } q < \beta \\ \frac{p!}{\alpha!}\frac{q!}{\beta!} z^{p-\alpha} \overline{z}^{q-\beta}, & \text{if } p \geq \alpha \text{ and } q \geq \beta. \end{cases} \tag{2.7}$$

After applying the operator $\frac{\partial^k}{\partial z^\alpha \partial \overline{z}^\beta}$ to both sides and evaluating at $z = 0$, one has

$$\alpha!\beta! A_{\alpha\beta} = 0,$$

which implies that $A_{\alpha\beta} = 0$. Proof of part (b) is complete. ∎

2.3 Rules of differentiations

(i) Some rules for differentiations:

$$\frac{\partial}{\partial z}(f+g) = \frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}, \quad \frac{\partial}{\partial \bar{z}}(f+g) = \frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}};$$

$$\frac{\partial}{\partial z}(fg) = \frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}, \quad \frac{\partial}{\partial \bar{z}}(fg) = \frac{\partial f}{\partial \bar{z}}g + f\frac{\partial g}{\partial \bar{z}};$$

$$\frac{\partial}{\partial z}\Big(\frac{f}{g}\Big) = \frac{\frac{\partial f}{\partial z}g - f\frac{\partial g}{\partial z}}{g^2}, \quad \frac{\partial}{\partial \bar{z}}\Big(\frac{f}{g}\Big) = \frac{\frac{\partial f}{\partial \bar{z}}g - f\frac{\partial g}{\partial \bar{z}}}{g^2};$$

(ii) The chain rule:

$$\frac{\partial h \circ f}{\partial z} = \frac{\partial h}{\partial w}\frac{\partial f}{\partial z} + \frac{\partial h}{\partial \bar{w}}\frac{\partial \bar{f}}{\partial z}$$
$$\frac{\partial h \circ f}{\partial \bar{z}} = \frac{\partial h}{\partial w}\frac{\partial f}{\partial \bar{z}} + \frac{\partial h}{\partial \bar{w}}\frac{\partial \bar{f}}{\partial \bar{z}}.$$

EXAMPLE 4 *Let* $g = e^{|z|^2+z^2}$. *Find* $\frac{\partial g}{\partial z}$ *and* $\frac{\partial g}{\partial \bar{z}}$

Solution. We can write: $h(w) = e^w$, $f = |z|^2 + z^2$ and $g = h \circ f$. Then

$$\frac{\partial g}{\partial z} = \frac{\partial h}{\partial w}\frac{\partial(|z|^2+z^2)}{\partial z} + \frac{\partial h}{\partial \overline{w}}\frac{\partial\overline{(|z|^2+z^2)}}{\partial z} = e^{|z|^2+z^2}(\overline{z}+2z)$$

and

$$\frac{\partial g}{\partial \overline{z}} = e^{|z|^2+z^2}\frac{\partial(|z|^2+z^2)}{\partial \overline{z}} = e^{|z|^2+z^2}z.$$

Homework 1: Complex numbers and maps.

1. Prove that, for any complex numbers z and w.
   (i) $|z+w|^2 = |z|^2 + |w|^2 + 2\text{Re}\,(z\overline{w})$;
   (ii) $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$;
   (iii) $1 - \left|\frac{z-w}{1-z\overline{w}}\right|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2}$.
2. Prove the following Lagrange's identity:

$$\Big|\sum_{j=1}^{n} z_j w_j\Big|^2 = \sum_{j=1}^{n} |z_j|^2 \sum_{j=1}^{n} |w_j|^2 - \sum_{1\le j\le k\le n} |z_j\overline{w}_k - \overline{w}_j z_k|^2.$$

(This identity implies Cauchy-Schwarz's inequality).
3. Prove

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

4. Let $z_1, \cdots, z_n \in \mathbb{C}$ be n complex numbers such that

$$\Big|\sum_{j=1}^{n} z_j w_j\Big| \le 1$$

for all $w_1, \cdots, w_n \in \mathbb{C}$ with $\sum_{j=1}^{n} |w_j|^2 \le 1$. Prove $\sum_{j=1}^{n} |z_j|^2 \le 1$.
5. Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$. Let $z_0 \in \mathbb{C}$ be a zero of $p(z)$ ($p(z_0) = 0$.)
   (i) If all $a_0, \cdots, a_n$ are real numbers, then $p(\overline{z}_0) = 0$;
   (ii) Provide a counter example shows that (i) fails if not all $a_0, \cdots, a_n$ are real.
6. Prove the Cayley transform:

$$\phi(z) = i\frac{1-z}{1+z}$$

maps the unit disc $D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ one-to-one onto the upper half plane $\mathbb{R}^2_+ = \{z \in \mathbb{C} : \text{Im}\,(z) > 0\}$.
7. Let

$$\psi(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers and $\alpha\delta - \beta\gamma > 0$. Prove $\psi : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is one-to -one and onto. Conversely, if

$$u(z) = \frac{az+b}{cz+d} : \mathbb{R}^2_+ \to \mathbb{R}^2_+$$

is one-to-one and onto, then a, b, c, d are real (after multiplying numerator and denominator by a constant) and $ad - bc > 0$.
8. Let $A = \{z \in \mathbb{C} : 1/2 \leq |z| \leq 2\}$, and $\phi(z) = z + \frac{1}{z}$. Compute $\phi(A)$.
9. Evaluate $\frac{\partial^3}{\partial z \partial \overline{z}^2}(xy^2)$.
10. Let $F : \mathbb{C} \to \mathbb{C}$ be a polynomial. Suppose that

$$\frac{\partial^2 F}{\partial z^2} = 0, \quad z \in \mathbb{C}.$$

Prove that

$$F(z, \overline{z}) = zG(\overline{z}) + H(\overline{z}).$$

11. If $f : \mathbb{C} \to \mathbb{C}$ is C^1 function such that $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \overline{z}} = 0$, then f is a constant.
12. If f is a real-valued holomorphic polynomial, then $f(z)$ must be a constant.

3 Holomorphic, harmonic functions

3.1 Holomorphic functions and C-R equations

Definition 3.1 *Let $f \in C^1(D)$ be a complex-valued function. We say that f is holomorphic in D if $\frac{\partial f(z)}{\partial \bar{z}} = 0$ on D. We call $\frac{\partial f}{\partial \bar{z}}(z) = 0$ the Cauchy-Riemann equation.*

EXAMPLE 5 *$f(z, \bar{z}) = x = \frac{1}{2}(z + \bar{z})$ is not a holomorphic function on $\mathbb{C}$.*

Proof. Since for all $z \in \mathbb{C}$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial z}{\partial \bar{z}} + \frac{\partial \bar{z}}{\partial \bar{z}}\right) = \frac{1}{2} \neq 0,$$

x is not a holomorphic function anywhere on $\mathbb{C}$.

Proposition 3.2 *P_n is a holomorphic polynomial of degree n if and only if*

$$P_n(z) = \sum_{k=0}^{n} a_k z^k.$$

Proof. Without loss of generality,

$$P_n = \sum_{k=0}^{n} \sum_{\alpha+\beta=k} A_{\alpha\beta} z^\alpha \overline{z}^\beta.$$

Notice that

$$\frac{\partial P_n}{\partial \bar{z}} = \sum_{k=1}^{n} \sum_{\alpha+\beta=k,\beta>0} A_{\alpha\beta} z^\alpha \beta \overline{z}^{\beta-1} = 0$$

for all $z \in \mathbb{C}$ if and only if $A_{\alpha\beta} = 0$ for all $\beta > 0$. Therefore,

$$P_n = \sum_{\alpha=0}^{n} A_{\alpha 0} z^\alpha.$$

The proof of is complete. □

• **Cauchy-Riemann equations**

Proposition 3.3 *$f(z) = u(z) + iv(z) \in C^1(D)$ is holomorphic in D if and only if u and v satisfy the following equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{on } D. \tag{3.1}$$

These equations are also called the Cauchy-Riemann equations (CR-equations.)

Proof. Notice that since

$$\begin{aligned}
\frac{\partial f(z)}{\partial \bar{z}} &= \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big) \\
&= \frac{1}{2}\Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y})\Big) \\
&= \frac{1}{2}\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i(\frac{\partial v}{\partial x} + (\frac{\partial u}{\partial y})\Big),
\end{aligned}$$

one has

$$\frac{\partial f}{\partial \bar{z}} = 0 \text{ on } D \iff \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases} \text{ on } D. \tag{3.2}$$

The proof is complete. ▯

Definition 3.4 *For any domain D in $\mathbb{C}$, $\mathcal{O}(D)$ denotes the set of all holomorphic functions in D.*

THEOREM 3.5 *$(\mathcal{O}(D), +, \cdot)$ forms an algebra (i.e. for $f, g \in \mathcal{O}(D)$, $f + g \in \mathcal{O}(D)$, and $f \cdot g \in \mathcal{O}(D)$).*

Proof. If $f, g \in \mathcal{O}(D)$, then $f + g \in \mathcal{O}(D)$ and $fg \in \mathcal{O}(D)$. ▯

3.2 Harmonic functions

Definition 3.6 *The Laplace operator or Laplacian, denoted Δ, is a second order elliptic operator defined by*

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{3.3}$$

Let $u \in C^2(D)$, we say that u is harmonic in D if and only if $\Delta u = 0$ on D.

• **Laplacian in z and $\overline{z}$**

We express Δ in terms of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$. Notice that since $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$, one has

$$4\frac{\partial^2}{\partial z \partial \overline{z}} = (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta. \tag{3.4}$$

Proposition 3.7 *If $f = u + iv$ is holomorphic in D then u and v are harmonic in D. Here v is called the harmonic conjugate of u.*

Proof. Since $f = u + iv$ is holomorphic, u and v satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Thus, without loss of generality, we may assume that $u, v \in C^2(D)$ (since we will prove $u, v \in C^\infty(D)$). Hence we can take second order partials and get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

and

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} = 0.$$

Using the expression of the Laplacian in terms of z and $\bar{z}$, we have

$$0 = \frac{\partial^2 f}{\partial z \partial \overline{z}} = \frac{\partial^2 u}{\partial z \partial \overline{z}} + i\frac{\partial^2 v}{\partial z \partial \overline{z}}$$

This implies that $\frac{\partial^2 u}{\partial z \partial \overline{z}} = \frac{\partial^2 v}{\partial z \partial \overline{z}} = 0$ since Δ is a real operator. □

Corollary 3.8 *Let $f = u + iv$ be holomorphic in D. Then u is a constant implies that f is constant.*

Proof. Let f be holomorphic in D. Using that u and v satisfy the Cauchy-Riemann equations and that u is a constant, we have

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = 0, \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0$$

Therefore, v is a constant. Thus, $f = u + iv$ is a constant. □

Proposition 3.9 *If f is holomorphic in D then*

(i) $\Delta|f(z)|^2 = 4|\frac{\partial f}{\partial z}|^2$ *on* D*;*

(ii) If $|f|^2$ *is harmonic in* D*, then* f *is a constant.*

Proof. (i) Since f is holomorphic, one has

$$0 = \Delta|f(z)|^2 = 4\frac{\partial^2 f(z)\overline{f}(z)}{\partial z\partial \overline{z}} = 4\frac{\partial}{\partial z}(f(z)\frac{\partial \overline{f}(z)}{\partial \overline{z}}) = 4\frac{\partial f(z)}{\partial z}\frac{\partial \overline{f}(z)}{\partial \overline{z}} = 4\Big|\frac{\partial f(z)}{\partial z}\Big|^2$$

(ii) If $|f|^2$ is harmonic then $\Delta|f|^2 = 0$. The previous identity implies that $\frac{\partial f}{\partial z} = 0$ and since f is holomorphic, $\frac{\partial f}{\partial \overline{z}} = 0$. Therefore, the gradient of f $\nabla f =: (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (0,0)$. So f is a constant. □

3.3 Translation formula for Laplacian

THEOREM 3.10 *If* $f : D \to G$ *is holomorphic and* $u \in C^2(G)$ *is a function in* G*, then*

(i) The invariant property for Laplacian

$$\Delta_z u \circ f(z) = (\Delta u)(f(z))\left|\frac{\partial f(z)}{\partial z}\right|^2. \tag{3.5}$$

(ii) If u *is harmonic in* G *then* $u \circ f$ *is harmonic in* D*.*

Proof. (i) We compute $\Delta(u \circ f)$ on D. Since

$$\begin{aligned}
\Delta(u\circ f) &= 4\frac{\partial^2(u\circ f)}{\partial z\partial\overline{z}} \\
&= 4\frac{\partial}{\partial z}\Big(\frac{\partial u}{\partial w}\circ f\frac{\partial f(z)}{\partial \overline{z}} + \frac{\partial u}{\partial \overline{w}}\circ f\frac{\partial \overline{f}}{\partial \overline{z}}\Big) \\
&= 4\frac{\partial}{\partial z}\Big(\frac{\partial u}{\partial \overline{w}}\circ f\frac{\partial \overline{f}(z)}{\partial \overline{z}}\Big) \\
&= 4\Big(\frac{\partial^2 u}{\partial w\partial\overline{w}}\circ f\Big|\frac{\partial f(z)}{\partial z}\Big|^2 + \frac{\partial^2 u}{\partial \overline{w}^2}\frac{\partial \overline{f}}{\partial z}\frac{\partial \overline{f}(z)}{\partial \overline{z}}\Big) \\
&= (\Delta u)\circ f\left|\frac{\partial f(z)}{\partial z}\right|^2.
\end{aligned}$$

(ii) If u is harmonic in G then $(\Delta u)\circ f = 0$ on D. Therefore, $u \circ f$ is harmonic in D. □

4 Line integral and cohomology group

4.1 Line integrals

• Wedge product. We need the following definitions and notation on the wedge product and differential forms

$$dx \wedge dx = dy \wedge dy = 0, \quad dx \wedge dy = -dy \wedge dx \tag{4.1}$$

• Differential 1-form is:

$$u = u_1 dx + u_2 dy, \tag{4.2}$$

where u_1 and u_2 are functions of x and y. We define

$$du =: du_1 \wedge dx + du_2 \wedge dy = (\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}) dx \wedge dy. \tag{4.3}$$

Definition 4.1 *Let $\gamma(t) = (x(t), y(t)) : [0,1] \to \mathbb{R}^2$ be a piecewise C^1 curve, and $u = u_1 dx + u_2 dy$ a 1-form where u_1 and u_2 are integrable functions on γ. Then the line integral of u on γ is defined by*

$$\int_\gamma u = \int_0^1 u_1((x(t), y(t))x'(t) + u_2((x(t), y(t))y'(t)dt. \tag{4.4}$$

EXAMPLE 6 *Let $\gamma(t) = (a\cos t, b\sin t) : [0,\pi] \to \mathbb{R}^2$. Evaluate the line integral*

$$\int_\gamma (x^3 - y^2)dx + 2xydy.$$

Solution. Since

$$\begin{aligned}
&\int_\gamma (x^3 - y^2)dx + 2xydy \\
&= \int_0^\pi (a^3 \cos^3 t - b^2 \sin^2 t) d(a\cos t) + 2ab\cos t \sin t d(b \sin t) \\
&= \int_0^\pi (a^4 \cos^3 t - ab^2 \sin^2 t)(-\sin t)dt + 2ab^2 \cos^2 t \sin t dt \\
&= [\frac{a^4}{4}\cos^4 t - ab^2(\cos t - \frac{\cos^3 t}{3}) - \frac{2ab^2}{3}\cos^3 t]\Big|_0^\pi \\
&= 2[ab^2(1 - \frac{1}{3}) + \frac{2ab^2}{3}] \\
&= \frac{8ab^2}{3}.
\end{aligned}$$

EXAMPLE 7 *Let* $\gamma(t) = (a\cos t, b\sin t) : [0, 2\pi] \to \mathbb{R}^2$. *Show the line integral*

$$\int_\gamma x^3 dx - y^5 dy = 0.$$

Proof. Since

$$x^3 dx - y^5 dy = d(\frac{x^4}{4} - \frac{y^6}{6})$$

and $\gamma(0) = \gamma(2\pi)$, one has

$$\int_\gamma x^3 dx - y^5 dy = (\frac{x^4}{4} - \frac{y^6}{6})\Big|_{\gamma(0)}^{\gamma(2\pi)} = 0.$$

4.2 Cohomology group

• Exact forms and closed forms

Definition 4.2 *Let* $u = u_1 dx + u_2 dy$ *be a one-form in a domain* D *with* $u_j \in C(D)$. *Then*

(i) We say that $u_1 dx + u_2 dy$ *is exact in* D *if and only if there exists a differentiable function* v *on* D *such that* $dv = u_1 dx + u_2 dy$ *(i.e.* $\frac{\partial v}{\partial x} = u_1$ *and* $\frac{\partial v}{\partial y} = u_2$ *in* D*)*

(ii) We say that $u_1 dx + u_2 dy$ *(with* $u_j \in C^1(D)$*) is closed in* D *if and only if* $d(u_1 dx + u_2 dy) = 0$. *In other words,* $\frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial x}$ *on* D.

Proposition 4.3 *If* $u_1 dx + u_2 dy$ *is exact in* D *with* $u_j \in C^1(D)$ *then* $u_1 dx + u_2 dy$ *is closed in* D.

Proof. Notice that $u_1 dx + u_2 dy$ in D is exact implies there exists v such that $dv = u_1 dx + u_2 dy$ in D. Then

$$d(u_1 dx + u_2 dy) = d^2 v = (\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}) dx \wedge dy = 0.$$

Therefore, $u_1 dx + u_2 dy$ is closed in D. □

• Cohomology group

Definition 4.4 *Let* D *be a domain in* $\mathbb{C}$. *Then the cohomology group* $H^1(D)$ *on* D *is defined by*

$$\begin{aligned} H^1(D) &= \{\text{closed one-forms in } D\}/\{\text{exact one forms in } D\} \\ &=: \{f = u_1 dx + u_2 dy : u_j \in C^1(D), df = 0\}/\{dg : g \in C^2(D)\}. \end{aligned}$$

Question: When is $H^1(D) = 0$?

Definition 4.5 *Let D be a domain in $\mathbb{C} = \mathbb{R}^2$. We say that D is simply connected if any loop based any point $p \in D$ can be continuously shrunk to the point p.*

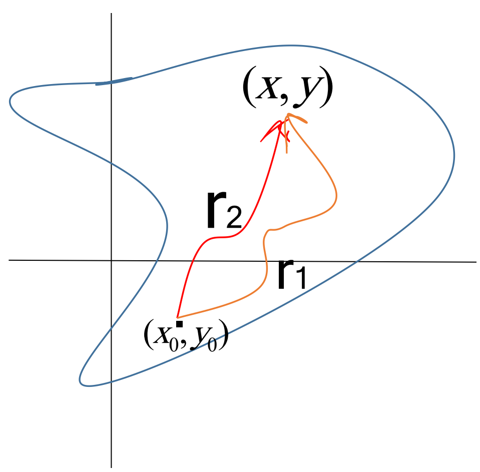
THEOREM 4.6 *Let D be a simply connected domain in $\mathbb{R}^2$. Then $H^1(D) = \{0\}$.*

Proof. Let $f = u_1dx + u_2dy$ with $u_j \in C(D)$ is closed one-form. We want to find a function g such that $dg = f$. Take a point $p_0 \in D$. Let $p = (x, y) \in D$ and $\gamma_1 : [0, 1] \to D$ be a piecewise C^1 curve with $\gamma_1(0) = p_0$ and $\gamma_1(1) = p$. Define a function g by

$$g(x, y) = \int_{\gamma_1} u_1dx + u_2dy. \tag{4.5}$$

Since D is simply connected, we claim that $g(x, y)$ is independent of the choice of γ_1 by Stokes' theorem. That is, we will show that for any other piecewise C^1 curve $\gamma_2 : [0, 1] \in D$ such that $\gamma(0) = p_0$ and $\gamma_2(1) = p = (x, y)$ that

$$g(x, y) = \int_{\gamma_2} u_1dx + u_2dy.$$



Let $\gamma : [0, 2] \to D$ be such that

$$\gamma(t) = \gamma_1(t), \quad 0 \le t \le 1, \quad \text{and} \quad \gamma(t) = \gamma_2(2 - t), \ 1 \le t \le 2.$$

Then γ is a closed, piecewise C^1 curve. Call the domain bounded by γ D_γ. Since D is simply connected, one has that $D_\gamma \subset D$. By Stokes' theorem, since f is closed one form ($df = 0$), one has

$$\int_\gamma u_1 dx + u_2 dy = \int_\gamma f = \int_{D_\gamma} df = \int_D 0 dx \wedge dy = 0.$$

This implies that

$$\int_{\gamma_1} u_1 dx + u_2 dy = \int_{\gamma_2} u_1 dx + u_2 dy.$$

Therefore, g is well-defined. Now let $\gamma_3 : [0, 1+h] \to D$ by $\gamma_3(t) = \gamma_1(t)$ when $0 \le t \le 1$ and $\gamma_3(t) = (x + (t-1), y)$ when $1 \le t \le 1+h$. When h is sufficiently small, $\gamma_3(t) \subset D$. Then

$$\begin{aligned}
\frac{\partial g}{\partial x}(x,y) &= \lim_{h\to 0} \frac{v(x+h,y) - v(x,y)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\Big(\int_{\gamma_3} u_1 dx + u_2 dy - \int_{\gamma_1} u_1 dx + u_2 dy\Big) \\
&= \lim_{h\to 0}[\frac{1}{h}\int_0^h u_1(x+t,y) - (x,y)dt + u_1(x,y) \\
&= u_1(x,y)
\end{aligned}$$

since $\frac{\partial u}{\partial y}$ is continuous in D. Similarly, $\frac{\partial g}{\partial y} = u_2$. Therefore, $dg = f$. Therefore, f is exact. So, $H^1(D) = 0$. The proof is complete. □

Note. The converse of the above theorem fails: $-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ is closed one-form in $D = \mathbb{R}^2\backslash\{(0,0)\}$, but it is not exact. The demonstration is given in next topic.

4.3 Harmonic conjugate

QUESTION 1 *Let D be a domain in $\mathbb{C}$ and u a harmonic function on D. Is there a harmonic function v on D such that $f(z) = u(z) + iv(z)$ is holomorphic in D? If such a v exists, then we call it a harmonic conjugate of u.*

In general, the answer is no as seen in the next example.

