Lecture Note for Math 220B
Complex Analysis of One Variable

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1 The Residue theorem applied to real integrals

We are going to apply the Residue theorem to evaluate integrals of real-valued functions over subsets of $\mathbb{R}$. We will call such integrals real integrals.

1.1 Several types of real integrals

I) Let $R(x, y)$ be a rational function in x and y. How does one evaluate

$$\int_0^{2\pi} R(\cos t, \sin t)dt?$$

Solution. Let

$$z = e^{it}, \quad t \in [0, 2\pi).$$

Then

$$\cos t = \frac{z + \frac{1}{z}}{2} = \frac{z^2+1}{2z}, \quad \sin t = \frac{z - \frac{1}{z}}{2i} = \frac{z^2-1}{2iz} \quad \text{and } dt = \frac{dz}{iz}.$$

Therefore,

$$\int_0^{2\pi} R(\cos t, \sin t)dt = \int_{|z|=1} R(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz})\frac{1}{iz}dz.$$

We can apply the Residue theorem to evaluate the right hand side.

EXAMPLE 1 *For $a > 1$, evaluate the integral*

$$\int_0^{2\pi} \frac{1}{a + \cos t}dt$$

Solution. Let

$$z =: e^{it}$$

Then

$$\cos t = \frac{1}{2}(z + \frac{1}{z}), \quad dz = izdt$$

Therefore,

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{a+\cos t}dt &= \int_{|z|=1} \frac{1}{a+\frac{1}{2}(z+1/z)}\frac{1}{iz}dz \\
&= \frac{2}{i}\int_{|z|=1} \frac{1}{2az+z^2+1}dz \\
&= 4\pi \mathrm{Res}\Big(\frac{1}{z^2+2az+1}; -a+\sqrt{a^2-1}\Big) \\
&= \frac{4\pi}{2(-a+\sqrt{a^2-1})+2a} \\
&= \frac{2\pi}{\sqrt{a^2-1}}.
\end{aligned}$$

II) How does one evaluate

$$\int_{-\infty}^{\infty} R(x)dx?$$

THEOREM 1.1 *Let $P_m(x)$ and $Q_n(x)$ be polynomials of degree m and n respectively and $(P_m, Q_n) = 1$. Let $R(x) = \frac{P_m(x)}{Q_n(x)}$. Suppose also that $Q_n(x) \neq 0$ for all $x \in \mathbb{R}$ and $n - m \geq 2$. Then*

$$\int_{-\infty}^{\infty} R(x)dx = 2\pi i (\sum_{k=1}^{\ell} Res(R; z_k))$$

where $z_1, \cdots, z_\ell$ are the zeros of Q_n not counting multiplicity in the upper half plane $\mathbb{R}^2_+$.

Proof. Let $\{z_1, ..., z_\ell\}$ be the zeros stated in the theorem's hypothesis and choose $r >> 1$ such that $\{z_1, \cdots, z_\ell\} \subset D(0, r)$. Apply the Residue theorem to $R(z)$ on $D_r = \{z \in \mathbb{R}^2_+ : |z| < r\}$. Then

$$\int_{-r}^{r} R(x) = -\int_{C_r} R(z)dz + 2\pi i \sum_{k=1}^{\ell} \mathrm{Res}(R, z_k)$$

where $C_r = \{z = re^{i\theta} : 0 \leq \theta \leq \pi\}$. Since $n \geq m+2$, we have

$$\lim_{r\to\infty} \int_{C_r} R(z)dz = 0.$$

The theorem is proved. ∎

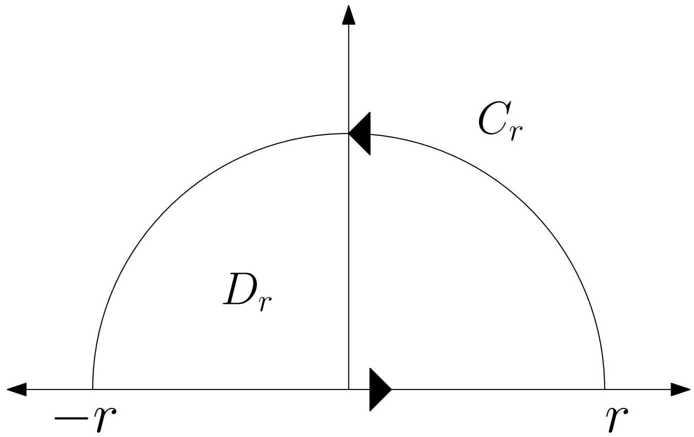

Figure 1: The nice contour

EXAMPLE 2 *Evaluate the integral*

$$\int_0^\infty \frac{1}{(1+x^2)^2} dx.$$

Solution. Let $f(z) = 1/(z^2+1)^2$. Then

$$\begin{aligned}
\int_0^\infty \frac{1}{(1+x^2)^2} dx &= \frac{1}{2}\int_{-\infty}^\infty \frac{1}{(1+x^2)^2} dt \\
&= \pi i \mathrm{Res}(f; i) = \pi i \frac{-2}{(z+i)^3}\bigg|_{z=i} \\
&= -\frac{2\pi i}{(2i)^3} = \frac{\pi}{4}
\end{aligned}$$

III) How does one evaluate

$$\int_{-\infty}^\infty f(x) e^{ix} dx?$$

Lemma 1.2 *Let $f(z)$ be meromorphic in $\mathbb{R}^2_+$ such that*

$$\lim_{r\to\infty} f(re^{i\theta}) = 0$$

uniformly for $\theta \in (0, \pi)$. Let $\{z_k : 1 \le k \le m\}$ be zero set of f in $\mathbb{R}^2_+$. Then

$$\lim_{R\to\infty} \int_{C_R} f(z) e^{iz} dz = 0$$

where $C_R = \{z = x + iy \in \mathbb{C} : |z| = R, y > 0\}$

Proof. Let $M_R = \max\{|f(Re^{i\theta})| : \theta \in [0, \pi]\}$. Then

$$\begin{aligned}
\Big|\int_{C_R} f(z)e^{iz}dz\Big| &\leq \int_0^\pi |f(Re^{i\theta})|e^{-R\sin\theta}Rd\theta \\
&\leq M_R \int_0^\pi e^{-R\sin\theta}Rd\theta \\
&= 2M_R \int_0^{\pi/2} e^{-R\sin\theta}Rd\theta \\
&\leq 2M_R \int_0^{\pi/2} e^{-2R\theta/\pi}Rd\theta \\
&\leq \pi M_R \int_0^\infty e^{-t}dt \\
&= \pi M_R \to 0 \quad \text{as } R \to \infty
\end{aligned}$$

where the third inequality follows because $sin(\theta)$ is concave on $[0, \frac{\pi}{2}]$; hence $sin(\theta) > \frac{2\theta}{\pi}$ on $[0, \frac{\pi}{2}]$. ▯

THEOREM 1.3 *Let $f(z)$ be meromorphic in $\mathbb{R}^2_+$ such that*

$$\lim_{r\to\infty} f(re^{i\theta}) = 0$$

uniformly for $\theta \in (0, \pi)$. Let $\{z_k : 1 \leq k \leq m\}$ be zero set of f in $\mathbb{R}^2_+$. Then

$$\int_{-\infty}^{\infty} f(x)e^{ix}dx = 2\pi i \sum_{k=1}^{m} Res(fe^{iz}; z_k)$$

Proof. Let $D_R = \{z \in \mathbb{R}^2_+ : |z| < R\}$. By the Residue theorem, one has

$$\int_{-R}^{R} f(x)e^{ix} + \int_{C_R} f(z)e^{iz}dz = \int_{\partial D_R} f(z)e^{iz}dz = 2\pi i \sum_{z_k \in D_R} \text{Res}(f(z)e^{iz}; z_k)$$

Let $R \to \infty$. By the previous lemma,

$$\int_{-\infty}^{\infty} f(x)e^{ix} = 2\pi i \sum_{k=1}^{m} \text{Res}(f(z)e^{iz}; z_k)$$

The proof is complete. ▯

EXAMPLE 3 *Evaluate the following integral:*

$$\int_0^\infty \frac{\cos x}{1+x^2}dx$$

Solution. Since

$$\begin{aligned}
\int_0^\infty \frac{\cos x}{1+x^2}dx &= \frac{1}{2}\int_{-\infty}^\infty \frac{\cos x}{1+x^2}dx \\
&= \frac{1}{2}\text{Re}\int_{-\infty}^\infty \frac{e^{ix}}{1+x^2}dx \\
&= Re\Big(\pi i \text{Res}(\frac{e^{iz}}{1+z^2}; i)\Big) \\
&= \frac{\pi}{2e}.
\end{aligned}$$