EXAMPLE 8 *Let* $u = \log |z|^2$, $z \in D = \mathbb{C} \setminus \{0\}$. *Then* u *is harmonic in* D *but it does not have a harmonic conjugate in* D.

Proof. If there is a v such that $f = u + iv$ is holomorphic in D, then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be defined by $\gamma(\theta) = e^{i\theta}$. Then

$$\begin{aligned} 0 &= \int_\gamma dv \\ &= \int_\gamma -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \\ &= 2\int_\gamma -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \\ &= 2\int_\gamma -ydx + xdy \\ &= 2\int_0^{2\pi}(-\sin\theta(-\sin\theta) + \cos\theta\, \cos\theta)d\theta \\ &= 4\pi. \end{aligned}$$

This is a contradiction. □

• **Existence of harmonic conjugate**

We will prove the following theorem.

Corollary 4.7 *If* D *is simply connected in* $\mathbb{C}$ *and if* u *is harmonic in* D, *then there is a harmonic function* v *on* D *such that* $f = u + iv$ *is holomorphic in* D.

Proof. Since u is harmonic in D, we have

$$-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

is a closed 1-form in D. By theorem 4.6, there is a $v \in C^1(D)$ such that

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

This implies

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}, \qquad \text{on } D.$$

Therefore, $u + iv$ is holomorphic in D and v is harmonic in D, which is a conjugate harmonic function of u on D. □

5 Complex line integrals

5.1 Definition and examples

A piecewise C^1 curve $\gamma : [a, b] \to \mathbb{C}$ is given by

$$\gamma(t) = x(t) + iy(t), \quad t \in [a, b] \tag{5.1}$$

Let

$$h = f dz + g d\overline{z}$$

be a continuous one-form. We define the complex line integral:

$$\int_\gamma h = \int_\gamma f dz + g d\overline{z} = \int_a^b f(x(t) + iy(t)) z'(t) dt + g(z(t)) \overline{z}'(t) dt. \tag{5.2}$$

EXAMPLE 9 *Evaluate*

$$\int_{|z|=1} (z^2 + \overline{z}^2) dz + z^2 \overline{z} d\overline{z}$$

Solution. We can parametrize the unit circle $|z| = 1$ as follows:

$$z(t) = \cos t + i \sin t = e^{it} : \quad t \in [0, 2\pi].$$

This parametrization gives the unit circle a counter-clockwise orientation. Then

$$\begin{aligned}
\int_{|z|=1} (z^2 + \overline{z}^2) dz + z^2 \overline{z} d\overline{z} &= \int_0^{2\pi} ((e^{it})^2 + (e^{-it})^2) i e^{it} dt + (e^{it})^2 e^{-it} (-i) e^{-it} dt \\
&= i \int_0^{2\pi} (e^{3it} + e^{-it} - 1) dt \\
&= i[\frac{1}{3i} e^{3it} - \frac{1}{i} e^{-it} - t]\Big|_0^{2\pi} \\
&= -2\pi i.
\end{aligned}$$

EXAMPLE 10 *Let f be complex-valued differentiable function in D. Let $\gamma(t) = x(t) + iy(t) : [a, b] \to D$ be a piecewise C^1 curve. Then*

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)). \tag{5.3}$$

Solution. Since

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z},$$

by the chain rule

$$\begin{aligned} d(f \circ \gamma(t)) &= \frac{\partial f}{\partial z}(\gamma(t))\gamma'(t)dt + \frac{\partial f}{\partial \overline{z}}(\gamma(t))\overline{\gamma'(t)}dt \\ &= (\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z})|_{z(t)=\gamma(t)} \end{aligned}$$

Therefore,

$$\int_\gamma df = \int_\gamma \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z} = \int_a^b d(f \circ \gamma(t)) = f(\gamma(b)) - f(\gamma(a)).$$

EXAMPLE 11 *Let* $\gamma(t) = (1-t) + i(1+t) : [0,1] \to \mathbb{C}$. *Evaluate*

$$\int_\gamma z^3 dz + (|z|^2 + \overline{z}^2) d\overline{z}$$

Solution. Since $\gamma(0) = 1 + i$ and $\gamma(1) = 2i$, we have

$$\begin{aligned} \int_\gamma z^3 dz + (|z|^2 + \overline{z}^2) d\overline{z} &= \int_\gamma (z^3 dz + \overline{z}^2 d\overline{z}) + |z|^2 d\overline{z} \\ &= \int_\gamma d(\frac{1}{4} z^4 + \frac{1}{3}\overline{z}^3) + |z|^2 d\overline{z} \\ &= (\frac{1}{4} z^4 + \frac{1}{3}\overline{z}^3)\Big|_{z=\gamma(0)}^{\gamma(1)} + \int_\gamma |z|^2 d\overline{z} \\ &= \frac{(2i)^4 - (1+i)^4}{4} + \frac{(-2i)^3 - (1-i)^3}{3} \\ &\quad + \int_0^1 [(1-t)^2 + (1+t)^2](-1-i)dt \\ &= \frac{16+4}{4} + \frac{8i + 2(1-i)}{3} - 2(1+i)\int_0^1 [1+t^2]dt \\ &= \frac{17}{3} + 2i - 2(1+i)\int_0^1 [1+t^2]dt \\ &= \frac{17}{3} + 2i - \frac{8(1+i)}{3} \\ &= \frac{17-8}{3} - i\frac{8-6}{3} \\ &= 3 - \frac{2}{3} i. \end{aligned}$$

5.2 Green's theorem for complex line integral

THEOREM 5.1 *Let D be a bounded domain in $\mathbb{C}$ with piecewise C^1 boundary. Let $f, g \in C(\overline{D}) \cap C^1(D)$. Then*

$$\int_{\partial D} f(z,\overline{z})dz + g(z,\overline{z})d\overline{z} = 2i\int_D \Big(\frac{\partial f(z,\overline{z})}{\partial \overline{z}} - \frac{\partial g(z,\overline{z})}{\partial z}\Big)dA. \tag{5.4}$$

Proof. By Stokes' theorem and $d\overline{z}\wedge dz = 2idx\wedge dy = 2idA$, one has

$$\begin{aligned}\int_{\partial D} f(z,\overline{z})dz + g(z,\overline{z})d\overline{z} &= \int_D d(f(z,\overline{z})dz + g(z,\overline{z})d\overline{z}) \\ &= \int_D df\wedge dz + dg\wedge d\overline{z} \\ &= \int_D \frac{\partial f}{\partial \overline{z}}d\overline{z}\wedge dz + \frac{\partial g}{\partial z}dz\wedge d\overline{z} \\ &= 2i\int_D \Big(\frac{\partial f(z,\overline{z})}{\partial \overline{z}} - \frac{\partial g(z,\overline{z})}{\partial z}\Big)dA.\end{aligned}$$

The proof is complete. □

6 Complex differentiation

6.1 Definition of complex differentiation

• **Linear Approximation.**

Let $f(x, y)$ be a continuously differentiable function on $D \subset \mathbb{R}^2$. Let $(x_0, y_0) \in D$. Then by Taylor's theorem,

$$f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) + o(|(x,y)-(x_0,y_0)|).$$

Similarly

$$f(z) = f(z_0) + \frac{\partial f}{\partial z}(z_0)(z-z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)\overline{(z-z_0)} + o(|z-z_0|). \tag{6.1}$$

• **Definition of complex differentiation.**

Definition 6.1 *If $f(z)$ is a complex-valued function in a domain $D \subseteq \mathbb{C}$ and $z_0 \in D$, we say that $f(z)$ is complex differentiable at z_0 if*

$$f'(z_0) =: \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{6.2}$$

exists in $\mathbb{C}$.

6.2 Properties of complex derivatives

Proposition 6.2 *Let f be a complex-valued function in a domain D and $z_0 \in D$. If $f'(z_0)$ exists, then*

i. $f'(z_0) = \frac{\partial f}{\partial x}(z_0)$;

ii. $f'(z_0) = -i\frac{\partial f}{\partial y}(z_0)$;

iii. $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$;

iv. $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$.

Proof. By hypothesis, $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.

Part 1 follows by taking the path of the limit $z = x + iy_0 \to z_0$.

Part 2 follows by taking the path of limit $z = x_0 + iy \to z_0$.

Part 3 and Part 4 are the combinations of Part 1 and Part 2;

□

THEOREM 6.3 *Let f be a complex-valued function on D. Then $f'(z) \in C(D)$ if and only if f is holomorphic in D.*

Proof. If $f'(z) \in C(D)$ then f is continuous on D and by the previous proposition $f \in C^1(D)$ and $\frac{\partial f}{\partial \bar{z}} = 0$. Therefore, f is holomorphic in D.

Conversely, if f is holomorphic in D then $f \in C^1(D)$ and $\frac{\partial f}{\partial \bar{z}} = 0$ on D. For any $z_0 \in D$, we have

$$\begin{aligned} f(z) - f(z_0) &= \frac{\partial f}{\partial z}(z_0)(z - z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)(\bar{z} - \bar{z}_0) + o(|z - z_0|) \\ &= \frac{\partial f}{\partial z}(z_0)(z - z_0) + o(|z - z_0|) \end{aligned}$$

Therefore,

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0)$$

exists for all $z_0 \in D$ and $f'(z) = \frac{\partial f}{\partial z} \in C(D)$. The proof of the theorem is complete. ▯

6.3 Complex anti-derivative

Definition 6.4 *Let $f, F \in \mathcal{O}(D)$. We say that F is an (complex) antiderivative of $f(z)$ on D if $F'(z) = f(z)$ or $\frac{\partial F}{\partial z} = f(z)$ on D.*

Recall: We know that if $f \in C[a, b]$ then f has an antiderivative on $[a, b]$ which is $F(x) = \int_a^x f(t)\, dt$.

Question: For any $f \in \mathcal{O}(D)$, does f have an (complex) anti-derivative $F \in \mathcal{O}(D)$?

Answer: No, in general.

EXAMPLE 12 *The function $f(z) = 1/z$ is holomorphic in $D =: \mathbb{C} \setminus \{0\}$, but it does not have anti-derivative on D.*

Proof. If there is an $F \in \mathcal{O}(D)$ with $F'(z) = f(z)$ on D, then

$$dF = \frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial \bar{z}} d\bar{z} = f(z)dz = \frac{1}{z} dz$$

Thus

$$0 = \int_{|z|=1} dF = \int_{|z|=1} \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = 2\pi i$$

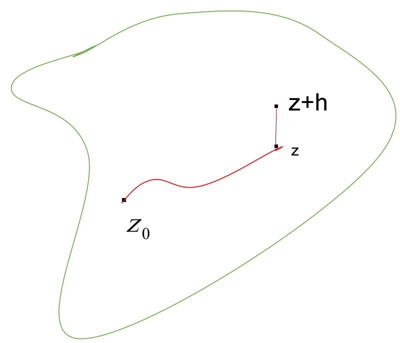
This is a contradiction. ∎

However, we have the following theorem.

THEOREM 6.5 *Let D be a simply connected domain in $\mathbb{C}$. Then for any $f \in \mathcal{O}(D)$ there is a $F \in \mathcal{O}(D)$ such that $F'(z) = f(z)$ on D*

Proof. Let D be a simply connected domain in $\mathbb{C}$ and $f \in \mathcal{O}(D)$. We want to find $F \in \mathcal{O}(D)$ such that $F'(z) = f(z)$. We define

$$F(z) = \int_{z_0}^{z} f(z)dz$$



The integral is interpreted as a line integral along a piecewise C^1 curve from z_0 to z in D. Since D is simply connected, by Stokes' theorem or Green's theorem, $F(z)$ is well-defined on D. Then

$$\begin{aligned}
\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} &= \lim_{h\to 0} \frac{1}{h} \int_{z}^{z+h} f(z)dz \\
&= \lim_{h\to 0} \frac{1}{h} \int_{0}^{1} f(z+th)hdt \\
&= \lim_{h\to 0} \int_{0}^{1} f(z+th)dt \\
&= f(z).
\end{aligned}$$

Therefore, $F'(z) = f(z)$. ∎

EXAMPLE 13 *Let $f(z) = z^n$ for $n \in \mathbb{Z}$. For $n \neq -1$, f has a holomorphic anti-derivative on $D = \mathbb{C} \setminus \{0\}$.*

Proof. For any integer $n \neq -1$, we let $F(z) = \frac{1}{1+n} z^{n+1}$. $F(z)$ is holomorphic in D and $F'(z) = z^n = f(z)$ on D. ∎

HW 2: Holomorphic/harmonic functions

1. (i) Prove that if f is holomorphic on $U \subset \mathbb{C}$ and $f(z) \neq 0$ on U, then

$$\Delta(|f|^p) = p^2|f|^{p-2}\Big|\frac{\partial f}{\partial z}\Big|^2, \quad p > 0.$$

(ii)If f is harmonic and real-valued on $U \subset \mathbb{C}$ and nonvanishing then

$$\Delta(|f|^p) = p(p-1)|f|^{p-2}|\nabla f(z)|^2 = p(p-1)|f|^{p-2}(|\frac{\partial f}{\partial x}|^2 + |\frac{\partial f}{\partial y}|^2)$$

2. Let U be a connected open set U on $\mathbb{C}$. Suppose that G_1, G_2 are holomorphic on U and

$$\frac{\partial G_1}{\partial z} = \frac{\partial G_2}{\partial z}.$$

Prove $G_1 - G_2 \equiv$constant.

3. Prove $f(z) = 1/z$ does not have a holomorphic anti-derivative on $A(0;1,2) = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

4. Let f be holomorphic function on an open set $U \subset \mathbb{C}$ and assume that f has a holomorphic antiderivative F. Does F have a holomorphic antiderivative on U?

5. If f has a holomorphic anti-derivative on the domain U in $\mathbb{C}$. If $\{z \in \mathbb{C} : z = e^{it}, 0 \leq t \leq 2\pi\} \subset U$ then

$$\int_0^{2\pi} f(e^{it})e^{it}dt = 0$$

6. Let f be holomorphic in the disc $D(0,r)$ (centered at 0 with radius r). For any $z \in D(0,r)$, we let

$$F(z) = \int_0^1 f(tz)zdt$$

Prove $F'(z) = f(z)$ on $D(0,r)$.

7. Compute the following integrals:

(a) $\qquad \oint_\gamma \overline{z}+z^2 dz, \quad \gamma$ is the unit square clockwise orientation (centered at 0)

(b) $\qquad \oint_\gamma \frac{z}{8+z^2}dz, \quad \gamma$ is the triangle with vertices$1+0i, 0+i, 0-i$ clockwise orientation .

(c) $\displaystyle\oint_\gamma z^j dz$, γ is the unit circle counter clockwise orientation (centered at 0)

where j is any integer.
8. Let U be an open set in $\mathbb{C}$. If $f : U \to \mathbb{C}$ has complex derivatives at each point of U, then f is continuous in U.
9. Let U be a simply connected domain in $\mathbb{C}$ and let $u \in C^1(U)$ and u is harmonic in $U \setminus \{z_0\}$ for some $z_0 \in U$. Prove there is holomorphic function f on U such that $\operatorname{Re} f = u$.
10. Compute the following line integrals:

(a) Let γ be the circle centered at 0 with radius 3 with counterclockwise orientation. Then evaluate

$$\oint_\gamma \frac{z^2}{z-1} dz$$

(b) Let γ be the circle centered at 0 with radius 1 with counterclockwise orientation. Then evaluate

$$\oint_\gamma \frac{z}{(z+4)(z-1+i)} dz$$

(c) Let γ be the circle centered at $2+i$ with radius 3 with clockwise orientation. Then evaluate

$$\oint_\gamma \frac{z(z+3)}{(z-8)(z+i)} dz$$

11. Let D be a bounded domain in $\mathbb{C}$ with C^1 boundary. Let γ be a parameterization of ∂D counterclockwise. If f is a continuous function on ∂D. Then

$$F(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w-z} dw, \quad z \in D$$

is holomorphic in D.

II. Cauchy Integral and Applications

7 Cauchy's theorem and Morera's theorem

The main purpose of this chapter is to develop a reproducing formula for holomorphic functions on a bounded domain. We will apply the formula to study entire holomorphic functions, etc..

7.1 Cauchy's theorems

Let D be a bounded domain in $\mathbb{C}$ with piecewise C^1 boundary ∂D which is parametrized with a counter clockwise orientation.

EXAMPLE 14 *Let $D = \{z \in \mathbb{C} : 1 < |z| < 2\}$. Then*

$$\partial D = \partial D(0,2) \cup (-\partial D(0,1))$$

where the $-$ sign denotes that $\partial D(0,1)$ is parametrized clockwise. Moreover,

$$\begin{aligned}&\int_{\partial D} f(z)dz + g(z)d\overline{z} \\ &= \int_0^{2\pi} f(2e^{it})d(2e^{it}) + g(2e^{it}d(2e^{-it}) - \int_0^{2\pi} f(e^{it})d(e^{it}) + g(e^{it})d(e^{-it})\end{aligned}$$

In particular,

$$\int_{\partial D} \frac{1}{z} dz = 0.$$

Proof. The first part is a straight forward application of the definition of the complex line integral. We calculate the last part as follows:

$$\int_{\partial D} \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{2e^{i\theta}} i2e^{i\theta} d\theta - \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = 2\pi i - 2\pi i = 0.$$

□

Notice that $\frac{1}{z}$ is holomorphic in D and continuous on $\overline{D}$. In general, one has the following Cauchy theorem

THEOREM 7.1 *(Cauchy's Theorem) Let D be a bounded domain in $\mathbb{C}$ with piecewise C^1 boundary. Let $f \in \mathcal{O}(D) \cap C(\overline{D})$. Then*

$$\int_{\partial D} f(z)\, dz = 0.$$

Proof. Applying Stokes' theorem, we have

$$\int_{\partial D} f(z)dz = \int_D d(f(z)dz) = \int_D \frac{\partial f}{\partial z} dz \wedge dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz = 0$$

This gives the proof of the theorem. ▯

EXAMPLE 15 *Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise C^1 simple closed curve with positive orientation. Let D be the region bounded by γ and suppose $0 \in D$. Then*

$$\int_\gamma \frac{1}{z} dz = 2\pi i.$$

Proof. Choose $\epsilon > 0$ small enough so that $D(0, 2\epsilon) \subset D$. Let

$$D_\epsilon = D \setminus \overline{D(0, \epsilon)}$$

Then

$$\begin{aligned}
\int_\gamma \frac{1}{z} dz &= \int_{\partial D} \frac{1}{z} dz \\
&= \int_{\partial D_\epsilon} \frac{1}{z} dz + \int_{\partial D(0,\epsilon)} \frac{1}{z} dz \\
&= 0 + \int_{\partial D(0,\epsilon)} \frac{1}{z} dz \\
&= \int_0^{2\pi} \frac{1}{\epsilon e^{i\theta}} d(\epsilon e^{i\theta}) \\
&= \int_0^{2\pi} i d\theta \\
&= 2\pi i
\end{aligned}$$

The proof is complete. ▯

EXAMPLE 16 *Evaluate*

$$\int_\gamma \frac{1}{z^2} \, dz,$$

where γ is the unit square in $\mathbb{C}$: $\{z : z = x + \pm 2i$ or $z = \pm 2 + iy : -2 \leq x, y \leq 2\}$.

Solution. Since $\frac{1}{z^2} = \left(-\frac{1}{z}\right)'$, we have $d(-\frac{1}{z}) = \frac{1}{z^2} dz$. Since γ is a closed curve in $\mathbb{C} \setminus \{0\}$, we have

$$\int_\gamma \frac{1}{z^2} \, dz = \int_\gamma d\left(-\frac{1}{z}\right) = 0.$$

EXAMPLE 17 *Evaluate*

$$\int_\gamma \frac{1}{z(z-1)}\, dz$$

where γ is the unit square in $\mathbb{C}$.

Solution. Since 1 is in the domain bounded by γ, we can mimic the proof of example 2 to show that $\int_\gamma \frac{1}{z-1} dz = 2\pi i$. Thus,

$$\int_\gamma \frac{1}{z(z-1)}\, dz = \int_\gamma \frac{1}{(z-1)} - \frac{1}{z}\, dz = 2\pi i - 2\pi i = 0$$

□

7.2 Morera's theorem

The following statement of Morera's theorem is, in some sense, the converse of Cauchy's theorem.

THEOREM 7.2 *(Morera's Theorem) Let $f \in C(D(0,r))$. Suppose for any simple closed piecewise C^1 curve γ in $D(0,r)$ that $\int_\gamma f(z)dz = 0$. Then f is holomorphic in $D(0,r)$.*

Proof. Let

$$F(z) = \int_0^1 f(tz)zdt = \int_0^z f(w)dw, \quad z \in D(0,r)$$

We will show that $F'(z) = f(z)$. It then follows that F is holomorphic in $D(0,r)$ because it is complex differentiable everywhere in $D(0,r)$. Thus, its derivative f is also holomorphic. For any $h \in \mathbb{C}$ small enough so that $z + h \in D(0,r)$ we have

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = \lim_{h\to 0} \frac{1}{h} \int_{[z,z+h]} f(z)dz = \lim_{h\to 0} \frac{1}{h} \int_0^1 f(z+th)hdt = f(z).$$ □

As an application, we can state the above Morera's theorem in a more general setting.

Corollary 7.3 *(Morera Theorem) Let D be a domain in $\mathbb{C}$ and let $f \in C(D)$. If for any disc $D(z_0,r)$ in D and any simple closed curve piecewise C^1 curve γ in $D(z_0,r)$ one has $\int_\gamma f(z)dz = 0$. Then f is holomorphic in D.*

EXAMPLE 18 *Let D be a domain in $\mathbb{C}$ and let $f \in C^1(D)$. Suppose for any $D(z_0, r) \subset D$ one has $\int_{\partial D(z_0,r)} f(z)dz = 0$. Then f is holomorphic in D.*

Proof. By Taylor's theorem,

$$f(z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)(z - z_0) + \frac{\partial f}{\partial \overline{z}}(z_0)\overline{(z - z_0)} + o(|z - z_0|).$$

Thus,

$$\begin{aligned} 0 &= \int_{\partial D(z_0,r)} f(z) - f(z_0))dz \\ &= \frac{\partial f}{\partial z}(z_0) \int_{\partial D(z_0,r)} (z - z_0)dz + \frac{\partial f}{\partial \overline{z}}(z_0) \int_{\partial D(z_0,r)} \overline{(z - z_0)}dz + o(r^2) \\ &= 2\pi r^2 \frac{\partial f}{\partial \overline{z}}(z_0) + o(r^2). \end{aligned}$$

This implies that $\frac{\partial f}{\partial \overline{z}}(z_0) = 0$ for all $z_0 \in D$. So f is holomorphic in D. ∎

8 Cauchy integral formula

8.1 Integral formula for C^1 and holomorphic functions

THEOREM 8.1 *(Cauchy Integral Formula) Let D be a bounded domain in $\mathbb{C}$ with piecewise C^1 boundary. Let $f \in C(\overline{D})$ and $\frac{\partial f}{\partial \overline{z}} \in C(\overline{D})$. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} dw - \frac{1}{2\pi i} \int_D \frac{1}{w - z} \frac{\partial f(w)}{\partial \overline{w}} d\overline{w} \wedge dw, \quad z \in D.$$

In particular, if f is holomorphic in D then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} dw, \quad z \in D.$$

Note: The last reproducing formula for holomorphic functions is called the Cauchy integral formula. It is a very important tool for study holomorphic functions.

Proof. Choose $\epsilon > 0$ small enough so that $D(z, 2\epsilon) \subset D$. Let

$$D_\epsilon = D \setminus \overline{D(z, \epsilon)}$$

By Stokes' theorem

$$\begin{aligned}
\int_{\partial D}\frac{f(w)}{w-z}dw &= \int_{\partial D(z,\epsilon)}\frac{f(w)}{w-z}dw+\int_{\partial D_\epsilon}\frac{f(w)}{w-z}dw\\
&= \int_{\partial D(z,\epsilon)}\frac{f(w)}{w-z}dw+\int_{D_\epsilon}\frac{1}{w-z}\frac{\partial f(w)}{\partial \overline{w}}d\overline{w}\wedge dw\\
&= \int_0^{2\pi}\frac{f(z+\epsilon e^{i\theta})}{\epsilon e^{i\theta}}i\epsilon e^{i\theta}d\theta+\int_{D_\epsilon}\frac{1}{w-z}\frac{\partial f(w)}{\partial \overline{w}}d\overline{w}\wedge dw\\
&= i\int_0^{2\pi} f(z+\epsilon e^{i\theta})\,d\theta+\int_{D_\epsilon}\frac{1}{w-z}\frac{\partial f(w)}{\partial \overline{w}}d\overline{w}\wedge dw.
\end{aligned}$$

Letting $\epsilon\to 0^+$ yields

$$\int_{\partial D}\frac{f(w)}{w-z}dw = 2\pi i f(z)+\int_D\frac{1}{w-z}\frac{\partial f(w)}{\partial \overline{w}}d\overline{w}\wedge dw.$$

which proves the first formula after rearranging the terms. If f is holomorphic in D then $\frac{\partial f}{\partial \bar{z}}=0$ in D. Therefore, one gets the Cauchy integral formula and the proof of the theorem is complete. ∎

8.2 Examples of evaluating line integrals

EXAMPLE 19 *Evaulate*

$$\int_{|z-1|=5}\frac{z^2-z^{100}}{(z-2)(z+1)}dz.$$

Solution Since $z=2,-1\in D(1,5)$ and

$$\frac{1}{(z-2)(z+1)}=\frac{1}{3}[\frac{1}{z-2}-\frac{1}{z+1}],$$

we have

$$\begin{aligned}
\int_{|z-1|=5}\frac{z^2-z^{100}}{(z-2)(z+1)}dz &= \frac{1}{3}\int_{|z-1|=5}\frac{z^2-z^{100}}{z-2}-\frac{z^2-z^{100}}{z+1}dz\\
&= \frac{2\pi i}{3}[2^2-2^{100}-((-1)^2-(-1)^{100})\\
&= -\frac{8\pi}{3}(2^{98}-1)
\end{aligned}$$

where the second to last equality follows from the Cauchy integral formula. ∎

8.3 Cauchy integral for kth derivative $f^{(k)}(z)$

THEOREM 8.2 *(Cauchy Integral Formula) Let D be a bounded domain in $\mathbb{C}$ with piecewise C^1 boundary and $f \in \mathcal{O}(D) \cap C(\overline{D})$. Then for any non-negative integer k,*

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{k+1}} dz, \quad z \in D.$$

Proof. By Cauchy Integral Formula, one has

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z} dw, \quad z \in D.$$

For $z_0 \in D$, choose $\epsilon > 0$ small enough so that $D(z_0, 2\epsilon) \subset D$, then

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D} f(w) \Big(\frac{1}{w-z}\Big)' dw = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^2} dw, \quad z \in D(z_0, \epsilon).$$

Similary,

$$f^{(k)}(z) = \frac{1}{2\pi i} \int_{\partial D} f(w) \frac{\partial^k}{\partial z^k} \Big(\frac{1}{w-z}\Big) dw = \frac{k!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{k+1}} dw.$$

The proof is complete. ∎

9 Application of the Cauchy integral formula

9.1 Mean value properties

Corollary 9.1 *(Mean value property) Let D be a domain in $\mathbb{C}$ and $f \in \mathcal{O}(D)$. Then for all $r > 0$ so that $\overline{D(z,r)} \subset D$,*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta.$$

In particular, if $f = u + iv$ is holomorphic in D then

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta \quad \text{and } v(z) = \frac{1}{2\pi} \int_0^{2\pi} v(z + re^{i\theta}) d\theta$$

Proof. Notice that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta + \frac{i}{2\pi}\int_0^{2\pi} v(z+re^{i\theta})d\theta &= \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta \\ &= \frac{1}{2\pi i}\int_{|w-z|=r}\frac{f(w)}{w-z}dw \\ &= f(z) \\ &= u(z)+iv(z).\end{aligned}$$

This gives the first equation and equating real and imaginary parts, we get the last two equations as well. ▯

Corollary 9.2 *(Mean value property) Let u be harmonic in $D(0,1)$. Then*

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta, \quad 0<r<1.$$

Proof. Since $D(0,1)$ is simply connected, there is a harmonic function v on D such that $f = u+iv$ is holomorphic in D. By the previous theorem, the proof is complete. ▯

EXAMPLE 20 *Let D be a domain in $\mathbb{C}$ and let $u \in C^2(D)$. Then u is harmonic in D if and only if*

$$\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta = u(z)$$

for all $z \in D$ and all $r>0$ so that $\overline{D}(z,r) \subset D$.

Proof. Since by Taylor's theorem

$$\begin{aligned}u(w) &= u(z) + \frac{\partial u}{\partial z}(z)(w-z) + \frac{\partial u}{\partial \overline{z}}(z)\overline{(w-z)} + \frac{1}{2}\frac{\partial^2 u}{\partial z^2}(z)(w-z)^2 \\ &\quad + \frac{1}{2}\frac{\partial^2 u}{\partial \overline{z}^2}(z)\overline{(w-z)}^2 + \frac{\partial^2 u}{\partial z\partial\overline{z}}(z)|w-z|^2 + o(|w-z|^2),\end{aligned}$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta = u(z) + r^2\frac{\partial^2 u}{\partial z\partial\overline{z}}(z) + o(r^2).$$

Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta = u(z)$$

implies that $\frac{\partial^2 u}{\partial z\partial\overline{z}}(z) = 0$. By mimicking the proof of corollary 9.2 for a disc $\overline{D(z_0,r)} \subset D$, we get the other direction of this theorem. This completes the proof of the conclusion. ▯

9.2 Entire holomorphic functions

Definition 9.3 *A function f is said to be entire holomorphic if f is holomorphic in the whole complex plane $\mathbb{C}$.*

Examples

The followings are examples of entire holomorphic functions
1) $p_n(z) = \sum_{k=0}^{n} a_k z^k$

2) $f(z) = e^z$
3) $f(z) = \cos(z) = (e^{iz} + e^{-iz})/2$
4) $f(z) = \sin(z) = (e^{iz} - e^{-iz})/(2i)$.

9.3 Liouville's Theorem

THEOREM 9.4 *(Liouville's Theorem) Any bounded entire function on $\mathbb{C}$ must be a constant.*

Proof. It suffices to prove $f'(z) = 0$ on $\mathbb{C}$. Assume that $|f(z)| \leq M$ on $\mathbb{C}$ for some constant M. By the previous theorem, for any $|z| < R$, one has

$$\begin{aligned} |f'(z)| &= \Big|\frac{1}{2\pi i}\int_{|w|=R}\frac{f(w)}{(w-z)^2}dw\Big| \\ &\leq \frac{1}{2\pi}\int_{|w|=R}\frac{|f(w)|}{|w-z|^2}|dw| \\ &\leq \frac{M}{2\pi}\int_0^{2\pi}\frac{R}{(R-|z|)^2}d\theta \\ &\leq \frac{MR}{(R-|z|)^2} \end{aligned}$$

Fix z and let $R \to \infty$. Then one has $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore, f is a constant. ∎

9.4 Application of Liouville's theorem

Proposition 9.5 *If f is non-constant entire holomorphic, then $f(\mathbb{C})$ is dense in $\mathbb{C}$.*

Proof. Suppose that f is not dense in $\mathbb{C}$. Then there is a $w \in \mathbb{C}$ and an $r > 0$ such that

$$D(w, r) \cap f(\mathbb{C}) = \emptyset.$$

Therefore, if we let $g(z) = \frac{1}{f(z)-w}$, then g is entire holomorphic and $|g(z)| \leq \frac{1}{r}$ on $\mathbb{C}$. Thus, by Liouville's theorem, g is constant. Therefore, f is constant. □

EXAMPLE 21 *If f is entire holomorphic and* $\operatorname{Re} f(z) < 0$*, then f must be a constant.*

Proof. Let $g(z) = \frac{1}{f(z)-1}$. Then g is entire holomorphic and $|g(z)| \leq 1$. By Liouville's theorem, g is a constant; thus so is f. □

9.5 Entire functions with sub-linear growth

Definition 9.6 *Let $f \in C(\mathbb{C})$. We say that f has sub-linear growth if*

$$\lim_{z\to\infty} \frac{f(z)}{z} = 0$$

THEOREM 9.7 *Any entire holomorphic function with sub-linear growth must be a constant.*

Proof. Since

$$f(z) = o(|z|), \quad \text{as } z \to \infty$$

by the Cauchy integral formula, one has

$$f'(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{f(w)}{(w-z)^2} = o(R) \frac{R}{(R-|z|)^2}, \quad |z| < R.$$

Let $R \to \infty$, one has that $f'(z) = 0$ on $\mathbb{C}$. Therefore, f is a constant on $\mathbb{C}$. □

9.6 Polynomial growth entire holomorphic functions

Definition 9.8 *Let f be an entire holomorphic function. We say that f has polynomial growth if there is a constant C and positive integer k such that*

$$|f(z)| \leq C(1 + |z|^k), \quad z \in \mathbb{C}.$$

THEOREM 9.9 *Any entire holomorphic function with polynomial growth must be a polynomial.*

Proof. Suppose that

$$|f(z)| \le C(1+|z|^k) = o(|z|^{k+1}), \quad z \in \mathbb{C}.$$

By the Cauchy Integral Formula, for fixed $|z| < R$, one has

$$f^{(k+1)}(z) = \frac{(k+1)!}{2\pi i} \int_{|w|=R} \frac{f(w)}{(w-z)^{k+2}} dw = \frac{o(R^{k+1})}{(R-|z|)^{k+1}} = \frac{o(R^{k+1})}{R^{k+1}} = 0.$$

as $R \to \infty$. Therefore, f is a polynomial of degree at most k. The proof is complete. ∎

10 Fundamental theorem of algebra

10.1 Lower bound for holomorphic polynomials

A holomorphic polynomial of degree n is:

$$p_n(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \ne 0.$$

Lemma 10.1 *Let*

$$p_n(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \ne 0.$$

be holomorphic polynomial of degree $n \ge 1$. Then there is a constant M such that

$$|p_n(z)| \ge \frac{|a_n|}{2}|z|^2, \quad |z| \ge M.$$

Proof. For any $|z| \ge 1 + 2\frac{\sum_{j=0}^{n-1} |a_j|}{|a_n|}$, we have

$$\begin{aligned} |p_n(z)| &\ge |z|^n(|a_n| - \sum_{j=0}^{n-1} |a_j||z|^{j-n}) \\ &\ge |z|^n(|a_n| - \sum_{j=0}^{n-1} |a_j||z|^{-1}) \\ &\ge |z|^n(|a_n| - \frac{|a_n|}{2}) \\ &= \frac{|a_n|}{2}|z|^n. \end{aligned}$$

This proves the lemma. ∎

10.2 Fundamental theorem of algebra

THEOREM 10.2 *Any holomorphic polynomial of degree $n \geq 1$ must have a root.*

Proof. Let

$$p_n(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \neq 0.$$

be a holomorphic polynomial of degree $n \geq 1$. Suppose that for all $z \in \mathbb{C}$ that $p_n(z) \neq 0$. Then $1/p_n(z)$ is entire holomorphic and

$$\Big|\frac{1}{p_n(z)}\Big| \leq \frac{2}{|a_n||z|^n} \leq \frac{2}{|a_n|}, \qquad \text{when } |z| \geq 1 + \frac{\sum_{j=0}^{n-1} |a_j|}{2|a_n|} =: M$$

Since $\frac{1}{p_n}$ is continuous on $\overline{D(0,M)}$, it is bounded on $\overline{D(0,M)}$. Therefore, there is a $B \geq \frac{2}{|a_n|}$ such that

$$\Big|\frac{1}{p_n(z)}\Big| \leq B, \quad z \in \mathbb{C}.$$

By Liouville's theorem, one has $\frac{1}{p_n}$ is a constant; thus p_n is also a constant. This contradicts that p_n has degree $n \geq 1$. ▯

Corollary 10.3 *Any holomorphic polynomial of degree $n \geq 1$ has n roots counting multiplicity.*

Proof. If z_1 is a zero of p_n, then

$$p_n(z) = (z - z_1)p_{n-1}(z)$$

where p_{n-1} is a polynomial of degree $n-1$. Mathematical induction implies the conclusion. ▯

By the proof of the corollary, any holomorphic polynomial p_n can be expressed as

$$p_n(z) = a_n(z - z_1) \cdots (z - z_n)$$

So, p_n is uniquely determined by a_n and its roots.

Corollary 10.4 *Let p_n be a holomorphic polynomial of degree $n \geq 1$. Then $p_n(\mathbb{C}) = \mathbb{C}$.*

11 Homework 3

Homework 3: Cauchy theorem and Cauchy Integral formula

i. Let $f(z) = z^2$. Calculate the integral of f along the circle $\partial D(2,1)$ given by
$$\int_0^{2\pi} f(2+e^{i\theta})d\theta$$
is not zero. Yet
$$\oint_{\partial D(2,1)} f(z)dz = 0$$
Give an explanation.

ii. Let f be a continuous function on $\partial D(0,1)$. Define
$$F(z) = \begin{cases} f(z), & \text{if } z \in \partial D(0,1) \\ \frac{1}{2\pi i}\oint_{\partial D(0,1)} \frac{f(\xi)}{\xi - z}d\xi, & \text{if } z \in D(0,1). \end{cases}$$
Is F continuous on $\overline{D(0,1)}$?

iii. Let f be a holomorphic polynomial. If
$$\oint_{\partial D(0,1)} f(z)\overline{z}^j dz = 0, \quad j = 0,1,2,\cdots,$$
then prove $f \equiv 0$.

iv. Let $f \in C^1(\mathbb{C})$ and suppose that
$$\oint_{\partial D(z_0,r)} f(z)dz = 0, \qquad \text{for any } z_0 \in \mathbb{C} \text{ and } r > 0.$$
Show that f is holomorphic in $\mathbb{C}$.

v. Give an example of C^2 function f on $\overline{D}(0,1)$ such that
$$\oint_{\partial D(0,r)} f(z)dz = 0,$$
but f is not holomorphic in $D(0,1)$.

vi. Let $u \in C^2(\mathbb{C})$. Prove that u is harmonic in $\mathbb{C}$ if and only if
$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta = u(z_0), \qquad \text{for all } z_0 \in \mathbb{C}, r > 0.$$

vii. Let f be holomorphic in the annulus $D(0,2) \setminus \overline{D}(0,1)$. Show that for any $1 < r < R < 2$, one has

$$\oint_{\partial D(0,r)} f(z)dz = \oint_{\partial D(0,R)} f(z)dz.$$

viii. Evaluate

$$\oint_{\partial D(0,5)} \frac{1}{z(z-1)(z-2)(z-3)(z-4)} dz$$

ix. Let $p_n(z)$ be a holomorphic polynomial of degree $n \geq 2$. Assume that $p_n(z) \neq 0$ when $|z| \geq R$. Prove that

$$\oint_{\partial D(0,R)} \frac{1}{p_n(z)} dz = 0.$$

x. Evaluate

$$\frac{1}{2\pi i} \oint_{\partial D(1,5)} \frac{z^2+z}{(z-2i)(z+3)} dz.$$

xi. Let $f \in C(\overline{D}(0,1))$ be holomorphic in $D(0,1)$ such that $|f(z)| \leq 1$ when $|z| = 1$. Prove that $|f(z)| \leq 1$ when $|z| < 1$.

xii. State the following theorems:

(a) Cauchy's Theorem

(b) Morera's Theorem

(c) Cauchy Integral formula.

xiii. Let $f(z) \in C(D(0,1))$ be so that for any triangular closed path γ in $D(0,1)$ we have

$$\int_\gamma f(z)dz = 0.$$

Prove that f is holomorphic in D.

II. Analytic functions

12 Series of complex numbers

12.1 Definition of the series of complex numbers

Let $c_n = a_n + ib_n$, the series of complex numbers

$$\sum_{n=1}^{\infty} c_n =: \sum_{n=1}^{\infty} a_n + i \sum_{n=1}^{\infty} b_n$$

We say that
(i) the series $\sum_{n=1}^{\infty} c_n$ converges if the both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.

(ii) the series $\sum_{n=1}^{\infty} c_n$ converges absolutely if $\sum_{n=1}^{\infty} |c_n|$ converges.

12.1.1 Tests for convergent series

THEOREM 12.1 *(Cauchy Criteria) $\sum_{n=1}^{\infty} c_n$ converges if and only if for any $\epsilon > 0$ there is a $N \in \mathbb{N}$ such that if $m > n \geq N$, then we have*

$$\Big| \sum_{k=n}^{m} c_k \Big| < \epsilon.$$

Corollary 12.2 *If $\sum_{n=1}^{\infty} c_n$ converges, then $\lim_{n\to\infty} c_n = 0$*

THEOREM 12.3 *(Root Test) Let*

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = r$$

Then

i. *If $r < 1$, then $\sum_{n=1}^{\infty} c_n$ converges absolutely;*

ii. *If $r > 1$, then $\sum_{n=1}^{\infty} c_n$ diverges;*

iii. *If $r = 1$, then the test fails.*

THEOREM 12.4 *(Ratio Test) Assume that $|c_n| > 0$ and*

$$r =: \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}$$

exists, then

i. *If* $r < 1$, *then* $\sum_{n=1}^{\infty} c_n$ *converges absolutely;*

ii. *If* $r > 1$, *then* $\sum_{n=1}^{\infty} c_n$ *diverges;*

iii. *If* $r = 1$, *then the test fails.*

THEOREM 12.5 *(Dirichlet's test for convergence of a series) Let* $\{c_n\}_{n=1}^{\infty}$ *be a sequence of complex numbers such that* $\{s_n =: \sum_{k=1}^{n} c_k\}_{n=1}^{\infty}$ *is bounded. Let* $b_n > 0$ *be a decreasing sequence and* $\lim_{n\to\infty} b_n = 0$. *Then* $\sum_{n=1}^{\infty} c_n b_n$ *converges.*

- **Abel's identity:**

$$\sum_{k=1}^{n} c_k b_k = s_n b_n + \sum_{k=1}^{n-1} s_k (b_k - b_{k+1})$$

Proof. Define $s_0 = 0$. Then

$$\begin{aligned}\sum_{k=1}^{n} c_k b_k &= \sum_{k=1}^{n} (s_k - s_{k-1}) b_k \\ &= \sum_{k=1}^{n} s_k b_k - \sum_{k=1}^{n-1} s_k b_{k+1} \\ &= s_n b_n - s_0 b_1 + \sum_{k=1}^{n-1} s_k (b_k - b_{k+1}).\end{aligned}$$

Proof of Theorem 12.5

Since $\{s_n\}$ is bounded, there is an $M > 0$ such that $|s_n| \le M$ for all $n \in \mathbb{N}$. For any $m > n$, we have

$$\begin{aligned}\Big|\sum_{k=n}^{m} c_k b_k\Big| &= \Big|s_m b_m - s_{n-1} b_n + \sum_{k=n}^{m-1} s_k (b_k - b_{k+1})\Big| \\ &\le M(b_m + b_n) + M \sum_{k=n}^{m-1} (b_k - b_{k+1}) \\ &= M(b_m + b_{n-1} + b_n - b_m) \\ &= M(b_n + b_{n-1}) \\ &\to 0\end{aligned}$$

as $n, m \to \infty$. By the Cauchy Criteria, the proof of the theorem is complete. ▯

12.1.2 Examples of Dirichlet's test

EXAMPLE 22 *For* $|z| < 1$,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

converges.

EXAMPLE 23

$$\sum_{n=0}^{\infty} \frac{z^n}{n}$$

converges for all $z \in \overline{D}(0,1) \setminus \{1\}$.

Proof. Since

$$s_n(z) =: \sum_{k=1}^{n} z^k = \frac{z - z^{n+1}}{1-z}$$

we have, for each $z \in \overline{D}(0,1) \setminus \{1\}$, $\{s_n(z)\}$ is a bounded sequence and $\{1/n\}$ is a decreasing sequence approaching zero. Therefore, $\sum_{n=1}^{\infty} \frac{z_n}{n}$ converges. □

12.2 Power series

12.2.1 Radius of convergence for a power series

For a power series

$$\sum_{n=0}^{\infty} a_n z^n,$$

we define

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

We call R the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ because of the following theorem.