EXAMPLE 4 *Evaluate the following integral:*

$$\int_0^\infty \frac{\sin x}{x}dx$$

Solution. We would like to write

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin x}{x}dx = \frac{1}{2}\text{Im}\int_{-\infty}^\infty \frac{e^{ix}}{x}dx.$$

However, the last equality may cause trouble because the last integral does not converge. So, instead, we write

$$\int_{-\infty}^\infty \frac{\sin x}{x}dx = \lim_{R\to\infty, r\to 0}\Big[\int_{-R}^{-r}\frac{sin(x)}{x}dx + \int_r^R \frac{\sin x}{x}dx\Big]$$

Let

$$D_{r,R} = \{z \in \mathbb{R}^2_+ : |z| < R, |z| > r\}$$

and

$$\partial D_{r,R} = C_R \cup [-R,-r] \cup (-C_r) \cup [r,R].$$

Thus

$$\begin{aligned}
\int_{-\infty}^\infty \frac{\sin x}{x}dx &= \lim_{R\to\infty, r\to 0}\text{Im}\Big[\int_{-R}^{-r}\frac{e^{ix}}{x}dx + \int_r^R \frac{e^{ix}}{x}dx\Big] \\
&= \lim_{R\to\infty, r\to 0}\text{Im}\Big[\int_{\partial D_{r,R}}\frac{e^{iz}}{z}dz - \int_{C_R}\frac{e^{iz}}{z}dz + \int_{C_r}\frac{e^{iz}}{z}dz\Big] \\
&= 0 + 0 + \lim_{r\to 0}\text{Im}\int_{C_r}\frac{e^{iz}}{z}dz \\
&= \lim_{r\to 0}\text{Im}\int_0^\pi \frac{e^{ri\cos\theta - r\sin\theta}}{re^{i\theta}} ire^{i\theta}d\theta \\
&= \pi
\end{aligned}$$

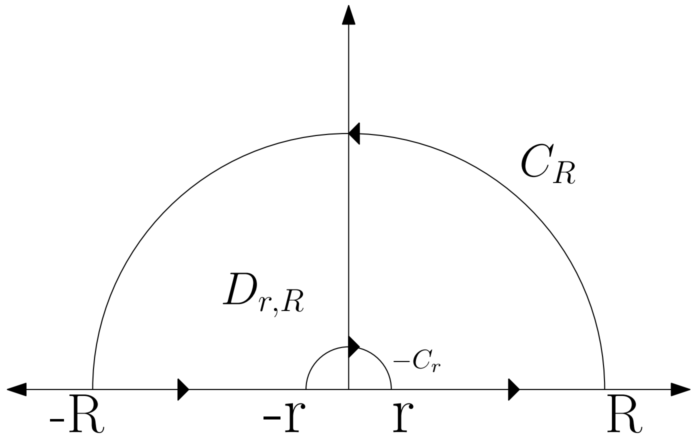

Figure 2: The semi-circular contour with a bump

where the third equality follows because $\frac{e^{iz}}{z}$ is holomorphic in a neighborhood of $D_{r,R}$ and lemma 1.2. Therefore,

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

∎

EXAMPLE 5 *Evaluate the following integral:*

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx.$$

Solution.

$$\begin{aligned} \int_0^\infty \frac{\sin^2 x}{x^2} dx &= -\frac{\sin^2 x}{x}\Big|_0^\infty + \int_0^\infty \frac{2 \sin x \cos x}{x} dx \\ &= \int_0^\infty \frac{\sin(2x)}{x} dx \\ &= \int_0^\infty \frac{\sin(2x)}{2x} d(2x) \\ &= \frac{\pi}{2}. \quad ∎ \end{aligned}$$

EXAMPLE 6 *Evaluate the following integral:*

$$\int_0^\infty \frac{1 - \cos x}{x^2} dx$$

Solution.

$$\int_0^\infty \frac{1-\cos x}{x^2}dx = -\frac{1-\cos x}{x}\Big|_0^\infty + \int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}. \qquad \square$$

IV) Choosing special integral path.

EXAMPLE 7 *Evaluate the following integral:*

$$\int_0^\infty \frac{1}{1+x^3}dx.$$

Solution. We know that $(e^{2\pi i/3})^3 = 1$. Choose

$$D_R = \{z = |z|e^{i\theta} : 0 < \theta < 2\pi/3, |z| < R\}$$

Then for $R > 2$,

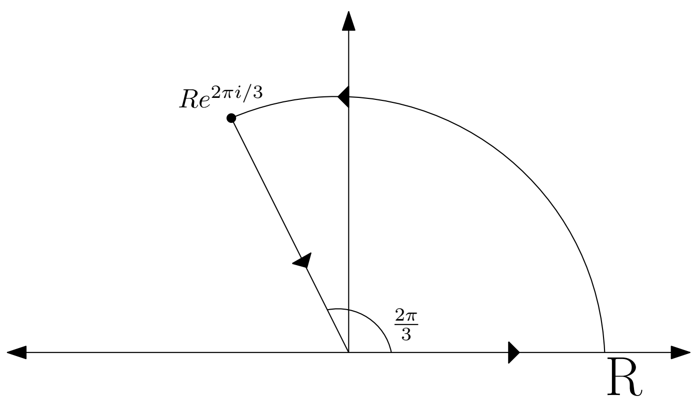

Figure 3: Another common contour

$$\int_{\partial D_R} \frac{1}{1+z^3}dz = 2\pi i \mathrm{Res}(\frac{1}{1+z^3}; e^{i\pi/3}) = \frac{2\pi i}{3e^{2\pi i/3}}$$

On the other hand,

$$\begin{aligned}\int_{\partial D_R} \frac{1}{1+z^3}dz &= \int_0^R \frac{1}{1+x^3}dx + \int_{C_R} \frac{1}{1+z^3}dz + \int_R^0 \frac{1}{1+x^3}e^{2\pi i/3}dx \\ &\to (1-e^{2\pi i/3})\int_0^\infty \frac{1}{1+x^3}dx\end{aligned}$$

as $R \to \infty$. Therefore,

$$\begin{aligned}\int_0^\infty \frac{1}{1+x^3}dx &= \frac{2\pi i}{3(1-e^{i2\pi/3})e^{2\pi i/3}} \\ &= \frac{2\pi i}{3}\frac{1}{(e^{2\pi i/3}-e^{-2\pi i/3})} \\ &= \frac{\pi}{3\sin\frac{2\pi}{3}} \\ &= \frac{2\pi\sqrt{3}}{9}. \quad \square\end{aligned}$$

EXAMPLE 8 *Integrate*

$$\int_0^\infty \frac{x}{(1+x^3)^2}dx.$$

You can do the same thing and get

$$(1-e^{4\pi i/3})\int_0^\infty \frac{x}{(1+x^3)^2}dx = 2\pi i \mathrm{Res}\Big(\frac{x}{(1+x^3)^2}; e^{\pi i/3}\Big)$$

It is left to the reader to finish the computation.

1.2 The Logarithm function

From calculus, $f(x) = e^x : (-\infty, \infty) \to (0, \infty)$ is one-to-one and onto; its inverse function can be denoted either *log* x or $\ln x$. The exponential function extends to an entire holomorphic function $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ defined by

$$f(z) = e^z = e^{x+iy} = e^x e^{iy}.$$

It is easy to see that

$$f(z + 2\pi i) = f(z)$$

and if $S_0 = \{x + iy : x \in \mathbb{R}, 0 < y < 2\pi\}$, then $f(z)|_{S_0} : S_0 \to \mathbb{C} \setminus [0, \infty)$ is one-to-one and onto. The inverse function of $f(z)|_{S_0}$ is called the principle log function, denoted by $\log : \mathbb{C} \setminus [0, \infty) \to S_0$ and it is defined by

$$\log z = \log |z| + i\theta, \quad -\pi < \theta < \pi.$$

For each integer k, let $S_k = \{2\pi ik\} + S_0$, then $\log : \mathbb{C} \setminus [0, \infty) \to S_k$ is a branch of Log defined by

$$\text{Log } z = \log z + 2k\pi i$$

The general *Log* z is an infinitely many valued function.

Definition 1.4 *Let D be a domain and f a continuous function on D. We say that f is a branch of Log on D if*

$$e^{f(z)} = z, \quad z \in D.$$

EXAMPLE 9 *Let $D = \mathbb{C} \setminus \{0\}$. Then Log cannot have a branch on D.*

Proof. Suppose there is a continuous function f on D such that $e^{f(z)} = z$ on D. Then f is holomorphic and

$$f'(z) = \frac{1}{z}, \quad z \in D.$$

Then

$$0 = \int_{|z|=1} f'(z)dz = \int_{|z|=1} \frac{1}{z}dz = \log z\Big|_1^{e^{2\pi i}} = 2\pi i$$

This is a contradiction. □

Remark: Let D be a simply connected domain in $\mathbb{C}$ such that $0 \notin D$. Then there is a branch of Log on D.

The function z^α is defined by

$$z^\alpha = e^{\alpha \text{Log} z}$$

EXAMPLE 10 *Find all z such that $z^{10} = -1$.*

Solution. Since $-1 = e^{i\pi}$, we have

$$Log(-1) = 0 + (2k+1)\pi i, \quad k \in \mathbb{Z}.$$

Thus

$$(-1)^{1/10} = e^{\frac{(2k+1)\pi}{10} i}, \quad k = 0, 1, \cdots, 9. \quad \Box$$

Let

$$R(z) = \frac{P_m(z)}{Q_n(z)}, \quad n - m - \alpha > 1$$

How does one integrate

$$\int_0^\infty R(x) x^\alpha dx?$$

When $\alpha \notin \mathbb{Z}$, by an argument similar to the one used in theorem 1.1,

$$(1 - e^{i\alpha 2\pi}) \int_0^\infty R(x) x^\alpha dx = 2\pi i \sum \text{Res}(R(z) z^\alpha, z_k)$$

EXAMPLE 11 *Integrate*

$$\int_0^\infty \frac{x^{1/3}}{1+x^3} dx$$

Solution. $\frac{1}{1+z^3}$ has three poles:

$$z_1 = e^{i\pi/3}, \quad z_2 = -1 \quad \text{and} \quad z_3 = e^{5\pi i/3}.$$

Therefore,

$$(1 - e^{2\pi i/3}) \int_0^\infty \frac{x^{1/3}}{1+x^3} dx = 2\pi i \left(\frac{e^{i\pi/9}}{3e^{2\pi i/3}} + \frac{e^{\pi i/3}}{3} + \frac{e^{5\pi i/9}}{3e^{10\pi i/3}} \right) = \frac{2\pi i}{3} (-e^{i4\pi/9} - e^{2\pi i/9} + e^{\pi i/3}).$$

Therefore,

$$\begin{aligned}
\int_0^\infty \frac{x^{1/3}}{1+x^3} dx &= \frac{2\pi i}{3} \left(-\frac{e^{i4\pi/9}}{(1-e^{2\pi i/3})} - \frac{e^{2\pi i/9}}{(1-e^{2\pi i/3})} + \frac{e^{\pi i/3}}{(1-e^{2\pi i/3})} \right) \\
&= \frac{\pi}{3} \left(-\frac{e^{i4\pi/9} + e^{2\pi i/9}}{\frac{(1-e^{2\pi i/3})}{2i}} + \frac{1}{\frac{e^{-\pi i/3} - e^{\pi i/3}}{2i}} \right) \\
&= \frac{\pi}{3} \left(-\frac{e^{i\pi/9} + e^{-\pi i/9}}{\frac{(e^{-\pi i/3} - e^{\pi i/3})}{2i}} - \frac{1}{\sin(\pi/3)} \right) \\
&= \frac{\pi}{3} \left(\frac{2\cos(\pi/9)}{\sin(\pi/3)} - \frac{1}{\sin(\pi/3)} \right).
\end{aligned}$$

Another way to compute is to use the argument preceeding this example:

$$(1 - e^{2\pi i/9}e^{2\pi i/3}) \int_0^\infty \frac{x^{1/3}}{1+x^3}dx = 2\pi i \frac{e^{\pi i/9}}{3e^{i\pi 2/3}} = -\frac{2\pi i}{3}e^{4\pi i/9}.$$

Therefore,

$$\int_0^\infty \frac{x^{1/3}}{1+x^3}dx = -\frac{\pi}{3}\frac{2i}{e^{-4\pi i/9} - e^{4\pi i/9}} = \frac{\pi}{3}\frac{1}{\sin(4\pi/9)}.$$

1.3 Real integrals involving $\ln x$

V. Integration involved in $\ln x$

Let $R = \frac{P}{Q}$ be rational with deg(Q)-deg(P) ≥ 2 and $Q(x) \neq 0$ on $[0, \infty)$. Evaluate

$$\int_0^\infty R(x) \ln x dx.$$

Discussion: Let $D_r = D(0, r) \setminus [0, r)$. We view $\partial D_r = (0, r) \cup C_r \cup [re^{2\pi i}, 0)$ where $C_r = \partial D(0, r)$. Choose $r >> 1$ such that all of the poles of R, say $z_1, \cdots, z_n$, are in $D(0, r)$. Then

$$\int_{\partial D_r} R(z) \ln z dz = 2\pi i \sum_{k=1}^{n} \text{Res}(R(z) \ln z; z_k)$$

On the other hand,

$$\begin{aligned}
\int_{\partial D_r} R(z) \ln z dz &= \int_0^r R(x) \ln x dx + \int_{C_r} R(z) \ln z dz + \int_r^0 R(x)(\ln x + 2\pi i) dx \\
&= \int_{C_r} R(z) \ln z dz - \int_0^r R(x) 2\pi i dx.
\end{aligned}$$

Since $n - m \geq 2$,

$$|R(z)| \leq C|z|^{m-n} \leq C|z|^{-2}, \quad \text{when } |z| >> 1$$

and

$$\begin{aligned}
|\int_{C_r} R(z) \ln z dz| &\leq \int_0^{2\pi} |R(Re^{i\theta})||\ln r + i\theta| r d\theta \\
&\leq C \int_0^{2\pi} r^{-2}(\ln r + 2\pi) r d\theta \\
&\leq 2\pi C r^{-1}(\ln r + 2\pi) \\
&\to 0
\end{aligned}$$

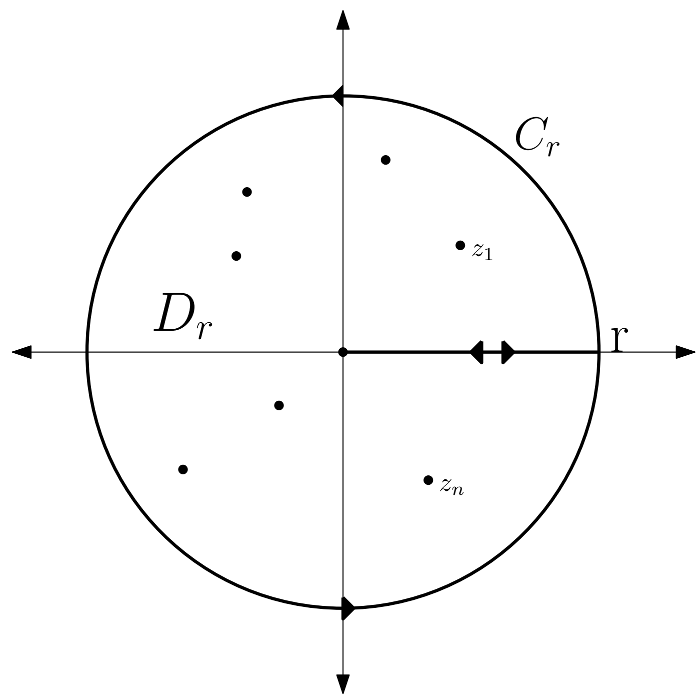

Figure 4: The contour that works well with $ln(x)$

as $r \to \infty$. Combining the above and letting $r \to \infty$, one has

$$-\int_0^\infty R(x)dx = \sum_{k=0}^{n} \text{Res}(R(z)\ln z, z_k)$$

We cannot get $\int_0^\infty R(x)\ln x\ dx$ with this calculation. Instead, we replace $R(z)\ln z$ by $R(z)(\ln z)^2$ and essentially repeat the above argument. On the one hand,

$$\int_{\partial D_r} R(z)(\ln z)^2 dz = 2\pi i \sum_{k=1}^{n} \text{Res}(R(z)(\ln z)^2; z_k)$$