THEOREM 12.6 *With the notation above, we have*

(i) $\sum_{n=1}^{\infty} a_n z^n$ *converges absolutely for all* $z \in D(0, R)$;

(ii) $\sum_{n=1}^{\infty} a_n z^n$ *converges absolutely and uniformly for all* $z \in \overline{D}(0, r)$ *with any* $0 < r < R$;

(iii) $\sum_{n=1}^{\infty} a_n z^n$ *diverges for all* $|z| > R$;

Proof. For part (i), since

$$\rho = \limsup_{n\to\infty} \sqrt[n]{|a_n||z|^n} = \frac{|z|}{R}$$

by the Root Test, $\sum_{n=0}^{\infty} a_n z^n$ converges if $\rho < 1$, which is equivalent to when $|z| < R$. So (i) is proved.

For part (ii), for any $\epsilon > 0$, since $\sum_{n=0}^{\infty} |a_n| r^n$ converges, there is an N such that if $m > n \geq N$, one has

$$\sum_{k=n+1}^{m} |a_k| r^k < \epsilon.$$

Therefore, for any $z \in \overline{D}(0, r)$, we have

$$\Big| \sum_{k=n+1}^{m} a_k z^k \Big| \leq \sum_{k=n+1}^{m} |a_k| r^k < \epsilon.$$

Therefore, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely and uniformly on $\overline{D}(0, r)$.

For part (iii), if $|z| > R$, then

$$\limsup_{n\to\infty} |a_n||z|^n > 1.$$

So, $\sum_{n=1}^{\infty} a_n z^n$ diverges. Therefore, the proof of the theorem is complete. ∎

12.2.2 Examples of power series

EXAMPLE 24 *Find the radius of convergence of the following series*

$$\textbf{1}.\ \sum_{n=0}^{\infty} \frac{z^n}{n!}; \qquad \textbf{2}.\ \sum_{n=1}^{\infty} \frac{z^n}{n^2}; \qquad \textbf{3}.\ \sum_{n=0}^{\infty} \frac{z^{n^2}}{3^n} \quad \textit{and} \ \ \textbf{4}.\ \sum_{n=0}^{\infty} \frac{n + i^n}{2^n} z^n.$$

Solution. For problem 1, since

$$\limsup_{n\to\infty} \sqrt[n]{1/n!} = 0, \quad R = \infty.$$

Thus, $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely on $\mathbb{C}$;

For problem 2, since

$$\limsup_{n\to\infty} \sqrt[n]{1/n^2} = 1, \quad R = 1.$$

Thus, $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$ converges absolutely in $|z| < 1$.

For problem 3. since

$$\limsup_{n\to\infty} \sqrt[n^2]{1/3^n} = 1, \quad R = 1.$$

Thus, $\sum_{n=0}^{\infty} \frac{z^n}{3^n}$ converges absolutely on $|z| < 1$.

For problem 4, since

$$\limsup_{n\to\infty} \sqrt[n]{|n + i^n|/2^n} = \frac{1}{2}, \quad R = 2.$$

Thus, $\sum_{n=0}^{\infty} \frac{n+i^n}{2^n} z^n$ converges absolutely on $|z| < 2$.

EXAMPLE 25 *Discuss the convergence of the following power series*

$$\mathbf{1}.\ \sum_{n=0}^{\infty} \frac{z^{n^2}}{3^n}; \qquad \mathbf{2}.\ \sum_{n=0}^{\infty} \frac{z^n}{n3^n} \quad \textit{and} \quad \mathbf{3}.\sum_{n=0}^{\infty} \frac{n+i^n}{2^n} z^n.$$

Solution.

i. The radius of convergence is 1 by the previous problem. It remains to check its convergence on $\partial D(0,1)$. Since $\sum_{n=0}^{\infty} \frac{1}{3^n}$ converges, $\sum_{n=0}^{\infty} \frac{z^{n^2}}{3^n}$ converges on $\overline{D(0,1)}$ and diverges for all z with $|z| > 1$.

ii. By the root test, the radius of convergence of $\sum_{n=0}^{\infty} \frac{z^n}{n\,3^n}$ is 3. The series diverges at $z = 3$ and for each $z \in \partial D(0,3) \setminus \{3\}$,

$$\sum_{k=0}^{n} \frac{z^k}{3^k} = \frac{(1 - (z/3)^{n+1})}{1 - z}$$

is bounded and $(1/n)_{n=1}^{\infty}$ is a decreasing sequence approaching zero. Therefore,

$$\sum_{k=0}^{\infty} \frac{z^n}{n3^n}$$

converges for all z with $|z| = 3$ except for $z = 3$. Thus, $\sum_{n=0}^{\infty} \frac{z^n}{n3^n}$ converges on $\overline{D}(0,3) \setminus \{3\}$.

iii. $\sum_{n=0}^{\infty} \frac{(n+i^n)}{2^n} z^n$ has a radius of convergence of 2 and for each $|z| = 2$, one has

$$\lim_{n\to\infty} \frac{n+i^n}{2^n} z^n = \infty.$$

Therefore, the series converges on $D(0,2)$ and diverges for all $z \in \mathbb{C}$ with $|z| \geq 2$.

∎

12.2.3 Examples of functions with power series expansions

1) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

2) $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$

3) $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$

12.2.4 Differentiation of power series

THEOREM 12.7 *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power with the radius of the convergence $R > 0$. Then*

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}, \quad z \in D(0,R)$$

In particular, f is holomorphic in $D(0,R)$.

Proof. Since the radius of convergence of $\sum_{n=1}^{\infty} a_n z^n$ is R, the radius of convergence of $\sum_{n=1}^{\infty} n a_n z^{n-1}$ is also R. Let $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. It suffices to prove that $f'(z) = g(z)$.

For each $z \in D(0,R)$, let $r = \frac{|z|+R}{2} < R$. For each h with $|h| < \frac{R-|z|}{2}$, we have $z, z+h \in D(0,r)$. For any $\epsilon > 0$, there is an N such that

$$\sum_{k=N}^{\infty} |a_k| k r^{k-1} < \epsilon$$

By the mean-value theorem from calculus,

$$|(z+h)^n - z^n| \leq |h| n r^{n-1}$$

Since

$$\lim_{h\to 0}\frac{1}{h}\Big(\sum_{k=0}^{N-1}(z+h)^k-\sum_{k=0}^{N-1}z^k\Big)=\sum_{k=1}^{N-1}a_kkz^{k-1}$$

There is $\delta>0$ such that $\delta<\frac{R-|z|}{2}$ and if $|h|<\delta$ then

$$\Big|\frac{1}{h}\Big(\sum_{k=0}^{N-1}(z+h)^k-\sum_{k=0}^{N-1}z^k\Big)-\sum_{k=1}^{N-1}a_kkz^{k-1}\Big|<\epsilon$$

Therefore, when $|h|<\delta$, we have

$$\begin{aligned}\Big|\frac{f(z+h)-f(z)}{h}-g(z)\Big| &\leq \Big|\frac{1}{h}\Big(\sum_{k=0}^{N-1}(z+h)^k-\sum_{k=0}^{N-1}z^k\Big)-\sum_{k=1}^{N-1}a_kkz^{k-1}\Big|\\ &\quad +2\sum_{k=N}^{\infty}|a_k|kr^{n-1}\\ &< 3\epsilon.\end{aligned}$$

Therefore,

$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=g(z).$$

The proof is complete. ▯

By an induction argument, one has

$$f^{(k)}(z)=\sum_{n=k}^{\infty}n(n-1)\cdots(n-k+1)a_nz^{n-k},\quad z\in D(0,R).$$

Evaluating at $z=0$, one has

$$a_k=\frac{f^{(k)}(0)}{n!}.$$

12.3 Analytic Functions

12.3.1 Definition of an analytic function

Let D be a domain in $\mathbb{C}$, f is a function on D. For $z_0\in D$, we say that f is analytic at z_0 if there is $\delta>0$ such that

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n,\quad \text{for all } z\in D(z_0,\delta).$$

We say that f is analytic in D if f is analytic at every point $z_0\in D$.

12.3.2 Connection between holomorphic and analytic functions

THEOREM 12.8 *Let D be a domain in $\mathbb{C}$ and f a function on D. Then f is analytic on D if and only if f is holomorphic in D.*

Proof. It is easy to see from the previous theorem that if f is analytic in D then $f'(z)$ exists for each $z \in D$. Thus f is holomorphic in D.

Conversely, assume that f is holomorphic in D. Without loss of generality, we may assume that D is bounded, ∂D is piecewise C^1 and $f \in C(\overline{D})$. Otherwise, we work on any domain D_1 with $\overline{D}_1 \subset D$ and then let $D_1 \to D$. Thus, for $z_0 \in D$ and $z \in D(z_0, d(z_0))$, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z} dw \\ &= \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z_0-(z-z_0)} dw \\ &= \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z_0)(1-\frac{(z-z_0)}{w-z_0})} dw \\ &= \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z_0)} \sum_{n=0}^{\infty} \Big(\frac{z-z_0}{w-z_0}\Big)^n dw \\ &= \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z_0)^{n+1}} dw (z-z_0)^n \\ &= \sum_{n=0}^{\infty} a_n (z-z_0)^n, \end{aligned}$$

where

$$a_n = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z_0)^{n+1}} dw = \frac{f^{(n)}(z_0)}{n!}.$$

This power series converges on $D(z_0, d(z_0))$. Therefore, f is analytic on D. $\square$

Remark. Complex analyticity is a global property while real analyticity is local property.

EXAMPLE 26

i) If f is entire holomorphic then

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad z \in \mathbb{C}.$$

ii) $f(x) = \frac{1}{1+x^2}$ *is real analytic in* $\mathbb{R}$*, but*

$$f(x) = \sum_{k=0}^{\infty} (-1)^k x^{2k}, \quad |x| < 1$$

and the series diverges when $|x| \geq 1$*.*

EXAMPLE 27 *Find the power series of* $f(z) = \frac{1}{(1-z)}$ *about* $z_0 = 0$ *and* $z_0 = 1/2$*.*

Solution.1) For $z_0 = 0$, we have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

2) For $z_0 = 1/2$, we have

$$\frac{1}{1-z} = \frac{1}{1-\frac{1}{2}-(z-\frac{1}{2})} = \frac{2}{1-2(z-\frac{1}{2})} = \sum_{n=0}^{\infty} 2^{n+1}(z-\frac{1}{2})^n, \quad |z-\frac{1}{2}| < 1/2.$$

$\square$

EXAMPLE 28 *Let* $f(z) = 1/(1-z)(z+i)(z-1-i)$*. Find the radius of the convergence of*

$$f(z) = \sum_{n=0}^{\infty} a_n (z+1+2i)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad z_0 = -1-2i$$

Solution. By theorem 12.8, the power series centered at z_0 converges in the largest disk centered at z_0 where f is holomorphic. Therefore

$$\begin{aligned} R &= \min\{\text{dist}(z_0, \{1, -i, 1+i\}) \\ &= \min\{|1-z_0|, |-i-z_0|, |1+i-z_0|\} \\ &= \min\{|2+2i|, |i+1|, |2+3i|\} \\ &= \sqrt{2}. \quad \square \end{aligned}$$

13 Homework 4

Homework 4: Power series and analytic functions

i. Let $\gamma : [0,1] \to \mathbb{C}$ be any C^1 curve. Define

$$f(z) = \oint_\gamma \frac{1}{\xi - z} d\xi.$$

Prove that f is holomorphic on $D = \mathbb{C} \setminus \{\gamma(t) : t \in [0,1]\}$.

In the case $\gamma(t) = t$, show that f cannot be extended to a continuous function on $\mathbb{C}$.

ii. Use the Morera's theorem to prove: *If $\{f_n\}$ is a sequence of holomorphic functions on a domain D and if the sequence converges uniformly on compact subsets of D to a limit function f, then f is holomorphic on D.*

iii. Let D be a domain in $\mathbb{C}$ and f be continuous on D. Suppose for any triangle Γ in D, one has

$$\oint_\Gamma f(z) dz = 0.$$

Prove that f is holomorphic in D.

iv. Find the (complex) power series for $z^2/(1 - z^2)^3$ about $z = 0$ and determine its radius of convergence.

v. Determine the disc of convergence for each of the following series. Then determine the points on the boundary of the disc of convergence where the series converges.

(a) $\sum_{k=3}^\infty kz^k$; (b) $\sum_{k=2}^\infty k^{\log k}(z+1)^k$; (c) $\sum_{k=2}^\infty (\log k)^{\log k}(z-3)^k$,

(d) $\sum_{k=0}^\infty p(k)z^k$ where p is a polynomial; (e) $\sum_{k=1}^\infty 3^k(z+2i)^k$;

(f) $\sum_{k=0}^\infty \frac{k}{k^2+1} e^{-k} z^k$; (g) $\sum_{k=1}^\infty \frac{1}{k!}(z-5)^k$; (h) $\sum_{k=1}^\infty k^{-k} z^k$.

vi. Consider the power series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n^2+2n+3}.$$

(i) Find the radius R of convergence; (ii) Discuss the convergence when $z = Re^{i2r\pi}$ where $r = p/q$ is a rational.

vii. Prove that the function

$$f(z) = \sum_{j=0}^{\infty} 2^{-j} z^{(2^j)}$$

is holomorphic in $D(0,1)$ and continuous on $\overline{D}(0,1)$.

viii. Prove or disprove: If f is holomorphic in $D(0,1)$ and f^2 is a holomorphic polynomial in $D(0,1)$ then f is a holomorphic polynomial.

ix. Let $\sum_{n=0}^{\infty} a_n z^n$ have a radius of convergence $r > 0$. What is the radius of convergence for $\sum_{n=1}^{\infty} (a_n + \epsilon) z^n$ where $\epsilon > 0$ is very small.

x. Let f be bounded holomorphic in a domain D. Let $z_0 \in D$ and $\delta(z)$ is the distance from z to ∂D. Prove that

$$\left| f^{(k)}(z_0) \right| \le \frac{k!}{\delta(z_0)^k} \sup_D |f|$$

xi. Suppose that $f : D(0,1) \to \mathbb{C}$ is holomorphic and that $|f| \le 2$. Derive an estimate for

$$\left| f^{(3)}(\frac{i}{3}) \right|.$$

xii. Find the power series expansion about $z = z_0$ for each of the following holomorphic functions and determine its radius of convergence.

(a) $f(z) = 1/(1+2z), \quad z_0 = 0$; (b) $f(z) = z^2/(4-z), \quad z_0 = i$;

(c) $f(z) = 1/z, \quad z_0 = 2-i$; (d)$f(z) = (z-1/2)/(1-z/2), \quad z_0 = 0$.

xiii. Let $f(z)$ be entire such that $\operatorname{Re} f(z) \neq 0$ on $\mathbb{C}$. Prove or disprove $f(z)$ is a constant.

13.1 Uniqueness for analytic functions

• Basic version of the uniqueness theorem

Proposition 13.1 *Let D be a domain in $\mathbb{C}$ and f holomorphic in D such that $f(z) \equiv 0$ on $D(p,\delta) \subset D$ for some $\delta > 0$ and some $p \in D$. Then $f \equiv 0$ on D.*

Proof. Let $Z_D(f)$ denote the zero set of f in D. Then $D(p,\delta) \subset Z_D(f)$. It is clear that $Z_D(f)$ is relatively closed set in D. It suffices to prove that $Z_D(f)$ is open in D. Suppose not. Since $\mathrm{Int}(Z_D(f)) \neq \emptyset$ and D is connected, there is a $z_0 \in D \setminus \mathrm{Int}(Z_D(f))$ which is a limit point of the interior of $Z_D(f)$. For each point $z \in Int(Z_D(f))$, $f^{(k)}(z) = 0$. Therefore, by continuity of the derivatives of f, $f^{(k)}(z_0) = 0$ for all $k \geq 0$. Since f is analytic in D, there is an $r > 0$ such that $D(z_0, r) \subset D$ and

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad z \in D(z_0, r)$$

where $a_n = \frac{f^{(n)}(z_0)}{n!}$. Then $f \equiv 0$ on $D(z_0, r)$. This contradicts that z_0 is not an interior point of $Z_D(f)$. So $Z_D(f)$ is open. Since D is connected, one has $Z_D(f) = D$. □

Proof. (Second proof) For any $z \in D$, since D is connected, we can choose a piecewise C^1 curve $\gamma : [0,1] \to D$ such that $\gamma(0) = p =: z_0$ and $\gamma(1) = z$. Let $\delta_0 = \mathrm{dist}(\gamma([0,1]) : \partial D) > 0$. Choose n-points in $\gamma([0,1])$ such that $\gamma([0,1]) \subset \cup_{j=0}^{n} D(z_j, \delta_1/4)$ with $\delta_1 = \min\{\delta_0, \delta\}$ and $D(z_j, \delta_1/4) \cap D(z_{j+1}, \delta_1/4) \neq \emptyset$ for $0 \leq j \leq n-1$.
We have $f(z) \equiv 0$ on $D(z_0, \delta)$ and

$$f(z) = \sum_{k=0}^{\infty} a_{j,k} (z - z_j)^k, \quad z \in D(z_j, \delta_1), \quad j = 0, 1, 2, \cdots, n.$$

If $D(z_j, \delta_1/4) \cap D(z_k, \delta_1/4) \neq \emptyset$ then $z_k \in D(z_j, \delta_1/2)$ and $z_j \in D(z_k, \delta_1/2)$. In particular, since $f \equiv 0$ on $D(z_0, \delta_1)$ and $D(z_0, \delta_1/4) \cap D(z_1, \delta_1/4) \neq \emptyset, f^{(k)}(z_1) = 0$ for all k. By the power series expansion of f centered at z_1, $f \equiv 0$ on $D(z_1, \delta_1)$. Repeating this argument $n-1$ times, $f \equiv 0$ on $D(z_n, \delta_1)$. Therefore, $f(z) = 0$ □.

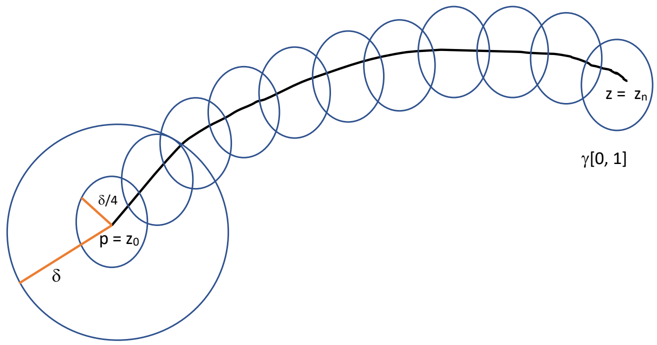


Figure 1: Disc Chain

• Factorization theorem

THEOREM 13.2 *Let D be a domain in $\mathbb{C}$ and $f \in \mathcal{O}(D)$ a non-constant function. If $z_0 \in D$ such that $f(z_0) = 0$, then there is a holomorphic function g on D and positive integer k such that $g(z_0) \neq 0$ and*

$$f(z) = (z - z_0)^k g(z)$$

Proof. Since f is not identically zero on D, we can let k be the smallest positive integer such that

$$f^{(k)}(z_0) \neq 0.$$

Then $k \geq 1$ and

$$f(z) = \sum_{n=k}^{\infty} a_n(z - z_0), \quad z \in D(z_0, d(z_0))$$

where $d(z_0) = dist(z_0, \partial D)$. Then

$$g(z) = \frac{f(z)}{(z - z_0)^k}$$

is holomorphic in D, $g(z_0) \neq 0$, and $f(z) = (z - z_0)^k g(z)$. □

• Uniqueness theorem of holomorphic functions

Corollary 13.3 *Let f be a non-zero holomorphic function on a domain D. Let $Z_D(f)$ be the zero set of f in D. Then* *i) Every point $z_0 \in Z_D(f)$ is an isolated point of $Z_D(f)$;*

ii) $Z_D(f)$ is a discrete set.

Proof. If $z_0 \in Z_D(f)$ then there is a holomorphic function g on D and a positive integer k such that

$$f(z) = (z - z_0)^k g(z)$$

with $g(z_0) \neq 0$. Then there is $r > 0$ such that $D(z_0, r) \subset D$ and $g(z) \neq 0$ in $D(z_0, r)$. Thus, $Z_D(f) \cap D(z_0, r) = \{z_0\}$. ▯

EXAMPLE 29 *Let f be an entire function such that $|f(z)| \geq 1$ when $|z| > 17$. Prove that f is a polynomial.*

Proof. First suppose $|Z_{D(0,17)}(f)| = \infty$. Then since $\overline{D(0,17)}$ is compact, there is a sequence $\{z_n\} \subset Z_{D(0,17)}(f)$ which converges to a $z_\infty \in \overline{D(0,17)}$. By continuity, $f(z_\infty) = 0$. But then z_∞ is a non-isolated zero of f, which contradicts Corollary 13.3. So without loss of generality, the zero set of f on $|z| < 17$ is a finite set, say, $z_1, \cdots, z_n$ counting multiplicity. We will show that f is a polynomial of degree at most n. Then

$$g(z) = \frac{f(z)}{\prod_{j=1}^n (z - z_j)} \neq 0, \quad z \in \mathbb{C}$$

and $h(z) = 1/g(z)$ is holomorphic in $\mathbb{C}$ and

$$|h(z)| \leq \prod_{j=1}^n |z - z_j| \leq (|z| + 17)^n \leq 2^n |z|^n, \quad |z| \geq 17.$$

Therefore,

$$|h(z)| \leq C(1 + |z|^n)$$

is an entire function of polynomial growth .Thus, h is a polynomial p_k of degree $k \leq n$. Therefore,

$$f(z) = \frac{\prod_{j=1}^n (z - z_j)}{p_k(z)}.$$

Since $z_1, \cdots, z_n$ are the zeros of f counting multiplicity and f is entire, p_k must be a constant, Thus, f is a polynomial of degree n. ▯

14 Uniqueness theorem and applications

Uniqueness Theorem

THEOREM 14.1 *Let D be a domain in $\mathbb{C}$, and f a holomorphic function on D with zero set $Z_D(f)$ in D. If $Z_D(f)$ has an accumulation (or limit) point in D, then f must be identically zero on D.*

Proof. Assume $Z_D(f)$ has an accumulation point $z_0 \in D$. If $f \not\equiv 0$ then there is $\delta > 0$ such that $f(z) \neq 0$ on $D(z_0, \delta) \setminus \{z_0\}$. This contradicts that z_0 is an accumulation point of $Z_D(f)$. Therefore, $f \equiv 0$ on D since D is connected. ▯

Corollary 14.2 *Let D be a domain in $\mathbb{C}$. If f and g are holomorphic functions on D such that $f(z) = g(z)$ on a set K which has an accumulation (or limit) point in D, then $f \equiv g$.*

Proof. Apply the previous theorem to $f - g$. Then $f - g \equiv 0$, i.e. $f \equiv g$ on D. ▯

EXAMPLE 30 *Let f be holomorphic in $D(0,1)$ such that $f(i/n) = \frac{1}{n^2}$, $\quad n = 1, 2, 3, \cdots$. What is f exactly?*

Solution. Let $g(z) =: -z^2$ is holomorphic in $D(0,1)$. Notice that $f(i/n) = g(i/n)$ for $n = 1, 2, 3, \cdots$. and $\{i/n : n \in \mathbb{N}\}$ has an accumulation point $z_0 = 0 \in D(0,1)$. By the corollary to the uniqueness theorem, $f(z) \equiv z^2$ on $D(0,1)$. ▯

EXAMPLE 31 *Let f be holomorphic in $D(0,1)$ such that*

$$f''(1/n) = f(1/n), \quad n \in \mathbb{N}.$$

What is f exactly?