On the other hand,

$$\begin{aligned} \int_{\partial D_r} R(z)(\ln z)^2 dz &= \int_0^r R(x)(\ln x)^2 dx + \int_{C_r} R(z)(\ln z)^2 dz + \int_r^0 R(x)(\ln x + 2\pi i)^2 dx \\ &= \int_{C_r} R(z)\ln z dz - 2\int_0^r R(x) 2\pi i \ln x dx - (2\pi i)^2 \int_0^r R(x)dx. \end{aligned}$$

Since $n-m\geq 2$,

$$|R(z)|\leq C|z|^{m-n}\leq C|z|^{-2},\quad \text{when } |z|>>1$$

and

$$\begin{aligned}
|\int_{C_r} R(z)(\ln z)^2 dz| &\leq \int_0^{2\pi} |R(Re^{i\theta})||\ln r+i\theta|^2 r d\theta \\
&\leq C\int_0^{2\pi} r^{-2}(\ln r+2\pi)^2 r d\theta \\
&\leq 2\pi C r^{-1}(\ln r+2\pi)^2 \\
&\to 0
\end{aligned}$$

as $r\to\infty$. Combining the above and letting $r\to\infty$, one has

$$-2\int_0^\infty R(x)\ln x dx - 2\pi i\int_0^\infty R(x)dx = \sum_{k=0}^n \text{Res}(R(z)(\ln z)^2, z_k)$$

Therefore, we have

THEOREM 1.5 *For any rational function $R(z)=P_m(z)/Q_n(z)$ with $n-m\geq 2$ and $Q_n(x)\neq 0$ on $[0,\infty)$, one has*

$$\int_0^\infty R(x)\ln x dx = -\frac{1}{2}\sum_{k=0}^n Res(R(z)(\ln z)^2, z_k) - \pi i\int_0^\infty R(x)dx$$

where $z_0,\cdots,z_n$ are the poles of $R(z)$ in $\mathbb{C}$.

EXAMPLE 12 *Evaluate*

$$\int_0^\infty \frac{\ln x}{1+x^2}dx.$$

Solution. We will provide a few methods to solve this problem.

Method 1. We can use the above formula to evaluate it.

Since $1+z^2=0$ if and only if $z=\pm i$, and

$$\text{Res}(\frac{(\ln z)^2}{1+z^2}, i) = \frac{(\ln i)^2}{2i} = \frac{(\pi i/2)^2}{2i} = -\frac{\pi^2}{8i}$$

and

$$\text{Res}(\frac{(\ln z)^2}{1+z^2}, -i) = \frac{(\ln(-i))^2}{2i} = \frac{(3\pi i/2)^2}{-2i} = \frac{9\pi^2}{8i}$$

Thus

$$\mathrm{Res}(\frac{(\ln z)^2}{1+z^2}, i) + \mathrm{Res}(\frac{(\ln z)}{1+z^2}, -i) = -\frac{\pi^2}{8i} + \frac{9\pi^2}{8i} = -\pi^2 i.$$

Therefore,

$$\int_0^\infty \frac{\ln x}{1+x^2} dx = \pi^2 i - 2\pi i \int_0^\infty \frac{1}{1+x^2} dx = \pi^2 i - 2\pi i \frac{\pi}{2} = 0.$$

Method 2. Let $D_R = \{z = re^{i\theta} \in \mathbb{C} : |z| < R, 0 < \theta < \pi\}$ with $R > 1$. Then

$$\int_{\partial D_R} \frac{\ln z}{1+z^2} dz = 2\pi i \frac{\ln i}{2i} = \pi(\pi i/2) = \frac{\pi^2}{2} i$$

and

$$\int_{\partial D_R} \frac{\ln z}{1+z^2} dz \quad \frac{\rightarrow}{R \rightarrow \infty} \quad \int_0^\infty \frac{\ln x}{1+x^2} dx + \int_\infty^0 \frac{\ln r + \pi i}{1+r^2}(-1) dr + 0 = 2\int_0^\infty \frac{\ln x}{1+x^2} dx + \pi i \frac{\pi}{2}$$

Therefore,

$$\int_0^\infty \frac{\ln x}{1+x^2} dx = 0.$$

Remark. The above argument can be applied to certain rational functions R satisfying $R(-x) = R(x)$. More precisely, we have

THEOREM 1.6 *For any rational function $R(z) = P_m(z)/Q_n(z)$ with $n - m \geq 2$, $Q_n(x) \neq 0$, and $R(-x) = R(x)$ on $[0, \infty)$,*

$$\int_0^\infty R(x) \ln x dx = \pi i \Big[\sum_{k=0}^n Res(R(z)(\ln z), z_k) - \frac{1}{2} \int_0^\infty R(x) dx \Big]$$

where $z_0, \cdots, z_n$ are the poles of $R(z)$ in $\mathbb{R}^2_+$.

Method 3. In fact, using the substitution $x \mapsto 1/x$,

$$\begin{aligned} \int_0^\infty \frac{\ln x}{1+x^2} dx &= \int_0^1 \frac{\ln x}{1+x^2} dx + \int_1^\infty \frac{\ln x}{1+x^2} dx \\ &= \int_\infty^1 \frac{\ln x}{1+x^2} dx + \int_1^\infty \frac{\ln x}{1+x^2} dx \\ &= 0. \end{aligned}$$

The arguments are done. □

1.4 Homework 1

- Applications of Residue theorem to real integrals; branches of the logarithm

1. Evaluate
$$\int_0^{2\pi} \frac{1}{1+\sin^2\theta} d\theta$$
2. Evaluate
$$\int_0^{\infty} \frac{1}{1+x^5} dx$$
3. Evaluate
$$\int_0^{\infty} \frac{x^4}{1+x^{10}} dx$$
4. Evaluate
$$\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$$
5. Evaluate
$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx$$
6. Evaluate
$$\int_0^{\infty} \frac{\cos^2 x}{1+x^2} dx$$
7. Evaluate
$$\int_0^{\infty} \frac{x^{1/4}}{1+x^3} dx$$
8. Evaluate
$$\int_0^{\infty} \frac{x}{\sinh x} dx$$
9. Evaluate
$$\int_{-\infty}^{\infty} \frac{e^{\frac{x}{3}}}{8+e^x} dx$$
10. Evaluate
$$\int_0^{\infty} \frac{\ln x}{1+x^2} dx, \quad \int_0^{\infty} \frac{\ln x}{1+x^3} dx$$
11. Let $\gamma : [0,\infty) \to \mathbb{C}$ be defined by $\gamma(t) = t^2 + it$. Prove that there is a branch $f(z)$ of the logarithm on $\mathbb{C} \setminus \gamma([0,\infty))$.
12. Prove or disprove: If f is holomorphic on $D(0,1)$ such that $f(z)^3$ is a polynomial, then f is a polynomial.

2 The Zero Set of Holomorphic Functions

Let D be a domain in $\mathbb{C}$ and let f be holomorphic function on D. Let $Z_D(f)$ denote the zero set of f on D counting multiplicity and $\#(Z_D(f))$ denote the cardinality of $Z_D(f)$. Then the following hold:

1. If $f \not\equiv 0$ then $Z_D(f)$ is at most countable.
2. If D_1 is any compact subset of D, then $\#(Z_D(f) \cap D_1)$ is finite.

Question: How can one determine $\#(Z_D(f) \cap D_1)$?

2.1 The Argument Principle and its Applications

EXAMPLE 13 *Consider $f(z) = z^{10}$ on $D(0, R)$ for any $R > 0$. We know that $\#Z_{D(0,R)}(f) = 10$.*

To understand the intuition behind the argument principle, consider these formal calculations:

$$\begin{aligned} 10 &= \frac{1}{2\pi i} \log z^{10} \Big|_R^{Re^{2\pi i}} \\ &= \frac{1}{2\pi i} \int_{|z|=R} (\log z^{10})' dz \\ &= \frac{1}{2\pi i} \int_{|z|=R} \frac{(z^{10})'}{z^{10}} dz. \end{aligned}$$

Let

$$f(z) = a \prod_{j=1}^{n} (z - z_j).$$

be a polynomial of degree n. Then

$$\frac{f'(z)}{f(z)} = (\log(f(z))' = \sum_{j=1}^{n} \frac{1}{z - z_j}.$$

If D is a bounded domain with piecewise C^1 boundary in $\mathbb{C}$ such that $z_1, \cdots, z_k \in D$ and $z_j \notin \overline{D}$ when $j > k$ then

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{k} \frac{1}{2\pi i} \int_{\partial D} \frac{1}{z - z_j} dz = k$$

In general, we have the following theorem.

THEOREM 2.1 *(Argument Principle) Let D be a bounded domain in $\mathbb{C}$ with piecewise C^1 boundary. Let $f(z)$ be holomorphic in D and $f \in C(\overline{D})$ and $f(z) \neq 0$ on ∂D. Then*

$$\#(Z_D(f)) = \frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)} dz$$

where $\#(Z_D(f))$ is the number of the zeros of f in D counting multiplicity.

Proof. Let $Z_D(f) = \{z_1, \cdots, z_n\}$ be the set of zeros of f in D counting multiplicity. Then

$$f(z) = \prod_{j=1}^{n}(z - z_j)H(z)$$

where $H(z)$ is holomorphic in D and $H(z) \neq 0$ on $\overline{D}$. Then

$$\frac{f'(z)}{f(z)} = \frac{H'(z)}{H(z)} + \sum_{j=1}^{n} \frac{1}{z - z_j}.$$

Thus,

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i}\int_{\partial D} \frac{H'(z)}{H(z)} dz + \frac{1}{2\pi i}\sum_{j=1}^{n}\int_{\partial D} \frac{1}{z - z_j} dz = 0 + n. \quad \square$$

EXAMPLE 14 *Let $p(z)$ be a polynomial such that $p(z) \neq 0$ when $\operatorname{Re} z \leq 0$. Prove that $p'(z) \neq 0$ when $\operatorname{Re} z \leq 0$.*

Proof. Let $z_1, \cdots, z_n$ be the zeros of p counting multiplicity. Then

$$p(z) = a_n \prod_{j=1}^{n}(z - z_j)$$

Then

$$\frac{p'(z)}{p(z)} = \sum_{j=1}^{n} \frac{1}{z - z_j}.$$

Since $\operatorname{Re} z_j > 0$, if $\operatorname{Re} z \leq 0$, then

$$\operatorname{Re} \frac{p'(z)}{p(z)} = \sum_{j=1}^{n} \operatorname{Re} \frac{1}{z - z_j} < 0$$

Therefore, $p'(z) \neq 0$ on $\operatorname{Re} z \leq 0$. $\square$

2.2 The Open Mapping theorem

THEOREM 2.2 *(Open Mapping Theorem) Let D be an open set in $\mathbb{C}$ and let f be holomorphic on D. Then $f(D)$ is open.*

Proof. It suffices to show that for any $w_0 \in f(D)$, there is an $\epsilon > 0$ such that $D(w_0, \epsilon) \subset f(D)$. Let $z_0 \in D$ such that $f(z_0) = w_0$. Choose $\delta > 0$ such that

$$D(z_0, \delta) \subset D, \quad f(z) \neq w_0, \quad \text{for } z \in \partial D(z_0, \delta).$$

Let

$$\epsilon = \frac{1}{2} \min\{|f(z) - w_0| : z \in \partial D(z_0, \delta)\}.$$

For $w \in D(w_0, \epsilon)$, define

$$N(w) = \frac{1}{2\pi i} \int_{\partial D(z_0,\delta)} \frac{f'(z)}{f(z) - w} dz.$$

Then $N(w)$ is well-defined on $D(w_0, \epsilon)$ and is continuous on $D(w_0, \epsilon)$. By the argument principle, $N(w)$ is an integer-valued function. Thus $N(w) \equiv N(w_0) \geq 1$. Therefore, $D(w_0, \epsilon) \subset f(D)$. □

2.3 Rouché Theorems

THEOREM 2.3 *(Rouché Theorem I) Let D be a bounded domain in $\mathbb{C}$ with piecewise C^1 boundary. Let $f, g \in C(\overline{D})$ be two holomorphic functions on D such that*

$$|f(z) + g(z)| < |f(z)| + |g(z)|, \quad z \in \partial D.$$

Then $\#(Z_D(f)) = \#Z_D(g)$.

Proof. Notice that

$$\Big|\frac{f(z)}{g(z)} + 1\Big| < 1 + \Big|\frac{f(z)}{g(z)}\Big|, \quad z \in \partial D.$$

Thus $\frac{f(z)}{g(z)} \notin [0, \infty)$ for $z \in \partial D$. Then $h(z) =: \log \frac{f(z)}{g(z)}$, where log is the principle branch, is holomorphic in a neighborhood of ∂D. Then

$$0 \;=\; \frac{1}{2\pi i} \int_{\partial D} dh(z)$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{\partial D} h'(z)dz \\
&= \frac{1}{2\pi i}\int_{\partial D} (\log\frac{f(z)}{g(z)})'dz \\
&= \frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}dz \\
&= \#(Z_D(f)) - \#(Z_D(g))
\end{aligned}
$$

where the last equality follows by the argument principle. This implies that $\#(Z_D(f)) = \#(Z_D(g))$. ▯

THEOREM 2.4 *(Rouché Theorem II) Let D be a bounded domain in $\mathbb{C}$ with piecewise C^1 boundary. Let $f, g \in C(\overline{D})$ be two holomorphic functions on D such that*

$$|f(z)| < |g(z)|, \quad z \in \partial D.$$

Then $\#(Z_D(g-f)) = \#Z_D(g)$.

Proof. On ∂D,

$$|g + (f-g)| = |f| < |g| \le |g| + |f-g|.$$

By Rouché Theorem I,

$$\#(Z_D(g)) = \#(Z_D(f-g)) = \#(Z_D(g-f)).$$

The proof is complete. ▯

2.4 Applications and Examples

EXAMPLE 15 *Find the number of zeros of*

$$f(z) = z^{10} + 3z + 1$$

on the annulus $A(0;1,2)$.

Solution. Notice that

$$\#(Z_{A(0;1,2)}) = \#(Z_{D(0,2)}) \setminus \#(Z_{\overline{D(0;1)}}).$$

Let $g(z) = -z^{10}$. Then

$$|f(z) + g(z)| = |3z + 1| \leq 7 < 2^{10} = |g(z)|, \quad z \in \partial D(0, 2)$$

By the argument principle,

$$\#(Z_{D(0,2)}(f)) = \#(Z_{D(0,2)}(g)) = 10.$$

On the other hand,

$$|g(z)| = 1 < 2 \leq |3z + 1|, \quad z \in \partial D(0, 1)$$

Then by the argument principle,

$$\#(Z_{D(0,1)}(3z + 1 + z^{10})) = \#(Z_{D(0,1)}(3z + 1)) = 1.$$

Therefore,

$$\#(Z_{A(0;1,2)}(f)) = 10 - 1 = 9. \quad \square$$

EXAMPLE 16 *Find the number of zeros of*

$$f(z) = z^{10} - 10z + 9$$

on the unit disc $D(0, 1)$.

Solution. Since $f(1) = 0$ and

$$f(z) = (z - 1)(z^9 + z^8 + \cdots + z - 9),$$

when $|z| < 1$,

$$|\sum_{k=1}^{9} z^k - 9| > 9 - 9|z| > 0$$

Therefore, $\#(Z_{D(0,1)}(f)) = 0$. $\square$

EXAMPLE 17 *Find the number of zeros of*

$$f(z) = z^2 e^z - z$$

in the disk $D(0, 2)$.

Solution. Since

$$f(z) = z^2 e^z - z = z e^z (z - e^{-z}),$$

$\#(Z_{D(0,2)}(f)) = \#(Z_{D(0,2)}(z - e^{-z})) + 1$. Let

$$g(z) = z - e^{-z}.$$

Then

$$g(z) = x + iy - e^{-x}\cos y + ie^{-x}\sin y = (x - e^{-x}\cos y) + i(y + e^{-x}\sin y)$$

and

$$g(z) = 0 \iff \begin{cases} x - e^{-x}\cos y = 0 \\ y + e^{-x}\sin y = 0. \end{cases}$$

Notice that $g(x) = x - e^{-x}$ with $g'(x) > 0$, $g(-2) < 0$ and $g(2) > 0$. Thus, there is only one $x_0 \in (-2, 2)$ such that $g(x_0) = 0$. Notice that since $2 < \pi$,

$$y + e^{-x}\sin y \neq 0 \quad \text{if } y \neq 0, \text{and } y \in (-2, 2).$$

Therefore,

$$\#(Z_{D(0,2)}(f)) = 1 + 1 = 2. \quad \square$$

2.5 Hurwitz's Theorem and Applications

THEOREM 2.5 *(Hurwitz's Theorem) Let D be domain in $\mathbb{C}$ and let f_n, f be holomorphic in D such that $f_n \to f$ uniformly on any compact subset of D as $n \to \infty$. Then*

a) If $f_n(z) \neq 0$ on D for all n, then either $f(z) \neq 0$ on D or $f(z) \equiv 0$ on D;

b) If $f(z) \neq 0$ on D then for any compact subset K of D there is an $N = N_K$ such that $f_n(z) \neq 0$ on K when $n > N$.