Solution. f and f'' are holomorphic in $D(0,1)$ and are equal on $\{1/n : n \in \mathbb{N}\}$, which has an accumulation point $0 \in D(0,1)$. By the uniqueness theorem,

$$f(z) \equiv f''(z), \quad z \in D(0,1).$$

Method 1. Consider $z = x \in (-1, 1)$ and the equation

$$y(x) = y''(x).$$

We know that all of the solutions to this ODE are

$$y(x) = C_1 e^x + C_2 e^{-x}, \quad x \in (-1, 1).$$

Therefore,

$$f(z) = C_1 e^z + C_2 e^{-z}, \quad z = x \in (-1, 1).$$

By the uniqueness theorem,

$$f(z) = C_1 e^z + C_2 e^{-z}, \quad z \in D(0, 1).$$

Method 2. Without loss of generality,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = f''(z) = \sum_{n=2}^{\infty} a_n n(n-1) z^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2) z^n, \quad z \in D(0, 1)$$

This implies that

$$a_n = (n+1)(n+2)a_{n+2}, \quad n - 0, 1, 2, 3, \cdots$$

Thus

$$a_{2n} = \frac{a_0}{(2n)!}, \quad a_{2n+1} = \frac{a_1}{(2n+1)!}, \quad n = 0, 1, 2, 3, \cdots$$

Therefore,

$$\begin{aligned} f(z) &= a_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} + a_1 \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \\ &= a_0 \frac{e^z + e^{-z}}{2} + a_1 \frac{e^z - e^{-z}}{2} \\ &= \frac{a_0 + a_1}{2} e^z + \frac{a_0 - a_1}{2} e^{-z} \end{aligned}$$

on $D(0, 1)$. ∎

Corollary 14.3 *Let f be a non-constant holomorphic function on a domain D in $\mathbb{C}$. Then $Z_D(f)$ is at most countable.*

Proof. If $z \in Z_D(f)$, then there is a $\delta_z > 0$ such that $Z_D(f) \cap D(z, \delta_z) = \{z\}$. Choose Let D_n be a sequence of compact sets in D such that $D = \cup_{n=1}^{\infty} D_n$. Since $Z_{D_n}(f) = Z_D(f) \cap D_n$ is compact and $\{D(z, \delta_z) : z \in Z_{D_n}(f)\}$ is an open cover of $Z_{D_n}(f)$, so $Z_{D_n}(f)$ is a finite set. Thus,

$$Z_D(f) = \cup_{n=1}^{\infty} Z_{D_n}(f)$$

is at most countable. □

EXAMPLE 32 *Let f be a holomorphic function on $D(0,1)$ such that*

$$(-1, 1) \subset \cup_{k=1}^{\infty} Z_{D(0,1)}(f^{(k)}).$$

Prove that f is a polynomial.

Proof. Since $(-1, 1)$ is uncountable and since

$$(-1, 1) \subset \cup_{k=1}^{\infty} Z_{D(0,1)}(f^{(k)}),$$

we have that there is a $k \in \mathbb{N}$ such that $Z_{D(0,1)}(f^{(k)})$ is uncountable. By the previous corollary, for that k, $f^{(k)} \equiv 0$ on $D(0,1)$. Then

$$f(z) = a_0 + a_1 z + \cdots + a_k z^k$$

is a polynomial of degree at most k. □

EXAMPLE 33 *Let f be entire such that*

$$\mathbb{Q}^c \subset \cup_{k=0}^{\infty} Z_{\mathbb{C}}(f^{(k)})$$

What can one say about f ?

15 Homework 5

Homework 5: Analytic functions, zero set and uniqueness theorem

i. Let $f(z)$ be an entire holomorphic function such that $|f(z)| \leq |z|^{11/2}$ when $|z| \geq 100$. Prove that f is a polynomial of degree at most 5.

ii. Suppose that f and g are two entire holomorphic functions such that $|f(z)| \leq |g(z)|$ on $\mathbb{C}$. Prove that $f(z) = cg(z)$ for all $z \in \mathbb{C}$ and some constant c.

iii. Let $f(z)$ be an entire holomorphic function such that

$$(Ref(z))^2 \neq (Imf(z))^2, \quad z \in \mathbb{C}.$$

Show that f(z) must be a constant.

iv. Prove or disprove there is a holomorphic function $f(z)$ on the unit disk $D(0,1)$ such that

$$f(\frac{i}{n}) = \frac{1}{n^2}, \text{ for all } n = 1, 2, 3, \cdots.$$

v. Let $f(z)$ be holomorphic in a domain D such that

$$(-\infty, \infty) \subset \cup_{k=1}^{\infty} \{z \in D : f^{(k)}(z) = 0\}$$

Prove that $f(z)$ is a polynomial.

vi. (a) Prove or disprove that if $f(z)^2$ is holomorphic in $D(0,1)$ then $f(z)$ is holomorphic in $D(0,1)$;

(b)Prove or disprove that if $f \in C^1(D(0,1))$ and $f(z)^2$ is holomorphic in $D(0,1)$ then $f(z)$ is holomorphic in $D(0,1)$.

vii. Find the radius of the convergence of $f(z) = \frac{1}{z^2+1}$ as a power series around $z_0 = 2$ (Hint: you don't need to write down the power series to solve this problem).

viii. Given an example of a power series $f(z) =: \sum_{n=0}^{\infty} a_n z^n$ converges for every $z \in \mathbb{C}$ and has infinite many zeros, but $f(z) \not\equiv 0$.

ix. Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of holomorphic functions on $D(0,1)$ such that $|f_j| < j^{-2}$ on $D(0,1)$. Prove

$$\sum_{j=1}^{\infty} f_j(z)$$

defines a bounded holomorphic function on $D(0,1)$.

x. Let $\phi : D(0,1) \to D(0,1)$ be holomorphic and satisfy that $\phi(0) = 0$ and $\phi'(0) = 1$. Let

$$\phi_1(z) = \phi(z),\ \phi_2(z) = \phi \circ \phi(z), \cdots, \phi_j(z) = \phi \circ \phi_{j-1}(z), \cdots.$$

Suppose that $\{\phi_j\}$ converges uniformly on any compact subset of $D(0,1)$. What can you say about ϕ?

xi. Let f be an entire holomorphic function such that

$$\lim_{n\to\infty} f^{(n)}(z) = g(z)$$

uniformly on any compact subset of $\mathbb{C}$. Find g (all possible g).

xii. If f is holomorphic in $D(0,1)$ such that

$$f''(\frac{1}{n}) + f'(\frac{1}{n}) - 6f(\frac{1}{n}) = 0, \quad n = 1, 2, 3, \cdots.$$

Find f (all possible f).

16 Midterm Review

Midterm Review:

1. Holomorphic, Analytic Functions

i. **Holomorphic functions:** For $f = u + iv \in C^1(D)$, f is holomorphic in D if and only if

$$\frac{\partial f}{\partial \overline{z}} = 0 \quad \text{on} \quad D \iff \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases} \quad \text{on} \quad D.$$

ii. **Harmonic functions:** $u \in C^2(D)$ is harmonic in D if and only if $\Delta u = 0$ on D. Here,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \overline{z}}$$

iii. **Existence of harmonic conjugate:** If $f = u + iv$ is holomorphic in a domain D then u and v are harmonic in D; the converse may not be true.

- Let u be a real-valued harmonic function on D. Then
  (a) if D is simply-connected, then there is a harmonic function v (called a harmonic conjugate of u) such that $f = u + iv$ is holomorphic.
  (b) $u = \log |z|^2$ is harmonic in $\mathbb{C} \setminus \{0\}$, but it does not have a harmonic conjugate.

iv. **Existence of complex anti-derivatives:** Let f be a function in D. If the complex derivative $f'(z)$ exists for all $z \in D$ then f is holomorphic. The converse is also true.

(a) Theorem: Let D be a domain in $\mathbb{C}$ and f holomorphic in D. If D is simply-connected, then f has an anti-derivative on D.

(b) Counter example: In general, the theorem fails if D is not simply-connected. For example, let $D = \mathbb{C} \setminus \{0\}$ and $f(z) = \frac{1}{z}$.

v. **Analytic functions**

2. Some examples for holomorphic maps

i. Cayley transform: $S(z) = \frac{z-i}{z+i}$ maps the upper half plane to the unit disc $D(0,1)$.

ii. Let $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$ with $a \in D(0,1)$. Then

(a) $\phi_a : D(0,1) \to D(0,1)$ is one-to-one and onto.

(b) $\phi_a : T \to T$ is one-to-one and onto.

3. Cauchy Integral formula

1) Cauchy Theorem
2) Cauchy integral formula
3) Mean value property of holomorphic/harmonic functions
4) Cauchy estimates
5) Liouville's Theorems
6) Fundamental theorem of algebra
7) Entire functions with polynomial growth

EXAMPLE 34 *If p_n is a polynomial of degree $n > 1$ and p_n has n zeros in $D(0,R)$, then*

$$\int_{|z|=R} \frac{1}{p_n(z)} dz = 0.$$

EXAMPLE 35 *Let $f(z) = u(z) + iv(z)$ be entire holomorphic such that $v(z) \neq u(z)$ for all $z \in \mathbb{C}$. Find all such f.*

4. Power Series

Consider a power series about $z_0 \in \mathbb{C}$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

1) What is radius of convergence of this power series?
2) Concepts for absolutely converges, uniformly converges
3) Find the largest set so that the series converges

EXAMPLE 36 *Find the largest set on* $\mathbb{C}$ *such that the following power series converges*

$$\sum_{n=1}^{\infty} \frac{z^{2n+1}}{n}.$$

EXAMPLE 37 *What is the radius of convergence of the power series of* $f(z) = \frac{1}{z-1}\sin(\frac{1}{z})$ *about* $z_0 = i$*?*

5. Factorization Theorem and Uniqueness Theorem

i. If $f \in \mathcal{O}(D)$, $f(z_0) = 0$ and $f \not\equiv 0$, then there is a $g \in \mathcal{O}(D)$ such that $g(z_0) \neq 0$ and a positive integer k such that

$$f(z) = (z - z_0)^k g(z), \quad z \in D.$$

ii. If $f \not\equiv 0$, then $Z_D(f)$ is discrete; i.e. every point in $Z_D(f)$ is an isolated point.

EXAMPLE 38 *Let* f *be continuous on a domain* D *such that* $f(z)^2$ *is holomorphic on* D*. Prove or disprove that* f *is holomorphic on* D*.*

EXAMPLE 39 *Let* f *be holomorphic in* $D(0,1)$ *such that*

$$f''(1/n) = f(1/n), \quad n \in \mathbb{N}.$$

What is f *exactly?*

Solution. f and f'' are holomorphic in $D(0,1)$ and they are equal on $\{1/n : n \in \mathbb{N}\}$, which has an accumulation point $0 \in D(0,1)$. By the uniqueness theorem,

$$f(z) \equiv f''(z), \quad z \in D(0,1).$$

Consider $z = x \in (-1,1)$ and the equation:

$$y(x) = y''(x).$$

We know all of the solutions to this ODE are

$$y(z) = C_1 e^x + C_2 e^{-x}, \quad x \in (-1,1).$$

Therefore,

$$f(z) = C_1 e^z + C_2 e^{-z}, \quad z = x \in (-1,1).$$

By the uniqueness theorem,

$$f(z) = C_1 e^z + C_2 e^{-z}, \quad z \in D(0,1).$$

EXAMPLE 40 *Let f be holomorphic function on $D(0,1)$ such that*

$$(0,1) \subset \cup_{k=1}^{\infty} Z_{D(0,1)}(f^{(k)})$$

Prove that f is a polynomial.

Proof. Since $(0,1)$ is uncountable and

$$(0,1) \subset \cup_{k=1}^{\infty} Z_{D(0,1)}(f^{(k)}),$$

we have that there is a $k \in \mathbb{N}$ such that $Z_{D(0,1)}(f^{(k)})$ is uncountable. For that k, $f^{(k)} \equiv 0$ on $D(0,1)$. Thus

$$f(z) = a_0 + a_1 z + \cdots + a_k z^k$$

is a polynomial of degree at most k. $\square$

17 Uniqueness Theorem and Applications

Uniqueness Theorem

THEOREM 17.1 *Let D be a domain in $\mathbb{C}$, f is a holomorphic function on D with zero set $Z(f, D)$ in D. If $Z(f, D)$ has an accumulation (or limit) point in D, then f must be zero on D.*

Proof. Assume $Z(f, D)$ has an accumulation point $z_0 \in D$, if $f \not\equiv 0$ then there is $\delta > 0$ such that $f(z) \neq 0$ on $D(z_0, \delta) \setminus \{z_0\}$. This contradicts with $Z(f) \cap (D(z_0, \delta) \setminus \{z_0\}) \neq \emptyset$. Therefore, $f \equiv 0$ on D since D is connected. ▯

Corollary 17.2 *Let D be a domain in $\mathbb{C}$, f and g are holomorphic functions on D such that $f(z) = g(z)$ on a set K which has an accumulation (or limit) point in D, then $f \equiv g$.*

Proof. Apply the previous theorem to $f - g$, we have $f - g \equiv 0$, i.e.. $f \equiv g$ on D. ▯

EXAMPLE 41 *Let f be holomorphic in $D(0,1)$ such that $f(i/n) = \frac{1}{n^2}, \quad n = 1, 2, 3, \cdots$. What is f exactly?*

Solution. Let $g(z) = -z^2$ is holomorphic in $D(0,1)$. Then $f(i/n) = g(i/n)$ for $n = 1, 2, 3, \cdots$. and $\{i/n : n \in \mathbb{N}\}$ has an accumulation point $z_0 = 0 \in D(0,1)$. By uniqueness theorem, we have $f(z) \equiv z^2$ on $D(0,1)$. ▯

EXAMPLE 42 *Let f be holomorphic in $D(0,1)$ such that*

$$f''(1/n) = f(1/n), \quad n \in \mathbb{N}.$$

What is f exactly?

Solution. Since f and f'' are holomorphic in $D(0,1)$ and they equal on $\{1/n : n \in \mathbb{N}\}$ which has an accumulation point $0 \in D(0,1)$. By uniqueness theorem, we have

$$f(z) \equiv f''(z), \quad z \in D(0,1).$$

Method 1. Consider $z = x \in (-1, 1)$ and the equation:

$$y(x) = y''(x)$$

we know all solutions are

$$y(z) = C_1 e^x + C_2 e^{-x}, \quad x \in (-1, 1)$$

Therefore,

$$f(z) = C_1 e^z + C_2 e^{-z}, \quad z = x \in (-1, 1)$$

By uniqueness theorem, we have

$$f(z) = C_1 e^z + C_2 e^{-z}, \quad z \in D(0, 1).$$

Method 2. We write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = f''(z) = \sum_{n=2}^{\infty} a_n n(n-1) z^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2) z^n, \quad z \in D(0, 1)$$

This implies that

$$a_n = (n+1)(n+2)a_{n+2}, \quad n - 0, 1, 2, 3, \cdots$$

Thus

$$a_{2n} = \frac{a_0}{(2n)!}, \quad a_{2n+1} = \frac{a_1}{(2n+1)!}, \quad n = 0, 1, 2, 3, \cdots$$

Therefore,

$$\begin{aligned} f(z) &= a_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} + a_1 \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \\ &= a_0 \frac{e^z + e^{-z}}{2} + a_1 \frac{e^z - e^{-z}}{2} \\ &= \frac{a_0 + a_1}{2} e^z + \frac{a_0 - a_1}{2} e^{-z} \end{aligned}$$

on $D(0, 1)$. □

Corollary 17.3 *Let f be a non-constant holomorphic function on a domain D in $\mathbb{C}$. Then $Z(f; D)$ is at most countable.*

Proof. If $z \in Z(f;D)$, then there is a $\delta_z > 0$ such that $Z(f;D) \cap D(z,\delta_z) = \{z\}$. Choose Let D_n be a sequence of compact sets in D such that $D = \cup_{n=1}^{\infty} D_n$. Since $Z(f;D) \cap D_n$ is compact and $\{D(z,\delta_z) : z \in Z(f;D_n)\}$ is an open cover of $Z(f;D_n)$, so it is finite set, we have

$$Z(f;D) = \cup_{n=1}^{\infty} Z(f;D_n)$$

is at most countable. ▯

EXAMPLE 43 *Let f be holomorphic function on $D(0,1)$ such that*

$$(-1,1) \subset \cup_{k=1}^{\infty} Z(f^{(k)}; D(0,1))$$

Prove f is a polynomial.

Proof. Since $(-1,1)$ is uncountable and since

$$(-1,1) \subset \cup_{k=1}^{\infty} Z(f^{(k)}; D(0,1)),$$

we have that there is a $k \in NN$ such that $Z(f^{(k)}; D(0,1)$ is uncountable. By the previous corollary, we have $f^{(k)} \equiv 0$ on $D(0,1)$. Then

$$f(z) = a_0 + a_1 z + \cdots + a_k z^k$$

is a polynomial of degree at most k. ▯

IV. Meromorphic Functions

18 Isolated Singularities

18.1 Definition of an isolated singularity

Definition 18.1 *A point $z_0 \in \mathbb{C}$ is said to be an isolated singularity of f if there is $\delta > 0$ such that f is holomorphic in*

$$D(z_0, \delta) \setminus \{z_0\}$$

Definition 18.2 *(Classification of isolated singularities) Let f be holomorphic in $D(z_0, \delta) \setminus \{z_0\}$ for some $\delta > 0$. We say that*

- *z_0 is a removable singularity if $\lim_{z \to z_0} f(z)$ exists;*
- *z_0 is a pole if $\lim_{z \to z_0} f(z) = \infty$;*
- *z_0 is an essential singularity if $\lim_{z \to z_0} f(z)$ does not exist.*

REMARK 1 *If z_0 is a removable singularity, then f is holomorphic in $D(z_0, \delta) \setminus \{z_0\}$ and we may define $f(z_0) = \lim_{z \to z_0} f(z)$. Then f is continuous on $D(z_0, \delta)$. In fact, f is holomorphic in $D(z_0, \delta)$.*

Proof. Since

$$f(z) = \frac{1}{2\pi i}\Big[\int_{|w-z_0|=\delta} \frac{f(w)}{w-z} dw - \int_{|w-z_0|=\epsilon} \frac{f(w)}{w-z} dw\Big], \quad z \in D(z_0, \delta) \setminus D(z_0, \epsilon),$$

if we let $\epsilon \to 0$, then

$$f(z) = \frac{1}{2\pi i} \int_{|w-z_0|=\delta} \frac{f(w)}{w-z} dw, \quad z \in D(z_0, \delta) \setminus \{z_0\}.$$

Since f is continuous at z_0, the above formula remains true when $z = z_0$. Therefore, f is holomorphic in $D(z_0, \delta)$.

EXAMPLE 44 *Let*

$$f(z) = \frac{z^2 - 1}{z + 1}.$$

Then $z = -1$ is a removable singularity.

Solution. Since

$$\lim_{z\to -1} f(z) = \lim_{z\to -1} \frac{z^2-1}{z+1} = \lim_{z\to -1}(z-1) = -2,$$

$z = -1$ is a removable singularity. In fact,

$$f(z) = (z-1), \quad z \in \mathbb{C} \setminus \{-1\}$$

can be extend to be holomorphic on $\mathbb{C}$. ▯

EXAMPLE 45 *Let*

$$f(z) = \frac{e^z}{z-1}.$$

Then $z = 1$ is a pole.

Proof. Since

$$\lim_{z\to 1} f(z) = \lim_{z\to 1} \frac{e^z}{z-1} = \infty$$

and f is holomorphic in $\mathbb{C} \setminus \{1\}$, $z = 1$ is a pole. ▯

EXAMPLE 46 *Let*

$$f(z) = \sin\left(\frac{1}{z}\right)$$

Then $z = 0$ is an essential singularity.

Proof. Since f is holomorphic in $\mathbb{C} \setminus \{0\}$ and

$$\lim_{z=\frac{1}{2n\pi}\to 0} f(z) = \lim_{z=\frac{1}{2n\pi}\to 0} \sin\left(\frac{1}{z}\right) = 0$$

and

$$\lim_{z=\frac{1}{(2n\pi+\frac{\pi}{2})}\to 0} f(z) = \lim_{z=\frac{1}{(2n\pi+\frac{\pi}{2})}\to 0} \sin\left(\frac{1}{z}\right) = 1,$$

$\lim_{z\to 0} f(z)$ does not exist. So, $z = 0$ is an essential singularity. ▯

18.2 Riemann removable singularity lemma

THEOREM 18.3 *If z_0 is an isolated singularity for f and if*

$$\lim_{z \to z_0} (z - z_0) f(z) = 0,$$

then z_0 is a removable singularity.