Proof.

a) If $f \equiv 0$ on D, then we are done. Assume that $f \not\equiv 0$ on D. If there is a $z_0 \in D$ such that $f(z_0) = 0$, then there is a $\delta > 0$ such that $\overline{D}(z_0, \delta) \subset D$ and $f(z) \neq 0$ on $\overline{D}(z_0, \delta) \setminus \{z_0\}$. Let

$$\epsilon = \min\{|f(z)| : |z - z_0| = \delta\}.$$

Then $\epsilon > 0$. Since $f_n \to f$ uniformly on $\partial D(z_0, \delta)$, there is a N such that if $n \geq N$ then

$$|f(z) - f_n(z)| < \epsilon/2, \quad z \in \partial D(z_0, \delta)$$

By Rouché's theorem,

$$1 \leq \#(Z_{D(z_0,\delta)}(f)) = \#(Z_{D(z_0,\delta)}(f - (f - f_n)) = \#(Z_{D(z_0,\delta)}(f_n)) = 0.$$

This is a contradiction.

b) Choose a bounded domain D_1 with smooth boundary such that $K \subset D_1 \subset \overline{D}_1 \subset D$. Let

$$\epsilon = \min\{|f(z)| : z \in \overline{D}_1\} > 0.$$

Since $f_n \to f$ uniformly, there is N such that if $n \geq N$ one has

$$|f_n(z) - f(z)| < \frac{\epsilon}{2}.$$

Then with the same argument as above

$$0 = \#(Z_{D_1}(f)) = \#(Z_{D_1}(f - (f - f_n)) = \#(Z_{D_1}(f_n)).$$

The proof is complete. □

2.6 Examples/Applications

EXAMPLE 18 *Prove that there is N such that $\sum_{k=0}^{N} \frac{z^k}{k!} \neq 0$ in $D(0,3)$.*

Proof. We know

$$\sum_{k=0}^{N} \frac{z^k}{k!} \to e^z$$

uniformly on $\overline{D}(0,3)$ as $N \to \infty$ and $e^z \neq 0$ on $\mathbb{C}$. Therefore, there is N such that

$$|\sum_{k=0}^{N} \frac{z^k}{k!} - e^z| \leq \frac{1}{2} \min\{|e^z| : z \in D(0,3)\}, \quad z \in \overline{D}(0,3)$$

Therefore, $\sum_{k=0}^{N} \frac{z^k}{k!}$ has same number of zeros in $D(0,3)$ as the e^z has on $D(0,3)$. Thus

$$\sum_{k=0}^{N} \frac{z^k}{k!} \neq 0, \quad z \in D(0,3). \quad □$$

EXAMPLE 19 *For any compact subset $K \subset D(0,1)$, there is $N = N_K$ such that $\sum_{k=0}^{N}(k+1)z^k \neq 0$ in K.*

Proof. There is a $0 < r < 1$ such that $K \subset D(0,r)$. We know $\sum_{k=0}^{N}(k+1)z^k \to (1-z)^{-2}$ uniformly on $\overline{D}(0,r)$ as $k \to \infty$. Therefore, there is an $N = N_r$ such that

$$\#(Z_{D(0,r)}(\sum_{k=0}^{N}(k+1)z^k)) = \#(Z_{D(0,r)}(\frac{1}{(1-z)^2})) = 0.$$

□

2.7 Homework 2

• Zeros of holomorphic functions/ Argument principle and Rouché's theorem.

1. Let $P(z)$ be a polynomial of dregree n. Let $z_0 \in \mathbb{C}$ be any fixed point. Find

$$\lim_{R\to\infty} \int_{|z-z_0|=R} \frac{P'(z)}{P(z)} dz$$

2. Let D be a bounded domain in $\mathbb{C}$ with piecewise C^1 boundary. Let $f(z) \in C(\overline{D})$ be holomorphic in D with all zeros $\{z_1, \cdots, z_n\} \subset D$ counting multiplicity. Let g be holomorphic in D and continuous on $\overline{D}$. Evaluate

$$\int_{\partial D} \frac{f'(z)}{f(z)} g(z) dz$$

3. Let $P(z)$ be a holomorphic polynomial of degree at least 1 so that $P(z) \neq 0$ in the upper half plane $\mathbb{R}^2_+ = \{z \in \mathbb{C} : \text{Im}\, z > 0\}$. Prove that $P'(z) \neq 0$ on $\mathbb{R}^2_+$

4. Find the number of zeros of the following given functions in the given regions.

   (a) $f(z) = z^8 + 5z^7 - 20, \quad D = D(0,6);$

   (b) $f(z) = z^3 - 3z^2 + 2, \quad D = D(0,1);$

(c) $f(z) = z^{10} + 10z + 9, \quad D = (D(0,1)$

(d) $f(z) = z^{10} + 10ze^{z+1} - 9, \quad D = D(0,1)$

(e) $f(z) = z^2 e^z - z, \quad D = D(0,2)$.

5. Let $f(z)$ be holomorphic in $D(0,1)$ so that $f'(0) = 0$. Prove that f is not one-to-one in $D(0,\delta)$ for any $\delta > 0$.

6. Prove that for any $0 < r < 1$, there is $N = N(r)$ such that $m \geq N(r)$

$$\sum_{n=0}^{m} z^n \neq 0, \quad z \in D(0,r)$$

7. Let $f : D(0,r) \to \mathbb{C}$ be holomorphic and one-to-one such that f is continuous on $\overline{D(0,r)}$. Prove that

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\partial D(0,r)} \frac{\xi f'(\xi)}{f(\xi) - w} d\xi, \quad w \in f(D(0,r)).$$

and

$$(f^{-1})'(w) = \frac{1}{2\pi i} \int_{\partial D(0,r)} \frac{1}{f(\xi) - f(0)} \Big(1 - \frac{w - f(0)}{f(\xi) - f(0)}\Big)^{-1} d\xi, \quad w \in f(D(0,r)).$$

8. Show that if f is a polynomial of degree ≥ 1, then the zeros of $f'(z)$ are contained in the closed convex hull of the zeros of f. (The convex hull of $Z(f)$ is the smallest convex set containing $Z(f)$.)

9. Suppose that f is holomorphic and has n zeros, counting multiplicities in a domain D. Can you conclude that f' has $(n-1)$ zeros in D? Can you conclude anything about the zeros of f'?

10. Let $f : D(0,1) \to D(0,1)$ be holomorphic so that $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. Prove that

$$|f(z)| \leq |z|^n, \quad z \in D(0,1).$$

Show that if there is a $z_0 \in D(0,1) \setminus \{0\}$ so that $|f(z_0)| = |z_0^n|$ then $f(z) = e^{i\theta} z^n$ for some $\theta \in [0, 2\pi)$.

3 The Geometry of Holomorphic Mappings

3.1 The Maximum Modulus Theorem

As we know, a nonconstant holomorphic function is an open mapping. An application of this fact is the following theorem:

THEOREM 3.1 *(Maximum Modulus Theorem) Let D be a domain in $\mathbb{C}$ and f a holomorphic function in D. If there is a $z_0 \in D$ such that*

$$|f(z)| \leq |f(z_0)|, \quad z \in D, \tag{3.1}$$

then f must be a constant.

Proof. **Method 1.** If $f \not\equiv$ constant on D, then f is an open map. Hence, $f(D)$ is an open set and $f(z_0)$ is an interior point of $f(D)$. This contradicts (3.1).

Method 2. Let $D(z_0, r) \subset D$. Then

$$f(z_0) = \frac{1}{\pi r^2} \int_{D(z_0,r)} f(z) dA(z).$$

Then

$$|f(z_0)| \leq \frac{1}{\pi r^2} \int_{D(z_0,r)} |f(z)| dA(z)$$

and

$$0 \leq \frac{1}{\pi r^2} \int_{D(z_0,r)} (|f(z)| - |f(z_0)|) dA(z) \leq 0.$$

Since the integrand is continuous and $|f(z)| - |f(z_0)| \leq 0$ on $D(z_0, r)$,

$$|f(z)| - |f(z_0)| \equiv 0, \quad z \in D(z_0, r).$$

This implies that

$$4|f'(z)|^2 = \Delta |f(z)|^2 = \Delta |f(z_0)|^2 = 0.$$

So, f is a constant on $D(z_0, r)$. To finish the theorem we can use either of the following two arguments:

- *Argument 1:* The proof thus far implies that $\{z \in D : f(z) = f(z_0)\}$ is an open and closed set in D. Since D is connected, $f(z) \equiv f(z_0)$ on D.

- *Argument 2:* For any point $z \in D$, choose finitely many points $z_1, \cdots, z_n$ and positive numbers $r_1, \cdots, r_n$ such that $z = z_n, r_0 = r$ and

$$z_j \in D(z_{j-1}, r_{j-1}), \quad j = 1, \cdots, n.$$

  Since $f \equiv f(z_0)$ on $D(z_0, r_0)$ implies that $f(z_1) = f(z_0)$, $|f(z_1)| \geq |f(z)|$ in $D(z_1, r_1)$. By an induction argument, $f(z) = f(z_n) = f(z_0)$.

□

Corollary 3.2 *(Minimum Modulus Theorem) Let D be a domain in $\mathbb{C}$ and f holomorphic in D with $f(z) \neq 0$ on D. If there is a $z_0 \in D$ such that*

$$|f(z)| \geq |f(z_0)|, \quad z \in D, \qquad (*)$$

then f must be a constant.

Proof. Apply the maximum modulus theorem to $1/f(z)$. □

EXAMPLE 20 *Let f be holomorphic in $D(0,1)$ and continuous on $\overline{D}(0,1)$ such that $|f(z)| = 1$ when $|z| = 1$. Then f must be rational.*

Proof. It is clear that f has only finitely many zeros, say $\{z_1, \cdots, z_n\}$ counting multiplicity. Recall that $\phi_{z_j}(z) = \frac{z_j - z}{1 - \bar{z}_j z}$ is a bijective, continuous self-map of $\bar{D}(0,1)$, which is a holomorphic and bijective self-map of $D(0,1)$. Moreover, since $\phi_{z_j}(z)$ is one-to-one and holomorphic on $D(0,1)$, $\phi'_{z_j}(z) \neq 0$ on $D(0,1)$; hence z_j is a simple zero of ϕ_{z_j}. Consider

$$g(z) = \frac{f(z)}{\prod_{j=1}^n \phi_{z_j}(z)} \neq 0, \quad z \in D(0,1),$$

which is holomorphic in $D(0,1)$ and continuous on $\overline{D}(0,1)$. Moreover, $|g(z)| = 1$ when $|z| = 1$. By the maximum and minimum modulus theorem, $g \equiv e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Therefore,

$$f(z) = e^{i\theta} \prod_{j=1}^n \phi_{z_j}(z)$$

is rational. □

3.2 Schwarz Lemma

THEOREM 3.3 *(Schwarz Lemma) Let $f : D(0,1) \to D(0,1)$ be holomorphic such that $f(0) = 0$. Then*

(a) $|f(z)| \le |z|$ for all $z \in D(0,1)$ and equality holds at some point $z_0 \ne 0$ if and only if $f(z) \equiv e^{i\theta} z$;

(b) $|f'(0)| \le 1$ and $|f'(0)| = 1$ if and only if $f(z) = e^{i\theta} z$.

Proof.

(a) Let

$$g(z) = \frac{f(z)}{z}.$$

Then g is holomorphic in $D(0,1)$ and

$$|g(z)| = \frac{|f(z)|}{r} \le \frac{1}{r}, \quad |z| = r$$

By the maximum modulus theorem, $|g(z)| \le \frac{1}{r}$ on $D(0,r)$. Let $r \to 1^-$. Then $|g(z)| \le 1$ on $D(0,1)$. Therefore, $|f(z)| \le |z|$ on $D(0,1)$. If $|f(z)| = |z|$ holds at some $z_0 \ne 0$ then $|g|$ has an interior maximum if and only if $g \equiv$ constant of modulus 1; i.e. $g(z) \equiv e^{i\theta}$ for some $\theta \in [0, 2\pi)$.

(b) Since $g(0) = f'(0)$, (b) follows from the maximum modulus theorem.

□

3.2.1 Schwarz-Pick Lemma

In the Schwarz lemma, we needed to assume that $f(0) = 0$. What happens if $f(0) \ne 0$?

THEOREM 3.4 *(Schwarz-Pick lemma) Let $f : D(0,1) \to D(0,1)$ be holomorphic. Then for any $a \in D(0,1)$, we have*

(a)

$$\left|\frac{f(z) - f(a)}{1 - \overline{f}(a) f(z)}\right| \le \left|\frac{z - a}{1 - \overline{a} z}\right|, \quad z \in D(0,1)$$

and

(b)

$$|f'(a)| \leq \frac{1-|f(a)|^2}{1-|z|^2}, \quad a \in D(0,1).$$

(c) If equality holds at some point in $D(0,1) \setminus \{0, a\}$ in either part (a) or (b), then

$$f(z) = \phi_{f(a)}(e^{i\theta}\phi_a(z))$$

for some $a \in D(0,1)$.

Proof.

(a) We know that

$$\phi_a(z) = \frac{a-z}{1-z\overline{a}}$$

maps $D(0,1) \to D(0,1)$, is one-to-one and onto, and satisfies that $\phi_a(0) = a$, $\phi_a(a) = 0$ and $\phi_a(\phi_a(z)) = z$. Let

$$g(z) = \phi_{f(a)}(f \circ \phi_a(z)).$$

Then $g : D(0,1) \to D(0,1)$ is holomorphic and

$$g(0) = \phi_{f(a)}(f(a)) = 0$$

By the Schwarz lemma, $|g(z)| \leq |z|$. Thus

$$|\phi_{f(a)}(f(z))| \leq |\phi_a(z)|, \quad z \in D(0,1).$$

(b) By part (a),

$$\Big|\frac{f(z)-f(a)}{z-a}\Big| \leq \frac{|1-\overline{f}(a)f(z)|}{|1-\overline{a}z|}$$

Let $a \to z$. Then

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}, \quad z \in D(0,1).$$

If equality holds, then by Schwarz's lemma,

$$\phi_{f(a)}(f(\phi_a(z))) = e^{i\theta}z.$$

Therefore,

$$f(z) = \phi_{f(a)}(e^{i\theta}\phi_a(z)).$$

The proof is complete. ∎

EXAMPLE 21 *Find all entire holomorphic functions f such that $|f(z)| = 1$ when $|z| = 1$.*

Solution. It is clear that f has only finitely many zeros in $D(0,1)$, say $\{z_1, \cdots, z_n\}$ counting multiplicity. Then

$$g(z) = \frac{f(z)}{\prod_{j=1}^n \phi_{z_j}(z)} \neq 0, \quad z \in D(0,1)$$

is holomorphic in $D(0,1)$ and continuous on $\overline{D}(0,1)$. Moreover, $|g(z)| = 1$ when $|z| = 1$. By the maximum and minimum modulus theorems, $g \equiv e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Therefore,

$$f(z) = e^{i\theta} \prod_{j=1}^n \phi_{z_j}(z).$$

If $z_j \neq 0$ then f has a pole at $z = 1/\overline{z}_j$. However, f is entire; thus, $z_j = 0$ for all $j = 1, \cdots, n$. Therefore,

$$f(z) = e^{i\theta} z^n.$$

EXAMPLE 22 *Let $f : \mathbb{R}^2_+ \to D(0,1)$ be holomorphic. Prove that $|f'(i)| < 1 - |f(i)|$*

Proof. Let $C(z) = i\frac{1-z}{1+z} : D(0,1) \to \mathbb{R}^2_+$ with $C(0) = i$. We consider

$$g(z) = f \circ C(z) : D(0,1) \to D(0,1)$$

Then

$$g'(z) = f'(C(z))C'(z) = f'(C(z))\frac{2i}{(1+z)^2}$$

Therefore,

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}$$

Then

$$|f'(i)| = \frac{|g'(0)|}{2} \leq \frac{1 - |g(0)|^2}{2} = \frac{1 - |f(i)|^2}{2} < 1 - |f(i)|.$$

3.3 Homework 3

- Maximum Modulus theorem, Schwarz's lemma and holomorphic mapping.

1. Let $f : D(0,1) \to D(0,1)$ be holomorphic so that $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. Prove that

$$|f(z)| \leq |z|^n, \quad z \in D(0,1).$$

   Prove that if either there is a $z_0 \in D(0,1) \setminus \{0\}$ so that $|f(z_0)| = |z_0^n|$ or $|f^{(n)}(0)| = n!$, then $f(z) = e^{i\theta} z^n$ for some $\theta \in [0, 2\pi)$.