Proof. Define

$$g(z) = (z - z_0) f(z), \quad z \in D(z_0, \delta) \text{ with } g(z_0) = 0.$$

Then g is holomorphic in $D(z_0, \delta)$ and $g(z_0) = 0$. Thus, we can write

$$g(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n = (z - z_0) \sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n$$

Therefore,

$$f(z) = \sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n$$

is holomorphic in $D(z_0, \delta)$ by redefining $f(z_0) = a_1$. □

EXAMPLE 47 *Let f be holomorphic function on $D(0,1) \setminus \{0\}$ such that*

$$M =: \int_{D(0,1)\setminus\{0\}} |f(z)|^3 dA(z) < \infty$$

Prove that $z = 0$ is a removable singularity.

Proof. For $|z| < 1/4$ since $D(z, |z|) \subset D(0,1) \setminus \{0\}$, by the mean-value property,

$$|f(z)|^3 \pi |z|^2 \leq \int_{D(z,|z|)} |f(w)|^3 dA(w) \leq M.$$

Therefore,

$$|f(z)| \leq \Big(\frac{M}{\pi |z|^2}\Big)^{1/3}$$

This implies that $\lim_{z \to 0} z f(z) = 0$. By Riemann's lemma, $z = 0$ is a removable singularity. □

18.3 Poles and essential singularity

18.3.1 Characterization for poles

THEOREM 18.4 *A point z_0 is a pole for f on $D(z_0, \delta) \setminus \{z_0\}$ if and only if there is a holomorphic function g on $D(z_0, \delta)$ with $g(z_0) \neq 0$ and some positive integer k such that*

$$f(z) = \frac{g(z)}{(z - z_0)^k}.$$

Proof. Since $\lim_{z \to z_0} f(z) = \infty$, there is a $0 < \delta_0 \leq \delta$ such that $|f(z)| \geq 1$ when $z \in D(z_0, \delta_0) \setminus \{0\}$. Then z_0 is a removable singularity for $h(z) = \frac{1}{f(z)}$ on $D(z_0, \delta_0) \setminus \{z_0\}$ and $h(z_0) = 0$. Therefore, there is a holomorphic function $H(z)$ on $D(z_0, \delta_0) \setminus \{z_0\}$ with $H(z_0)$ and some positive integer k such that

$$h(z) = (z - z_0)^k H(z), \quad z \in D(z_0, \delta_0).$$

Since $h(z) \neq 0$ on $D(z_0, \delta_0) \setminus \{z_0\}$, we have $H(z) \neq 0$ on $D(z_0, \delta_0)$. Then

$$g(z) =: \frac{1}{H(z)}$$

is holomorphic in $D(z_0, \delta_0)$ with $g(z_0) \neq 0$. Thus

$$f(z) = \frac{g(z)}{(z - z_0)^k}, \quad z \in D(z_0, \delta_0) \setminus \{z_0\}$$

By uniqueness of holomorphic function, we have $g(z) = f(z)(z - z_0)^k$ can be extended to be holomorphic on $D(z_0, \delta)$. □

18.3.2 Order of the pole

Definition 18.5 *Let z_0 be a pole of f. We say that z_0 is a pole of f of order k if*

$$\lim_{z \to z_0} (z - z_0)^k f(z) \neq 0 \quad \textit{and} \quad \lim_{z \to z_0} (z - z_0)^{k+1} f(z) = 0.$$

EXAMPLE 48 *What is the order of the pole of the following function f at $z = 0$?*

$$f(z) = \frac{\sin^2 z}{z^{10}(z - 1)(z + 1)}$$

Solution. Since

$$\lim_{z\to 0} z^8 f(z) = \lim_{z\to 0} z^8 \frac{\sin^2 z}{z^{10}(z-1)(z+1)} = \lim_{z\to 0} \frac{\sin^2 z}{z^2(z-1)(z+1)} = -1$$

and

$$\lim_{z\to 0} z^9 f(z) = \lim_{z\to 0} z^9 \frac{\sin^2 z}{z^{10}(z-1)(z+1)} = \lim_{z\to 0} \frac{\sin^2 z}{z(z-1)(z+1)} = 0,$$

$z = 0$ is a pole of f of order 8. ∎

18.3.3 Behavior of f near an essential singularity

THEOREM 18.6 *(Cassorati-Weierstrass theorem) If z_0 is an essential singularity for f, then for any small $\delta > 0$, we have $f(D(z_0,\delta)\setminus\{z_0\})$ is dense in $\mathbb{C}$.*

Proof. Suppose the statement fails. Then there are $\delta > 0$, $\epsilon > 0$ and $w_0 \in \mathbb{C}$ such that

$$f(D(z_0,\delta)\setminus\{z_0\}) \cap D(w_0,\epsilon) = \emptyset.$$

Let

$$g(z) =: \frac{1}{f(z) - w_0}.$$

Then g is holomorphic in $D(z_0,\delta)\setminus\{z_0\}$ and $|g(z)| \le \frac{1}{\epsilon}$ on $D(z_0,\delta)\setminus\{z_0\}$. Therefore, $z = z_0$ is removable singularity by Riemann Lemma. Then g is holomorphic in $D(z_0,\delta)$. This implies, $z = z_0$ is a removable singularity or a pole for $f(z) = \frac{1}{g(z)} + w_0$. This is a contradiction. ∎

EXAMPLE 49 *Let f be holomorphic in $D(0,1)\setminus\{0\}$. For $1 \le p < \infty$, let*

$$M_p = \int_{D(0,1)} |f(z)|^p dA(z) < \infty.$$

Then

i. If $p \ge 2$ then $z = 0$ is removable;

ii. If $1 \le p < 2$, then $z = 0$ is a pole of order at most 1.

Remark: For this example, we need Jensen's inequality for convex functions:

Definition 18.7 *A function $f : (a,b) \to \mathbb{R}$ which is twice differentiable on (a,b) is convex on (a,b) if and only if*

$$f''(x) \geq 0, \quad x \in (a,b).$$

THEOREM 18.8 ***(Jensen's inequality)*** *Let ϕ is convex on an interval $I \subset \mathbb{R}$. Suppose that $w : D \to [0,\infty)$ satisfies*

$$\int_D w(x)dx = 1.$$

If $h : D \to I$, then

$$\phi(\int_D h(x)w(x)\ dx) \leq \int_D \phi(h(x))w(x)\ dx.$$

For a proof of Jensen's inequality, see, for instance, An Introduction to the Art of Mathematical Inequalities: The Cauchy-Schwarz Master Class by J. Michael Steele, p. 113.

Proof. For $0 < |z| < 1/2$, we have $D(z,|z|) \subset D(0,1) \subseteq \{0\}$. By the sub-mean value property

$$|f(z)| \leq \int_{D(z,|z|)} |f(w)| \frac{dA(w)}{\pi|z|^2}$$

and since x^p is convex on $\mathbb{R}_{>0}$ for $p \geq 1$, by Jensen's inequality,

$$|f(z)|^p \leq \int_{D(z,|z|)} |f(w)|^p \frac{dA(w)}{\pi|z|^2} \leq \frac{M_p}{\pi|z|^2}.$$

Therefore,

$$|f(z)| \leq \big(\frac{M_p}{\pi}\big)^{1/p}|z|^{-2/p}.$$

We have the following cases.

If $p > 2$, then $z = 0$ is a removable singularity by Riemann's lemma since $\lim_{z\to 0} zf(z) = 0$.

If $1 < p \leq 2$, $z = 0$ can a pole of order at most 1. In particular, if $p = 2$, then we claim that $z = 0$ is removable. If

$$f(z) = \frac{g(z)}{z}, \quad g(0) \neq 0$$

then by the continuity of g at 0, there is a $\delta > 0$ such that

$$|f(z)| \geq \frac{|g(0)|}{2|z|}, \quad z \in D(0, \delta).$$

Thus

$$M_2 \geq \int_{D(0,\delta)} |f(z)|^2 dA \geq \frac{|g(0)|^2}{4} \int_{D(0,\delta)} |z|^{-2} dA = +\infty$$

This is a contradiction.

If $p = 1$, $z = 0$ is a pole of order at most 2. By the same proof as in the claim of case 2, it is impossible for the order of the pole to be 2. Therefore, the proof is complete. □

18.4 Homework 6

Contents: Morera's theorem, isolated singularity, uniformly convergence

i. Let $f(z)$ be continuous on $\mathbb{C}$ such that $f(z)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$. Then $f(z)$ is entire, where $\mathbb{R}$ is the real line in $\mathbb{C}$.

ii. Let $p_j(z)$ be a sequence of holomorphic polynomial of degree less than or equal n. Assume that $p_j(z) \to f(z)$ as $j \to \infty$ uniformly on any compact subset of $\mathbb{C}$. Show that $f(z)$ is a holomorphic polynomial of degree $\leq n$.

iii. Let $f(z)^2$ and $f(z)^3$ be holomorphic on a domain D. Then $f(z)$ is holomorphic in D.

iv. Let $f(z)$ be entire satisfying $|f(z)| \leq |\sin z|$ for all $z \in \mathbb{C}$ and $f(0) = 0$. Find $f(z)$ explicitly

v. Consider the power series

$$f(z) = \sum_{n=1}^{\infty} \frac{z^{n!}}{n}$$

Prove that there is a dense subset K on the unit circle $T = \{|z| = 1\}$ so that $\lim_{r \to 1^-} f(rz) = +\infty$ for $z \in K$.

vi. Find the Taylor series of the following functions around $z = z_0$:

(a) $f(z) = \frac{z}{(z+1)^2}, \quad z_0 = i$

(b) $f(z) = \sin z, \quad z_0 = \pi/2$

vii. Let $f(z)$ be entire such that $\lim_{z\to\infty} \frac{f(z)}{z^n} = 0$. Use Cauchy's estimates to prove $f(z)$ is a polynomial of degree at most n.

viii. Let f be meromorphic in $\mathbb{C}$, and let $R(z)$ be a rational function such that

$$|f(z)| \leq |R(z)|, \quad \mathbb{C}(\text{except for poles of } R \text{ and } f).$$

Prove $f(z) = cR(z)$ for some $c \in \mathbb{C}$.

ix. Let f be holomorphic in $D(z_0, r) \setminus \{z_0\}$, and let $U = f(D(z_0, r) \setminus \{z_0\})$ be open. Let $g : U \to \mathbb{C}$ be non-constant holomorphic. Answer the following questions with a justification.

a) If z_0 is a removable singularity, what kind singularity does $g \circ f$ have at z_0;

b) If z_0 is a pole, what kind singularity does $g \circ f$ have at z_0;

c) If z_0 is an essential singularity, what kind singularity does $g \circ f$ have at z_0;

x. Classify each of the following as having a removable singularity, a pole, or an essential singularity at $z_0 = 0$:

(a) $\frac{z^2+2}{z(z+1)}$;

(b) $\sin \frac{1}{z}$;

(c) $\frac{1}{z^2} - \cos z$;

(d) $ze^{1/z}e^{-1/z^2}$;

(e) $\frac{\sum_{k=2}^{\infty} 2^k z^k}{z^3}$.

xi. If f is holomorphic in $D(z_0, r) \setminus \{z_0\}$ and $\int_{D(z_0,r)} |f(z)|^2 dA(z) < \infty$. Prove that f has removable singularity at $z = z_0$.

xii. Assume that $f(z)$ has an essential singularity at $z = 0$. Prove there are a sequence of points $\{z_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} z_n = 0 \quad \text{and} \quad |f(z_n)||z_n|^n \geq n, \quad n = 1, 2, 3, \cdots.$$

19 Laurent Series

19.1 Definition and examples of Laurent Series

A Laurent series about $z = z_0$ is an expression of the form

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \tag{1}$$

Let

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \quad \text{and} \quad r = \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}. \tag{2}$$

Then

THEOREM 19.1 *The Laurent series (1) converges for all*

$$z \in A(z_0; r, R) := \{z \in \mathbb{C} : r < |z - z_0| < R\}. \tag{3}$$

• We call $A(z_0; r, R)$ the convergent annulus for the Laurent series (1).

19.2 Examples of the convergent annulus

EXAMPLE 50 *Find the Laurent series for $f(z) = (z^2+1)e^{1/z}$ about $z_0 = 0$.*

Solution. Notice that

$$\begin{aligned} f(z) &= (z^2+1)e^{1/z} = z^2 e^{1/z} + e^{1/z} \\ &= z^2 \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{1}{z}\Big)^n + \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{1}{z}\Big)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n+2} + \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \end{aligned}$$

$$= z^2 + z + \sum_{n=0}^{\infty}\Big(\frac{1}{(n+2)!} + \frac{1}{(n)!}\Big)z^{-n}$$
$$= \sum_{n=-\infty}^{0}\Big(\frac{1}{(-n+2)!} + \frac{1}{(-n)!}\Big)z^{n} + z + z^2. \quad \square$$

EXAMPLE 51 *Find the Laurent series for* $f(z) = \frac{1}{z(z-1)(z-2)}$ *about* $z_0 = 1$.

Solution.

$$\begin{aligned} f(z) &= \frac{1}{z(z-1)(z-2)} \\ &= \frac{1}{z-1}\frac{1}{(z-1)^2-1} \\ &= -\frac{1}{z-1}\sum_{n=0}^{\infty}(z-1)^{2n} \\ &= -\sum_{n=-1}^{\infty}(z-1)^{2n+1}. \end{aligned}$$

19.3 Laurent series expansion

THEOREM 19.2 *Let* $f(z)$ *be holomorphic in* $D(z_0, R_0) \setminus \overline{D}(z_0, r_0)$ *with* $r_0 > 0$ *or* $r_0 = 0$. *Then*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \quad z \in D(z_0, R_0) \subseteq \overline{D}(z_0, r_0)$$

Proof. For $r_0 < r < R < R_0$, by the Cauchy Integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{\partial A(z_0;r,R)} \frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{\partial D(z_0,R)} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(w)}{w-z}dw$$

for all $z \in D(z_0, r) \subseteq \overline{D}(z_0, r_0)$.

Notice that

$$\begin{aligned} \frac{1}{2\pi i}\int_{\partial D(z_0,R)} \frac{f(w)}{w-z}dw &= \frac{1}{2\pi i}\int_{\partial D(z_0,R)} \frac{f(w)}{w-z_0-(z-z_0)}dw \\ &= \frac{1}{2\pi i}\int_{\partial D(z_0,R)} \frac{f(w)}{(w-z_0)(1-\frac{z-z_0}{w-z_0})}dw \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{\partial D(z_0,R)} \frac{f(w)}{(w-z_0)}\sum_{n=0}^{\infty}\Big(\frac{z-z_0}{w-z_0}\Big)^n dw \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\partial D(z_0,R)} \frac{f(w)}{(w-z_0)^{n+1}}dw(z-z_0)^n \\
&= \sum_{n=0}^{\infty} a_n(z-z_0)^n
\end{aligned}
$$

where

$$a_n =: \frac{1}{2\pi i}\int_{\partial D(z_0,R)} \frac{f(w)}{(w-z_0)^{n+1}}dw, \quad n \geq 0,$$

and

$$
\begin{aligned}
-\frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(w)}{w-z}dw &= -\frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(w)}{w-z_0-(z-z_0)}dw \\
&= \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(w)}{(z-z_0)(1-\frac{w-z_0}{z-z_0})}dw \\
&= \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(w)}{(z-z_0)}\sum_{n=0}^{\infty}\Big(\frac{w-z_0}{z-z_0}\Big)^n dw \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\partial D(z_0,r)} f(w)(w-z_0)^n dw(z-z_0)^{-(n+1)} \\
&= \sum_{n=-\infty}^{-1} a_n(z-z_0)^n
\end{aligned}
$$

where, by Cauchy's theorem,

$$a_n =: \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(w)}{(w-z_0)^{n+1}}dw = \frac{1}{2\pi i}\int_{\partial D(z_0,R)} \frac{f(w)}{(w-z_0)^{n+1}}dw$$

for all $n = -1, -2, -3, \cdots$.

REMARK 2 *a_n does not depend on the choices of $0 < r < R < R_0$.*

Therefore,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n, \quad z \in A(z_0; r, R_0) =: D(z_0, R_0) \setminus \overline{D}(z_0, r).$$

EXAMPLE 52 *Find the largest set in $\mathbb{C}$ such that the following Laurent series converges*

$$\sum_{n=-\infty}^{\infty} \frac{a_n}{n} z^n, \quad a_n = 2^n \text{ if } n \geq 0; \quad a_n = 3^n \text{ if } n < 0.$$

Solution. Since

$$\limsup_{n\to\infty} \sqrt{|a_n|} = \limsup_{n\to\infty} \sqrt{\frac{2^n}{n}} = 2,$$

$R = 1/2$. Also,

$$r = \limsup_{n\to\infty} \sqrt{|a_{-n}|} = \limsup_{n\to\infty} \sqrt{\frac{3^{-n}}{n}} = 1/3.$$

Using Dirichlet test, one can prove

$$\sum_{n=0}^{\infty} \frac{a_n}{n} z^n$$

converges for all $|z| = 1/2$ except $z = 1/2$, where the series is diverges. Similarly, one can prove

$$\sum_{n=-\infty}^{\infty} \frac{a_n}{n} z^n$$

converges for all $|z| = 1/3$ except $z = 1/3$, where the series is diverges. Thus, it converges on

$$\overline{A(0; 1/3, 2)} \setminus \{1/3, 2\}.$$

19.4 Laurent series and isolated singularities

THEOREM 19.3 *Let f be holomorphic in $D(z_0, \delta)$ with the Laurent series:*

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad z \in D(z_0, \delta) \setminus \{z_0\}$$

Then

(i) z_0 is a removable singularity for f if and only if $a_n = 0$ for all $n < 0$;

(ii) z_0 is a pole for f if and only if there is a positive integer k such that

$$a_n = 0 \text{ when } n < -k;$$

(iii) z_0 *is an essential singularity for* f *if and only if there is a infinite subsequence* $n_{k}{}_{k=1}^{\infty}$ *such that* $a_{-n_k} \neq 0$ *for all* $k \in \mathbb{N}$.

Proof.

(i) z_0 is removable if and only if f is holomorphic in $D(z_0, r)$. So the Laurent series for f is a Taylor series; that is, $a_n = 0$ when $n < 0$.

(ii) z_0 is pole if and only if there is $k \in \mathbb{N}$ such that $(z - z_0)^k f(z)$ is holomorphic in $D(z_0, r)$ if and only if

$$(z - z_0)^k f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n \iff f(z) = \sum_{n=-k}^{\infty} b_{k+n} (z - z_0)^n.$$

(iii) z_0 is essential singularity for f if and only if z_0 is neither removable nor pole. By (i) and (ii), f has a Laurent series

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where $a_n \neq 0$ for infinitely many $n < 0$. ∎

EXAMPLE 53 *Let* f *be holomorphic in* $D(0, 1) \setminus \{0\}$ *such that*

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leq 1, \quad 0 < r < 1$$

Prove that $z = 0$ *is a removable singularity.*

Proof. Without loss of generality, let

$$f(z) = \sum_{n=-\infty} a_n z^n, \quad 0 < |z| < 1$$

Then

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = 2\pi \sum_{n=-\infty}^{\infty} |a_n|^2 r^{2n} \leq 1$$

Let $r \to 0^+$. Then $a_n = 0$ when $n < 0$. ∎

20 Meromorphic functions

Definition 20.1 *Let D be a domain in $\mathbb{C}$, we say that f is meromorphic in D if f is holomorphic except for possibly isolated singularities and all of them are poles. The pole set is at most countable.*

EXAMPLE 54 *(i) $f(z) = \frac{1}{z(z-1)^3(z-2)^2}$ is meromorphic in $\mathbb{C}$;*
(ii) $f(z) = e^{1/z}$ is not meromorphic in $\mathbb{C}$ because $z = 0$ is an essential singularity, not a pole, for f.

EXAMPLE 55 *Let*

$$f(z) = \frac{1}{\sin(\frac{1}{z})}.$$

Then f is meromorphic in $\mathbb{C} \setminus \{0\}$, but it is not meromorphic in $\mathbb{C}$.

THEOREM 20.2 *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on a domain D. If for any compact subset $K \subset D$,*

$$f(z) =: \sum_{n=1}^{\infty} f_n(z)$$

converges uniformly on K, then $\sum_{n=1}^{\infty} f_n$ defines a holomorphic function on D.

Proof. For any $z_0 \in D$ and $r \in (0, \operatorname{dist}(z_0, \partial D))$, we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} f_n(z) \\ &= \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f_n(w)}{w-z} dw \\ &= \frac{1}{2\pi i} \int_{|w-z_0|=r} \sum_{n=1}^{\infty} \frac{f_n(w)}{w-z} dw \\ &= \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{w-z} dw \end{aligned}$$

for any $z \in D(z_0, r)$. Therefore, f is holomorphic in $D(z_0, r)$. This implies that f is holomorphic in D. □

20.1 Construction of certain meromorphic functions

EXAMPLE 56 *Prove that*

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

defines a meromorphic function in $\mathbb{C}$.

Proof. Let

$$f_n(z) = \frac{1}{(z-n)^2}$$

Then f_n is holomorphic on $D = \mathbb{C} \subseteq \mathbb{Z}$. For any compact subset K of D, there is an n_0 such that

$$K \subset D(0, n_0).$$

Thus

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \sum_{|n|\le n_0} \frac{1}{(z-n)^2} + \sum_{|n|>n_0} \frac{1}{(z-n)^2}$$

and

$$\sum_{|n|>n_0} \left|\frac{1}{(z-n)^2}\right| \le 2 \sum_{n=1+n_0} \frac{1}{(n-n_0)^2} = \sum_{k=1}^{\infty} \frac{2}{k^2}$$

converges uniformly on K. This implies that f defines a holomorphic function on $D = \mathbb{C} \setminus \mathbb{Z}$. For each $z = n$, f is holomorphic in $D(n, 1/2) \setminus \{n\}$ and

$$\lim(z-n)^2 f(z) = 1 \quad \text{and} \quad \lim(z-n)^3 f(z) = 0$$

Therefore, $z = n$ is a pole of f of order 2. Therefore, f is meromorphic in $\mathbb{C}$.
□

EXAMPLE 57 *Prove or disprove there is a non-constant entire holomorphic function f such that*

$$f(z+1) = f(z) \quad \textit{and} \quad f(z+i) = f(z), \quad z \in \mathbb{C}. \quad (1)$$

Solution. If f is entire holomorphic and satisfies equation (1), then $|f|$ is bounded by

$$M = \max\{|f(z)| : |z| \le 2\}$$

By Liouville's theorem, f must be a constant. □

EXAMPLE 58 *Prove or disprove there is a meromorphic function on* $\mathbb{C}$ *such that*

$$f(z) = f(z \pm 1) = f(z + \pm i), \quad z \in \mathbb{C}.$$

Solution. Let

$$f(z) =: \sum_{n,m=-\infty}^{\infty} \frac{1}{(z-n-mi)^3}$$

Then we claim:

(i) f is holomorphic in $D =: \mathbb{C} \setminus \{n+mi : n, m \in \mathbb{Z}\}|$

Proof. For any compact subset K in D, there is an $n_0 \in \mathbb{N}$ such that if $|z| < n_0 - 2$ for all $z = x + iy \in K$. Then

a) if $|n| > n_0$

$$|z-n-mi|^2 = (x-n)^2 + (y-m)^2 \geq (n-n_0)^2$$

Thus

$$\sum_{|m|<n_0} \sum_{|n|>n_0} \frac{1}{(z-n-mi)^3}$$

converges uniformly on K;

b) if $|m| > n_0$

$$|z-n-mi|^2 = (x-n)^2 + (y-m)^2 \geq (m-n_0)^2$$

Thus

$$\sum_{|n|<n_0} \sum_{|m|>n_0} \frac{1}{(z-n-mi)^3}$$

converges uniformly on K;

c) if $|n|, |m| > n_0$

$$|z-n-mi|^2 = (x-n)^2 + (y-m)^2 \geq (n-n_0)^2 + (m-n_0)^2$$

Thus

$$\begin{aligned}
\sum_{|n|>n_0} \sum_{|m|>n_0} \left|\frac{1}{(z-n-mi)^3}\right| &\leq \sum_{|n|>n_0} \sum_{|m|>n_0} \frac{1}{\Big((n-n_0)^2 + (m-n_0)^2\Big)^{3/2}} \\
&\leq \sum_{|n|>n_0} \sum_{|m|>n_0} \frac{1}{\Big(2(n-n_0)^2(m-n_0)^2\Big)^{3/2}} \\
&= \frac{1}{2^{3/2}} \sum_{|n|>n_0} \frac{1}{(n-n_0)^3} \sum_{|m|>n_0} \frac{1}{(m-n_0)^3} \\
&< +\infty
\end{aligned}$$

converges uniformly on K. (The second inequality follows from the trivial inequality $2ab \le a^2 + b^2$.) Therefore,

$$\begin{aligned} f(z) &= \sum_{|n|\le n_0}\sum_{|m|\le n_0}\frac{1}{(z-n-mi)^3} + \sum_{|n|\le n_0}\sum_{|m|> n_0}\frac{1}{(z-n-mi)^3} \\ &\quad + \sum_{|n|> n_0}\sum_{|m|\le n_0}\frac{1}{(z-n-mi)^3} + \sum_{|n|,|m|> n_0}\frac{1}{(z-n-mi)^3} \end{aligned}$$

converges uniformly on K. So it defines a holomorphic function on D.

It is easy to see that

(ii) For each $n_0, m_0 \in \mathbb{Z}$, one has

$$f(z + (n_0 + im_0)) = f(z)$$

(iii) $\lim_{z\to n_0+im_0} f(z) = \infty$

In conclusion, this function f satisfies the above conditions. ▯

EXAMPLE 59 *Prove or disprove there is a meromorphic function f on $\mathbb{C}$ such that the set of poles of f is $\{z = \ln n : n \ge 1\}$ and every pole is a simple pole.*

Proof. Let

$$f(z) = \sum_{n=1}^{\infty}\frac{(-1)^n}{(-z+\ln n)} = \sum_{n=1}^{\infty}(-1)^n\frac{\ln n - x}{(\ln n - x)^2 + y^2} + iy\sum_{n=1}^{\infty}\frac{(-1)^n}{(\ln n - x)^2 + y^2}$$

Notice that $\frac{\ln n - x}{(\ln n - x)^2 + y^2}$ and $\frac{1}{(\ln n - x)^2 + y^2}$ are decreasing in n when $\ln n > |z|$. For any compact set $K \subset D = \mathbb{C} \setminus \{\ln n : n \in \mathbb{N}\}$, choose n_0 such that

$$\max_K |z| < \ln n_0 - 1.$$

For any $z \in K$ and $m > n \ge n_0$, By Abel's identity, we have

$$\Big|\sum_{k=n}^{m}(-1)^k\frac{\ln k - x}{(\ln k - x)^2 + y^2}\Big| \le 6\frac{\ln n - x}{(\ln n - x)^2 + y^2} \le \frac{6}{\ln n - \ln n_0 + 1}$$

converges to 0 uniformly for $z \in K$ as $n \to \infty$. Therefore, it defines a holomorphic function on D. It is easy to see it is meromorphic in $\mathbb{C}$ with the satisfying properties. ▯

20.2 Singularity at ∞

Definition 20.3 *We say that $z = \infty$ is an isolated singularity of $f(z)$ if $z = 0$ is an isolated singularity for $f(1/z)$. Moreover $f(z)$ has a removable singularity (respectively pole of order k, essential singularity) at $z = \infty$ if $z = 0$ is a removable singularity (respectively pole of order k, essential singularity) for $f(\frac{1}{z})$.*

By the definition, one can easily see

EXAMPLE 60 *If f is a non-constant entire function then $z = \infty$ is an isolated singularity for f.*

THEOREM 20.4 *(Liouville's theorem) Let f be an entire function. Then $z = \infty$ is a removable if and only if f is a constant.*

Proof. Since f is entire, we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C}.$$

Then

$$f\Big(\frac{1}{z}\Big) = \sum_{n=0}^{\infty} a_n z^{-n}$$

has a removable singularity at $z = 0$ if and only if $a_n = 0$ when $n > 0$ if and only if $f \equiv a_0$. □

We will add ∞ into our consideration. Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We say that f is holomorphic at ∞ if ∞ is a removable singularity for $f(1/z)$. Then we can restate Liouville's theorem as follows:

THEOREM 20.5 *(Liouville's theorem) f is holomorphic on $\overline{\mathbb{C}}$ if and only if f is a constant.*

The theorem for entire functions with polynomial growth can be restated as follows:

THEOREM 20.6 *If f is entire holomorphic, then $z = \infty$ is a pole if and only if f is a polynomial.*

Proof. Let
$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
Then
$$f\Big(\frac{1}{z}\Big) = \sum_{n=0}^{\infty} a_n z^{-n}$$
has a pole at $z = 0$ of order k if and only if $a_n = 0$ when $n > k$ and $a_k \neq 0$. Thus,
$$f(z) = \sum_{n=0}^{k} a_n z^n,$$
which is a polynomial of degree k. ▯

20.3 Rational functions

Let $R(z)$ be a rational function of z; that is,
$$R(z) = \frac{p_m(z)}{q_n(z)}$$
where p_m and q_n are holomorphic polynomials of degree m and n respectively and have no common factors. Then
$$\lim_{z\to\infty} R(z) = A$$
where $A \in \mathbb{C}$ if $m \leq n$ and $A = \infty$ if $m > n$. Thus, $z = \infty$ is either pole or removable singularity for $R(z)$. Therefore, $R(z)$ is meromorphic in $\overline{\mathbb{C}}$. Conversely, we have

THEOREM 20.7 *f is meromorphic on $\overline{\mathbb{C}}$ if and only if f is a rational function.*

Proof. Since $z = \infty$ is either a pole or removable singularity, f has only finitely many poles in $\mathbb{C}$, say, $z_1, \cdots, z_n$. Then
$$F(z) = f(z) \prod_{j=1}^{n} (z - z_j)$$

is entire holomorphic and $z = \infty$ is either a pole or a removable singularity for F because there is a non-negative integer k such that $\lim_{z\to\infty} \frac{F(z)}{z^{n+k}} = 0$. Therefore, F is a polynomial p_m of degree $0 \le m < n + k$. Therefore,

$$f(z) = \frac{p_m(z)}{\prod_{j=1}^n (z - z_j)}.$$

The proof is complete. ▯

20.4 Stereographic projection

The topology of $\mathbb{C}$ and $\overline{\mathbb{C}}$ are completely different. The later one is compact; in fact, it is diffeomorphic to the unit sphere S^2 in $\mathbb{R}^3$.

Let $S^2 = \{p =: (x_1, x_2, x_3) \in \mathbb{R}^3 : \|p\| = 1\}$ be the unit sphere in $\mathbb{R}^3$. We define the stereographic projection $\pi : S^2 \to \overline{\mathbb{C}}$ by

$$(1) \qquad z =: \pi(x_1, x_2, x_3) = \frac{x_1}{1 - x_3} + i\frac{x_2}{1 - x_3}, \quad \pi(0, 0, 1) = \infty$$

Then

$$|z|^2 = \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{1 - x_3^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3}.$$

Thus

$$x_3 = \frac{|z|^2 - 1}{1 + |z|^2}, \quad 1 - x_3 = \frac{2}{1 + |z|^2}.$$

By (1),

$$x_1 = (1 - x_3)\frac{1}{2}(z + \overline{z}) = \frac{z + \overline{z}}{1 + |z|^2}, \quad x_2 = -i\frac{z - \overline{z}}{1 + |z|^2}.$$

Thus,

$$\pi^{-1}(z) = (\frac{z + \overline{z}}{1 + |z|^2}, -i\frac{z - \overline{z}}{1 + |z|^2}, \frac{|z|^2 - 1}{1 + |z|^2}).$$

For $p = (x_1, x_2, x_3)$ and $q = (y_1, y_2, y_3) \in S^2$, let $z = \pi(p)$ and $w = \pi(q)$. Then

$$\begin{aligned} &d(p, q)^2 \\ &\quad = \|p\|^2 + \|q\|^2 - 2\langle p, q\rangle \\ &\quad = 2(1 - \langle p, q\rangle) \end{aligned}$$

$$= \quad 2\frac{(1+|z|^2)(1+|w|^2) - \langle z+\overline{z}, w+\overline{w}\rangle - \langle z-\overline{z}, w-\overline{w}\rangle - (z|^2-1)(|w|^2-1))}{(1+|z|^2)(1+|w|^2)}$$

$$= \quad 2\frac{2|z|^2+2|w|^2-4\langle z,w\rangle}{(1+|z|^2)(1+|w|^2)}$$

$$= \quad \frac{4|z-w|^2}{(1+|z|^2)(1+|w|^2)}$$

Therefore,

$$d(\pi^{-1}(z), \pi^{-1}(w)) = \frac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}$$

20.5 Homework 7

Content: Isolated singularity, Laurent series, meromorphic functions

i. If f is holomorphic non-vanishing on $D(0,1)\setminus\{0\}$ with 0 as an essential singularity, then what type singularity of f^2 has at $z=0$?

ii. Calculate the annulus of convergence (including possible boundary points) for each of the following Laurent series:
(a) $\sum_{j=-\infty}^{2} 2^{-j}z^j$; (b) $\sum_{j=-\infty}^{\infty} z^j/j^2$; (c) $\sum_{j=0} 4^{-j}z^j + \sum_{j=-\infty}^{-1} 3^j z^j$.

iii. Let z_0 be an isolated singularity for f. Then z_0 is an isolated singularity for $1/f$. Find the relation of the type of singularity between f and $1/f$.

iv. Let f have an essential singularity at $z=z_0$, Prove $(z-z_0)^m f(z)$ has an essential singularity at $z=z_0$ for any positive integer m.

v. Let f_j be holomorphic in $D(0,1)\setminus\{0\}$ and has pole at $z=0$ for $j=1,2,3,\cdots$. Assume that f_j converges to f uniformly on any compact subset of $D(0,1)\setminus\{0\}$. Prove or disprove that f has pole at $z=0$.

vi. If f has non-removable singularity at $z=z_0$, then e^f has an essential singularity at z_0.

vii. Prove the Laurent series about $z=0$:

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}z^k$$

for some real number B_k, which is called Bernoulli number. Find B_1, B_2 and B_3.

viii. Find Laurent expansion of the given function about the given point. In each case, specify the annulus of convergence of expansion.

(a) $f(z) = \csc z$ about $z_0 = 0$; (b) $f(z) = z/(z+1)^3$ about $z = -1$;

(c) $f(z) = z/[(z-1)(z-3)(z-5)]$ about $z_0 = 1, 3, 5$ respectively.

(d) $f(z) = e^z/z^3$ about $z_0 = 0$.

ix. If $f(z)$ is holomorphic in $D = D(0,1) \setminus \{0\}$ such that

$$\int_D |f(z)|dA(z) = \int_D |f(x,y)|dxdy < \infty.$$

Prove or disprove $z = 0$ is either removable or simple pole for f.

x. Prove the following infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(z-n)}$$

defines a meromorphic function on the whole complex plane.

xi. Let $f(z)$ and $g(z)$ be two entire functions such that $|f(z)| \le |g(z)|$ for $z \in \mathbb{C}$. Find the relation between f and g.

xii. Let $f(z)$ and $g(z)$ be two entire functions such that $|f(z)| \le |g(z)|$ for $|z| \ge 1$. Find the relation between f and g.

xiii. Construct a meromorphic function f on $\mathbb{C}$ having (m,n) as its poles of order 3 for all $m, n \in \mathbb{Z}$.

xiv. Let

$$d(z,w) = \frac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}$$

Prove d defines a metric on $\mathbb{C}$, which is called the *spherical metric*.

21 Residues and the Residue Theorem

21.1 Definition of a Residue

Definition 21.1 *Let $f(z)$ be holomorphic in $D(z_0, \delta) \subseteq \{z_0\}$ with a Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

We call the number a_{-1} the residue of f at z_0.

By the formula for a_n in theorem 19.2,

$$a_{-1} = \frac{1}{2\pi i}\int_{|z-z_0|=r} f(z)dz, \quad 0 < r < \delta.$$

EXAMPLE 61 *Find the residue of f at $z = z_0$:*

i. $f(z) = (z^2+1)\sin(1/z)$ *at* $z_0 = 0$*;*

ii. $f(z) = (\cos z)e^{1/z}$ *at* $z_0 = 0$.

Solution. i. Notice that

$$f(z) = (z^2+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{-2n-1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{-2n+1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{-2n-1}$$

So,

$$\text{Res}(f,0) = a_{-1} = \frac{(-1)^1}{(2+1)!} + \frac{1}{1!} = \frac{5}{6}.$$

ii. Notice that

$$f(z) \quad = \quad (\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n})(\sum_{m=0}^{\infty}\frac{1}{m!}z^{-m}) = \sum_{k=-\infty}^{\infty}\sum_{2n-m=k}(\frac{(-1)^n}{(2n)!}\frac{1}{m!})z^k.$$

Therefore,

$$\text{Res}(f;0) = a_{-1} = \sum_{2n-m=-1}\frac{(-1)^n}{(2n)!}\frac{1}{m!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!(2n+1)!}.$$

21.2 Formulae for Computing Residue of f

THEOREM 21.2 *Let g and h be holomorphic in $D(z_0,\delta)$ with $g(z_0) \neq 0$ and $h(z_0) = 0$. Suppose further that $f(z) = g(z)/h(z)$ with an isolated singularity at $z = z_0$. Then*

i. If $h'(z_0) \neq 0$, *then* $Res(f, z_0) = \frac{g(z_0)}{h'(z_0)}$.

ii. If z_0 is a zero of h with multiplicity $k > 1$, then

$$Res(f, z_0) = \frac{1}{(k-1)!}\frac{\partial^k (z-z_0)^k f}{\partial z^{k-1}}(z_0)$$

Proof. Since

$$f(z) = \sum_{n=-k}^{\infty} a_n(z-z_0)^n,$$

we have

$$(z-z_0)^k f(z) = \sum_{n=-k}^{\infty} a_k(z-z_0)^{n+k}$$

is holomorphic. Then

$$\frac{1}{(k-1)!}\frac{\partial^{k-1}[(z-z_0)^k f(z)]}{\partial z^{k-1}}(z_0) = a_{-1} = \text{Res}(f; z_0)$$

When $k = 1$, $a_{-1} = g(z_0)/h'(z_0)$. ∎

EXAMPLE 62 *Find the residue of f at $z = z_0$:*

i. $f(z) = \frac{(z-1)^5(z+1)}{(z-2)^6}$ *at* $z_0 = 2$*;*

ii. $f(z) = \frac{z^2+1}{\cos^2(z+\frac{\pi}{2})}$ *at* $z_0 = 0$*.*

Solution.(i) Let $f(z) = g(z)/h(z)$ with $g(z) = (z-1)^5(z+1)$ and $h(z) = (z-2)^6$. Then $g(2) \neq 0$ and $z = 2$ is a zero of h of multiplicity 6. Then

$$\begin{aligned}
\text{Res}(f; 2) &= \frac{1}{5!}\frac{d^5(z-2)^6 f(z)}{dz^5}(2) \\
&= \frac{1}{5!}\frac{d^5[(z-1)^5(z+1)]}{dz^5}(2) \\
&= \frac{1}{5!}\frac{d^5(z-1)^6 + 2(z-1)^5}{dz^5}(2) \\
&= \frac{1}{5!}(6!(2-1) + 2\cdot 5!) \\
&= 8.
\end{aligned}$$

(ii) Write

$$\begin{aligned}
f(z) &= \frac{z^2+1}{\cos^2(z+\frac{\pi}{2})} \\
&= \frac{z^2+1}{\sin^2 z}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{z^2+1}{(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1})^2} \\
&= \frac{z^2+1}{z^2} \frac{1}{(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n})^2} \\
&= \Big(1+\frac{1}{z^2}\Big)\Big(1+\sum_{k=1}^{\infty} a_k z^{2k}\Big).
\end{aligned}
$$

Therefore, $\text{Res}(f;0) = 0$.

21.3 Residue at $z = \infty$

Definition 21.3 *If $z = \infty$ is an isolated singularity for f, we define the residue of f at ∞ by*

$$Res(f;\infty) = -\frac{1}{2\pi i}\int_{|z|=R} f(z)dz, \quad R >> 1$$

where R is chosen large enough so that ∞ is the only singularity in $\bar{\mathbb{C}} \setminus \bar{D}(0,R)$. Since

$$Res(\frac{1}{z^2}f(\frac{1}{z});0) = \frac{1}{2\pi i}\int_{|z|=\frac{1}{R}} \frac{1}{z^2}f(\frac{1}{z})dz$$

for R sufficiently large, after a change of variables,

$$Res(f;\infty) = Res(\frac{1}{z^2}f(1/z);0).$$

EXAMPLE 63 *Find the residue of f at $z = \infty$ if $f(z) = \frac{(z+1)}{(z-2)(z-1)}$.*

Solution. For $R > 2$, we have

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{|z|=R} f(z)dz &= \frac{1}{2\pi i}\int_{|z|=R} \frac{z+1}{(z-2)(z-1)}dz \\
&= \frac{1}{2\pi i}\int_{|z|=R} \frac{1}{(z-2)}dz + \frac{1}{2\pi i}\int_{|z|=R} \frac{2}{(z-2)(z-1)}dz \\
&= \frac{1}{2\pi i}\int_{|z|=R} \frac{1}{(z-2)}dz + 2\frac{1}{2\pi i}\int_{|z|=R} \frac{1}{(z-2)} - \frac{1}{(z-1)}dz \\
&= 1
\end{aligned}
$$

Therefore, $\text{Res}(f;\infty) = -1$.

EXAMPLE 64 *Prove or disprove that there is holomorphic function f in $\mathbb{C} \setminus \overline{D}(0,2)$ such that*

$$f'(z) = \frac{z+1}{(z-2)(z-1)}$$

Solution. The answer is no. If it were yes, then

$$0 = \frac{1}{2\pi i}\int_{|z|=R} f'(z)dz = \frac{1}{2\pi i}\int_{|z|=R} \frac{z+1}{(z-2)(z-1)}dz = 1,$$

which is a contradiction. ▯

THEOREM 21.4 *Let g be a holomorphic function on $|z| > R$. Then there is a holomorphic function F on $|z| > R$ such that*

$$F'(z) = g(z), \quad |z| > R$$

if and only if $Res(g;\infty) = 0$.

Proof. Write

$$g(z) = \sum_{j=-\infty}^{\infty} a_j z^j, \quad |z| > R.$$

Then $\text{Res}(g,\infty) = 0$ if and only if $a_{-1} = 0$. Thus, one can define

$$F(z) = \sum_{j=-\infty}^{-1} \frac{a_{j-1}}{j} z^j + \sum_{j=0}^{\infty} \frac{a_j}{j+1} z^{j+1}.$$

$F(z)$ is holomorphic on $|z| > R$ and $F'(z) = g(z)$. ▯

THEOREM 21.5 *Let p_m and q_n be two holomorphic polynomials such that $q_n \neq 0$ on $|z| \geq R$. If $n - m \geq 2$ then*

$$\int_{|z|=R} \frac{p_m(z)}{q_n(z)}dz = 0$$

Proof. Since $\frac{p_m}{q_n}$ is holomorphic on $|z| > R$, for any $R_1 > R$ we have

$$0 = \int_{\partial A(0;R,R_1)} \frac{p_m(z)}{q_n(z)}dz = \int_{|z|=R_1} \frac{p_m(z)}{q_n(z)}dz - \int_{|z|=R} \frac{p_m(z)}{q_n(z)}dz$$

Therefore,

$$\Big|\int_{|z|=R} \frac{p_m(z)}{q_n(z)}dz\Big| = \Big|\int_{|z|=R_1} \frac{p_m(z)}{q_n(z)}dz\Big| \leq C\int_0^{2\pi} R_1^{m-n} R_1 d\theta = 2\pi C R_1^{m+1-n} \to 0$$

as $R_1 \to \infty$. This proves the theorem. ▯

21.4 Residue Theorems

THEOREM 21.6 *(Residue Theorem I) Let D be a bounded domain in $\mathbb{C}$ with piecewise C^1 boundary. For $\{z_1, \cdots, z_n\} \subset D$, if f is holomorphic in $D \subseteq \{z_1, \cdots, z_n\}$ and $f \in C(\overline{D} \setminus \{z_1, \cdots, z_n\})$, then*

$$\frac{1}{2\pi i} \int_{\partial D} f(z)dz = \sum_{j=1}^{n} Res(f; z_j)$$

Proof. For any $0 < \epsilon < \frac{1}{3} \min\{|z_j - z_k|, \text{dist}(z_j, \partial D) : j \neq k, 1 \leq j, k \leq n\}$, let

$$D_\epsilon = D \setminus \cup_{j=1}^{n} \overline{D}(z_j, \epsilon).$$

Then

$$\frac{1}{2\pi i} \int_{\partial D} f(z)dz = \frac{1}{2\pi i} \int_{\partial D_\epsilon} f(z)dz + \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{|z-z_j|=\epsilon} f(z)dz = \sum_{j=1}^{n} \text{Res}(f; z_j)$$

This proves the theorem. □

THEOREM 21.7 *If f is holomorphic in $\mathbb{C} \subseteq \{z_1, \cdots, z_n\}$, then*

$$\sum_{j=1}^{n} Res(f; z_j) + Res(f; \infty) = 0$$

Proof. For any $R > r =: \max\{|z_1|, \cdots, |z_n|\}$

$$-\text{Res}(f; \infty) = \frac{1}{2\pi i} \int_{|z|=R} f(z)dz = \frac{1}{2\pi i} \int_{\partial D(0,R)} f(z)dz = \sum_{j=1}^{n} \text{Res}(f; z_j).$$

This proves the theorem. □

Examples

EXAMPLE 65 *Evaluate the integral*

$$\int_{\partial D(0,19)} \frac{z^2 + 1}{(z-2)(z-4)\cdots(z-20)} dz$$

Solution. Since

$$\begin{aligned}
0 &= \int_{\partial D(0,21)} \frac{z^2+1}{(z-2)(z-4)\cdots(z-20)}dz \\
&= \int_{\partial A(0;19,21)} \frac{z^2+1}{(z-2)(z-4)\cdots(z-20)}dz + \int_{\partial D(0,19)} \frac{z^2+1}{(z-2)(z-4)\cdots(z-20)}dz \\
&= 2\pi i \text{Res}(\frac{z^2+1}{(z-2)(z-4)\cdots(z-20)};20) + \int_{\partial D(0,19)} \frac{z^2+1}{(z-2)(z-4)\cdots(z-20)}dz \\
&= 2\pi i \frac{20^2+1}{(20-2)(20-4)\cdots(20-18)} + \int_{\partial D(0,19)} \frac{z^2+1}{(z-2)(z-4)\cdots(z-20)}dz \\
&= 2^{-8}\pi i \frac{401}{(10-1)(10-2)\cdots(10-9)} + \int_{\partial D(0,19)} \frac{z^2+1}{(z-2)(z-4)\cdots(z-20)}dz
\end{aligned}$$

Therefore,

$$\int_{\partial D(0,19)} \frac{z^2+1}{(z-2)(z-4)\cdots(z-20)}dz = -\frac{401}{2^8 9!}\pi i$$

EXAMPLE 66 *Let f be entire holomorphic such that $f(k\pi) = \frac{1}{k\pi}$ if $k > 0$ and $f(k\pi) = 0$ if $k \leq 0$. Evaluate the integral*

$$\int_{|z|=n\pi+\frac{\pi}{2}} \frac{f(z)}{\sin z}dz \quad \textit{and} \quad \lim_{n\to\infty} \int_{|z|=n\pi+\frac{\pi}{2}} \frac{f(z)}{\sin z}dz$$

Solution. Notice that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{|z|=n\pi+\frac{\pi}{2}} \frac{f(z)}{\sin z}dz &= \sum_{|k|\leq n} \text{Res}(\frac{f}{\sin z};k\pi) \\
&= \sum_{|k|\leq n} \frac{f(k\pi)}{\cos(k\pi)} \\
&= \frac{1}{\pi}\sum_{k=1}^{n} \frac{(-1)^k}{k}.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \int_{|z|=n\pi+\frac{\pi}{2}} \frac{f(z)}{\sin z}dz = 2i\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -2i\ln 2.$$

21.5 Winding number and Residue Theorem

Definition 21.8 *Let $\gamma : [0,1] \to \mathbb{C}$ be piecewise connected C^1 closed curve. Let $z_0 \in \mathbb{C} \subseteq \gamma([0,1])$. Then the winding number of γ around z_0 is*

$$n(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{w - z_0}dw.$$

EXAMPLE 67 *Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be defined by $\gamma(t) = e^{imt}$. Then*

$$n(\gamma; 0) = m.$$

Proof.

$$n(\gamma; 0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{e^{imt}}de^{imt} = \frac{2m\pi i}{2\pi i} = m.$$

The proof is complete. ▯

THEOREM 21.9 *Let $\gamma : [0,1] \to \mathbb{C}$ be piecewise connected C^1 closed curve. Then $n(\gamma; z)$ is a constant integer on each connected component of $\mathbb{C} \subseteq \gamma([0,1])$.*

Proof. It suffices to prove $n(\gamma; z)$ is a integer-valued function on $\mathbb{C} \subseteq \gamma([0,1])$ because $n(\gamma; z)$ is a continuous function of z on each connected component of $C \setminus \gamma([0,1])$. This is equivalent to proving that $e^{in(\gamma;z)2\pi} \equiv 1$ on $\mathbb{C} \subseteq \gamma([0,1])$. Notice that

$$e^{in(\gamma;z)2\pi} = e^{\int_\gamma \frac{1}{w-z}dw} = e^{\int_0^1 \frac{\gamma'(t)}{\gamma(t)-z}dt}.$$

Let

$$g(t) = (\gamma(t) - z)e^{-\int_0^t \frac{\gamma'(s)}{\gamma(s)-z}ds}.$$

Then

$$g(0) = (\gamma(0) - z), \quad g(1) = g(0)e^{-2\pi i n(\gamma,z)}$$

and

$$g'(t) = [\gamma'(t) - (\gamma(t) - z)\frac{\gamma'(t)}{\gamma(t) - z}]e^{-\int_0^t \frac{\gamma'(s)}{\gamma(s)-z}ds} \equiv 0$$

Therefore, $g(1) = g(0)$. This implies

$$e^{\int_0^1 \frac{\gamma'(s)}{\gamma(s)-z}ds} = 1$$

The proof is complete. ▯

Definition 21.10 *Let f be holomorphic in $D(z_0, \delta) \setminus \{z_0\}$ with Laurent series:*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

We call

$$s(z, z_0) = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n$$

the singular part of f around $z = z_0$.

REMARK 3 *If f is holomorphic in $D(z_0, \delta) \subseteq \{z_0\}$ with singular part $s(z, z_0)$ at z_0, then $f(z) - s(z, z_0)$ is holomorphic in $D(z_0, \delta)$.*

THEOREM 21.11 *(Residue Theorem II) Let D be a simply connected domain in $\mathbb{C}$. Let $\{z_1, \cdots, z_n\} \subset D$ and f be holomorphic in $D \subsetneq \{z_1, \cdots, z_n\}$. Let γ be any connected, piecewise C^1 closed curve in $D \setminus \{z_1, \cdots, z_n\}$. Then*

$$\frac{1}{2\pi i} \int_\gamma f(z) dz = \sum_{k=1}^{n} n(\gamma; z_k) Res(f; z_k)$$

Proof. Let s_k be the singular part of f around z_k. Then $g(z) = f(z) - \sum_{k=1}^{n} s_k(z)$ is holomorphic in D. By Cauchy's theorem, since D is simply connected, we have

$$\int_\gamma g(z) dz = 0$$

Thus

$$\begin{aligned} 0 = \frac{1}{2\pi i} \int_\gamma g(z) dz &= \frac{1}{2\pi i} \int_\gamma f(z) dz - \sum_{k=1}^{n} \frac{1}{2\pi i} \int_\gamma s_k(z) dz \\ &= \frac{1}{2\pi i} \int_\gamma f(z) dz - \sum_{k=1}^{n} \mathrm{Res}(f; z_k) \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_k} dz \\ &= \frac{1}{2\pi i} \int_\gamma f(z) dz - \sum_{k=1}^{n} \mathrm{Res}(f; z_k) n(\gamma; z_k) \end{aligned}$$

So the proof is complete. ▯

EXAMPLE 68 *Let $f(z) = \pi \cot(\pi z)$. Then*

(i) f is meromorphic in $\mathbb{C}$ with $\mathbb{Z}$ as the set of poles;
(ii) Every pole $z = k$ of f is simple and $Res(f; k) = 1$;
(iii) Singular part of f at $z = k$ is $\frac{1}{z-k}$
(iv) $f(z) - \frac{1}{z} - \sum_{k=1}^{\infty} \frac{2z}{(z-k)(z+k)}$ is entire.

Proof. (i) Since $\sin(\pi z) = 0$ if and only if $z = k \in \mathbb{Z}$.

(ii) $(\sin \pi(z))' = \pi \cos(\pi z) \neq 0$ when $z = k$. So, $z = k$ is a simple pole and

$$(f, k) = \pi \frac{\cos(k\pi)}{\pi \cos(k\pi)} = 1.$$

(iii) By Part (ii), one has

$$S_f(z, k) = \frac{1}{z - k}$$

(iv) By Parts (i), (ii) and (iii), one has

$$f(z) - \frac{1}{z} - \sum_{|k|<N+1} \frac{1}{z - k}$$

is holomorphic in $|z| < N + 1$. Moreover,

$$\sum_{|k|<N} \frac{1}{z-k} = \sum_{k=1}^{N} \frac{1}{z-k} + \frac{1}{z-k} = \frac{1}{z} + \sum_{k=1}^{N} \frac{2z}{(z-k)(z+k)}$$

and

$$g(z) =: \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{(z-k)(z+k)}$$

defines a meromorphic function on $\mathbb{C}$ and $f(z) - g(z)$ is holomorphic in $\mathbb{C}$.
□

THEOREM 21.12 *We have the functional identity:*

$$\pi \cot(\pi z) = \frac{1}{z} - \sum_{k=1}^{\infty} \frac{2z}{k^2 - z^2}$$

Proof. We sketch the proof: Let

$$\Gamma_n = \{z = x + iy : x = \pm(n + \frac{1}{2}), |y| \leq n + \frac{1}{2} \text{ or } y = \pm(n + \frac{1}{2}), |x| \leq n + \frac{1}{2}\}$$

Step 1. Show

$$|f(z)| = |\pi \cot(\pi z) - \frac{1}{z} - \sum_{k=1}^{\infty} \frac{2z}{(z-k)(z+k)}| \leq C(1 + |z|^{1/2}), \quad z \in \Gamma_n.$$

By Cauchy integral formula,

$$f(z) = \frac{1}{2\pi} \int_{\Gamma_n} \frac{f(w)}{w-z}, \quad |z| \leq n.$$

and

$$f'(z) = \frac{1}{2\pi} \int_{\Gamma_n} \frac{f(w)}{(w-z)^2} dw$$

converges to *zero* when $n \to \infty$. Therefore, f is constant.

Step 2. Show $f(0) = 0$.

Therefore, combining Steps 1 and 2, one has $f(z) \equiv 0$. ▯

21.6 Homework 8

Contents: Residues, Residue theorem and meromorphic functions

i. Compute each of the following residues:

(a) Res$(\frac{z^2}{(z-2i)(z+3)}, 2i)$; (b) Res$(\frac{e^z}{(z-i-1)^3}, 1+i)$; (c) Res$(\frac{cotz}{z(z+1)}, 0)$.

ii. Evaluate

(a) $\frac{1}{2\pi i}\int_{\partial D(0,5)} \frac{z}{(z+1)(z+2i)}dz$

(b) $\frac{1}{2\pi i}\int_{\gamma} \frac{e^z}{(z+3i)^2(z+3)^2(z+4)}dz$, where γ is the negatively oriented rectangle with vertices $2 \pm i$ and $-8 \pm i$.

iii. Given any k distinct points $z_1, \cdots, z_k \in D(0,1)$ and complex numbers $\alpha_1, \cdots, \alpha_k$. Construct a meromorphic function f on $D(0,1)$ with z_j as its poles and Res$(f; z_j) = \alpha_j$ for all $1 \le j \le k$.

iv. For any complex number α and positive integer k. Construct a holomorphic function f in $D(z_0, r) \setminus \{z_0\}$ such that z_0 is a pole of f of order k and Res$(f; z_0) = \alpha$.

v. Suppose that f and g are holomorphic on $\overline{D}(z_0, r)$ and g has simple zeros at $z_1, \cdots, z_n \in D(z_0, r)$. Compute

$$\frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(z)}{g(z)}dz$$

in terms of $f(z_j)$ and $g'(z_j)$, $1 \le j \le k$.

vi. Let $f(z) = e^{(z+1/z)}$. Prove

$$\text{Res}(f; 0) = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!}$$

vii. Calculate the residue of the following given functions at ∞:

(a) $f(z) = z^3 - 7z^2 + 8$; (b) $f(z) = p(z)e^z$, (c) $f(z) = \frac{e^z}{p(z)}$,

(d) $f(z) = \frac{\cos z}{4z^2 - \pi^2}$, (e) $f(z) = \sin z$, (f) $f(z) = \frac{p(z)}{q(z)}$,

where $p(z)$ and $q(z)$ are polynomials.

viii. When $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, prove

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+\alpha)^2} = \frac{\pi^2}{\sin^2(\pi\alpha)}$$

ix. Let $z = \infty$ is an isolated singularity for f, noticing the orientation with respect to ∞, we define the residue of f at ∞ as

$$\operatorname{Res}(f;\infty) = -\frac{1}{2\pi i}\int_{|z|=R} f(z)dz, \quad R >> 1.$$

a) Prove $\operatorname{Res}(f;\infty) = -\operatorname{Res}(\frac{1}{z^2}f(1/z);0)$

b) If $R(z)$ is a rational function $(P_n(z)/Q_m(z))$, then sum of all residues of f (including the residue at ∞) is zero.

c) Is the statement in b) true if R is not rational with $z = \infty$ as an isolated singularity ?

x. If f is entire and if

$$\lim_{z\to\infty} \frac{f(z)}{z} = 0$$

then f is constant

xi. Evaluate $\int_{|z|=2} \frac{1}{z^{10}+1} dz$

xii. Every convex domain D is holomorphic simply connected, i.e. $n(\gamma, z) = 0$ for $z \notin \overline{D}$ and any closed continuous, piecewise C^1 courve γ in D.)

xiii. If $f \in C(D(0,1))$ so that $\sin f(z)$ and $\cos f(z)$ are holomorphic in $D(0,1)$. Prove $f(z)$ is holomorphic in $D(0,1)$.

22 Final Review

- Material before the midterm, refer to Midterm Review (30%)
- Material after midterm examination (70%).

i. **Zeros of holomorphic functions**

a) Factorization theorem
b) Zero set of a non-constant holomorphic function is discrete; hence it is at most countable.
c) Uniqueness theorem for holomorphic functions
d) Uniqueness theorem for meromorphic functions

EXAMPLE 69 *Let f be entire holomorphic such that $|f(z)|^2 \leq |\cos z|$ on $\mathbb{C}$. Find all such f.*

EXAMPLE 70 *Let f be holomorphic in $D(0,1)$ such that*

$$f(\frac{i}{n}) = 1/n^2, \quad n \in \mathbb{N}.$$

Find all such f.

EXAMPLE 71 *Let f be holomorphic in $D(0,1)$ such that*

$$f''(1/n) + f'(1/n) = 0, \quad n \in \mathbb{N}$$

Prove that f is entire holomorphic.

Solution. The general solution of $y''(t) + y'(t) = 0$ is

$$y(t) = C_1 + C_2 e^{-t}.$$

Since $y(z) = C_1 + C_2 e^{-z}$ is entire holomorphic and $f(z) = y(z)$ on $\mathbb{R}$, by the Uniqueness theorem for holomorphic functions,

$$f(z) = C_1 + C_2 e^{-z}.$$

ii. **Isolated Singularities**

a) Riemann's Lemma

b) Factorization theore for meromorphic functions: $f(z) = \frac{g(z)}{(z-z_0)^k}$

c) Cassoratti-Weierstrass Theorem

d) Laurent series

EXAMPLE 72 *Let f be holomorphic in $D = D(0,1) \setminus \{0\}$ such that*

$$\int_D |f(z)| dA(z) < \infty.$$

Prove that $z = 0$ is either a removable singularity or a pole of order 1.

EXAMPLE 73 *Let*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad r_0 \leq |z| < R_0$$

Evaluate

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta, \quad r \in (r_0, R_0)$$

in terms of a_n and r.

The answer is

$$\int_0^{2\pi} |f(re^{i\theta})|^2 = 2\pi \sum_{n=-\infty}^{\infty} |a_n| r^{2n}$$

EXAMPLE 74 *Find the largest set in $\mathbb{C}$ such that the Laurent series*

$$\sum_{n=-\infty}^{\infty} a_n z^n, \quad a_n = \begin{cases} 3^n, & \textit{if } n \leq 0 \\ \frac{2^{-n}}{n}, & \textit{if } n < 0 \end{cases}$$

The answer is

$$\overline{D}(0,2) \setminus \{2\} \setminus \overline{D}(0,1/3).$$

EXAMPLE 75 *Let f be holomorphic on $D(0,1) \setminus \{0\}$. Classify the singularity of* $\cos(f(z))$ *at $z = 0$ based on the singularity that f has at $z = 0$; that is, suppose f has a removable singularity (resp. pole, essential singularity), what type of singularity does cos$(f(z))$ have at $z = 0$?*

iii. **Meromorphic functions**

How do you construct a meromorphic function?

EXAMPLE 76 *Construct a meromorphic function in $\mathbb{C}$ with $\mathbb{Z}$ as its set of the poles*

EXAMPLE 77

(a f is holomorphic in $\overline{\mathbb{C}}$ if and only if f is a constant;

(b) f is meromorphic on $\overline{\mathbb{C}}$ if and only if f is rational.

iv. **Residue and Residue Theorem**

(a) How to find the residue?
(b) Residue Theorem
(c) Winding number

EXAMPLE 78 *Find the Residue of f at z_0:*

(a) $f(z) = \frac{\cot z}{z}$, $z_0 = 0$. *Answer:* $Res(f;0) = 0$
(b) $f(z) = \frac{e^z}{sinz}$, $z_0 = \pi$. *Answer:* $Res(f;0) = -e^{\pi}$
(c) $\frac{\cos z}{z^4}$, $z_0 = 0$. *Answer:* $Res(f;0) = 0$

EXAMPLE 79 *Prove or disprove there is a holomorphic f $\mathbb{C} \backslash D(0,3)$ such that*

$$f'(z) = \frac{10}{(z-1)(z+2)}$$

The answer is yes because $\text{Res}(f;\infty) = 0$. Can you construct f?

$$\frac{10}{(z-1)(z+2)} = \frac{10}{3}(\frac{1}{z-1} - \frac{1}{z+2}) = \frac{10}{3}\sum_{n=1}^{\infty}[1+(-2)^n]z^{-n-1} = \frac{10}{3}(\sum_{n=1}^{\infty}\frac{1+(-2)^n}{-n}z^{-n})'.$$

EXAMPLE 80 *Let $\gamma(t) = a\cos(nt) + ib\sin(nt) : [0, \pi] \to \mathbb{C}$ for some $n \in \mathbb{N}$ and $a, b > 1$. Evaulate*

$$\int_\gamma \frac{z^2+2}{z(z-1)} dz.$$

The answer is

$$\begin{aligned}\int_\gamma \frac{z^2+2}{z(z-1)} dz &= 2\pi i(n(\gamma, 0)\text{Res}(f; 0) + n(\gamma, 1)\text{Res}(f; 1)] \\ &= 2\pi i[-2n + 3n] \\ &= 2n\pi i.\end{aligned}$$

REMARK 4 *If f is holomorphic in $D(z_0, \delta) \subseteq \{z_0\}$ with singular part $s(z, z_0)$ at z_0, then $f(z) - s(z, z_0)$ is holomorphic in $D(z_0, \delta)$.*

THEOREM 22.1 *(Residue Theorem II) Let D be a simply connected domain in $\mathbb{C}$. Let $\{z_1, \cdots, z_n\} \subset D$ and f is holomorphic in $D \subseteq \{z_1, \cdots, z_n\}$. Let γ be any connected, piecewise C^1 closed curve in $D \subseteq \{z_1, \cdots, z_n\}$. Then*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{k=1}^{n} n(\gamma; z_k)Res(f; z_k).$$

Exercise:

i. Let $f(z)$ and $g(z)$ be two entire functions so that $f(1/2) - g(1/2) = 1$ and $|f(z) - g(z)| \leq |\sin(\pi z)|$ on $\mathbb{C}$. What is the relation between f and g?

ii. Every convex domain D is holomorphic simply connected, i.e. $n(\gamma, z) = 0$ for $z \notin \overline{D}$ and any closed continuous, piecewise C^1 curve γ in D.)

iii. Let $f(z)$ be holomorphic in $D(0, 1)$ so that $f(re^{i\pi/4}) = (1 - i)r$ for $r \in (0, 1)$. Find all such f.

iv. If $f \in C(D(0, 1))$ so that $\sin f(z)$ and $\cos f(z)$ are holomorphic in $D(0, 1)$. Prove $f(z)$ is holomorphic in $D(0, 1)$.

v. Find the value

$$\sum_{j=-\infty}^{\infty} \frac{\tan(ni)}{n^2+1}$$

References

[1] L. Ahlfors, Complex Analysis, 3rd Edition, McGraw-Hill, New Yprk, 1979.

[2] J. Conway, Functions of One Complex Analysis, Springer, New York, 2012.

[3] Robert Greene and Steven Krantz, *Function Theory of One Complex Variable*, Graduate Study in Mathematics, Vol 40, AMS, Thord Edition, 2006.

[4] W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1996.