2. Use the Open Mapping Theorem to prove the Maximum Modulus Theorem of holomorphic function: Let f be holomorphic in a domain D so that $|f(z)| \leq |f(z_0)|$ for all $z \in D$ and some fixed $z_0 \in D$. Then $f(z)$ must be constant.

3. Let $f_1(z), f_2(z), \cdots, f_n(z)$ be holomorphic in a domain D. Suppose there is a $z_0 \in D$ so that

$$\sum_{j=1}^{n} |f_j(z)| \leq \sum_{j=1}^{n} |f_j(z_0)|, \quad z \in D.$$

   Prove $f_1, \cdots, f_n$ are constants on D.

4. Construct a conformal holomorphic function $f : D(0,1) \to \mathbb{C} \setminus \{0\}$.

5. Let $f(z) : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ be holomorphic. Give a statement of Schwarz-Pick type lemma and verify your statement.

6. Let $f(z) : \overline{D(0,1)} \to \overline{D(0,1)}$ be holomorphic so that $|f(z)| = 1$ when $|z| = 1$ and $f(0) = f'(0) = f''(0) = 0$ but $f'''(0) = 3!$. Find all such f.

7. Let f and g be holomorphic in a domain D and continuous on $\overline{D}$.
   (a) Assume that $|f(z)| = |g(z)|$ for all $z \in D$. What is the relation between f and g?
   (b) Assume that $|f(z)| = |g(z)|$ for $z \in \partial D$ and $|f(z)| \leq 10|g(z)|$ on D. What is the relation between f and g?
   (c) Assume that $|f(z)| = |g(z)|$ for $z \in \partial D$ and $\frac{1}{10}|g(z)| \leq |f(z)| \leq 10|g(z)|$ on D. What is the relation between f and g?

8. Let $f : D(0,1) \to D(0,1)$ be holomorphic such that $f(0) = 0$. Let $f^1 = f$, $f^j(z) = f^{j-1}(f(z))$ for $j \geq 2$. Prove that $f^j(z) \to 0$ uniformly on any compact subset K of $D(0,1)$ unless $f(z) = e^{i\theta}z$ for some $\theta \in [0, 2\pi)$.

3.4 Conformal and proper holomorphic function maps

Definition 3.5 *Let $f(z) = u(z) + iv(z) : D \to \mathbb{C}$ be a C^1 map. We say that f is a conformal map if f is an angle preserving map which means for any $z_0 \in D$ and any two curves $\gamma_j : (-1,1) \to D$ with $\gamma_j(0) = z_0$ we have the angle between $(f \circ r_1)'(0)$ and $(f \circ \gamma_2)'(0)$ (at $f(z_0)$) is the same as the angle between $\gamma_1'(0)$ and $\gamma_2'(0)$ (at z_0).*

THEOREM 3.6 *Let $f : D \to \mathbb{C}$ be holomorphic and $f'(z) \neq 0$ on D. Then f is a conformal map on D.*

Proof. For any $z_0 \in D$, consider any two C^1 curves $\gamma_j : (-1,1) \to D$ with $\gamma_j(0) = z_0$. Then the angle between $\gamma_1'(0) = |\gamma'(0)|e^{i\theta_1}$ and $\gamma_2'(0) = |\gamma_2'(0)|e^{i\theta_2}$ is $\theta_2 - \theta_1$. Notice that

$$\frac{\gamma_2'(0)}{\gamma_1'(0)} = \frac{|\gamma_1'(0)|}{|\gamma_1'(0)|} e^{i(\theta_2 - \theta_1)}$$

and

$$\frac{(f \circ \gamma_2)'(0)}{(f \circ \gamma_2)'(0)} = \frac{f'(z_0)\gamma_2'(0)}{f'(z_0)\gamma_1'(0)} = \frac{\gamma_2'(0)}{\gamma_1'(0)}.$$

This proves the statement. □

EXAMPLE 23 *$f(z) = e^z : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a conformal map since $f'(z) \neq 0$ on $\mathbb{C}$.*

EXAMPLE 24 *Find a conformal holomorphic map $f : \mathbb{R}^2_+ \to \mathbb{C} \setminus \{0\}$.*

Solution. Let $f(z) = e^{z^3} : \mathbb{R}^2_+ \to \mathbb{C} \setminus \{0\}$. It is easy to verify f is is a conformal map since $f'(z) \neq 0$ on $\mathbb{R}^2_+$.

Definition 3.7 *Let D_1 and D_2 be two domains in $\mathbb{C}$. Let $f(z) : D_1 \to D_2$ be a continuous map. We say that f is a proper map if $f^{-1}(K)$ is compact in D_1 for any compact subset K of D_2.*

THEOREM 3.8 *Let $f : D_1 \to D_2$ be a proper holomorphic. Then $f : \partial D_1 \to \partial D_2$. (Here, if D_i is unbounded, then we consider $\infty \in \partial D_i$.)*

Proof. For this proof, consider D_1 and D_2 as subsets of $\mathbb{C} \cup \{\infty\}$ with the topology given by the one-point compactification of $\mathbb{C}$. With this topology, $\bar{D}_1$ and $\bar{D}_2$ are compact subsets of $\mathbb{C} \cup \{\infty\}$.

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points in D_1 such that $\lim_{n\to\infty} z_n = z_0 \in \partial D_1$ or $\lim_{n\to\infty} z_n = \infty$. We will show $f(z_n) \to \partial D_2$. If it is not true, then there is a point $w_0 \in D_2$ such that there is a subsequence $\{f(z_{n_k})\}_{k=1}^{\infty}$ such that $f(z_{n_k}) \to w_0$ as $k \to \infty$. Then there is $\epsilon > 0$ such that $D(w_0, 2\epsilon) \subset D_2$. Then $f^{-1}(\overline{D}(w_0, \epsilon))$ is compact set in D_1. But $\{z_k\}_{k=N}^{\infty} \subset f^{-1}(\overline{D}(w_0, \epsilon))$ for some N. This contradicts with $\{z_{n_k}\}$ converges to boundary of ∂D_1.

Proposition 3.9 *If $f : D(0,1) \to D(0,1)$ is a proper holomorphic map such that $f \in C(\overline{D}(0,1))$, then f is a rational map.*

Proof. Since $f : D(0,1) \to D(0,1)$ is proper,

$$\lim_{|z|\to 1} |f(z)| = 1.$$

Since $f \in C(\overline{D}(0,1))$, f has only finitely many zeros in $D(0,1)$, say, $z_1, \cdots, z_n$ counting multiplicity. Consider

$$g(z) = \frac{f(z)}{\prod_{j=1}^{n} \phi_{z_j}(z)} \neq 0, \quad z \in D(0,1),$$

which is holomorphic in $D(0,1)$, continuous on $\overline{D}(0,1)$, and has no zeros in $D(0,1)$. Moreover, $|g(z)| = 1$ when $|z| = 1$. By the maximum and minimum modulus theorems, $g \equiv e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Therefore,

$$f(z) = e^{i\theta} \prod_{j=1}^{n} \phi_{z_j}(z).$$

The proof is complete. ▯

Definition 3.10 *$f : D_1 \to D_2$ is called a biholomorphism if it is a bijective holomorphic function with holomorphic inverse.*

Proposition 3.11 *If $f : D_1 \to D_2$ is a biholomorphism then f is proper holomorphic.*

Proof. Let K be any compact subset of D_2. Since f^{-1} is continuous, $f^{-1}(K)$ is compact in D_1.

3.5 Automorphism groups

Definition 3.12 *Let D be a domain in $\mathbb{C}$. Let $Aut(D) = \{f : D \to D$ is biholomorphic$\}$. We define an operation $\circ$ on $Aut(D)$ as follows: For any $f, g \in Aut(D)$, we define $f \circ g(z) = f(g(z))$.*

Proposition 3.13 *$(Aut(D), \circ)$ forms a group, which is called the automorphism group.*

Proof. It is easy to verify. □

Question: How does one find $Aut(D)$ explicitly?

EXAMPLE 25 $Aut(D(0,1)) = \{e^{i\theta}\phi_a : a \in D(0,1), \theta \in [0, 2\pi)\}$

Proof. Since $f : D(0,1) \to D(0,1)$ is one-to-one and onto, there is $a \in D(0,1)$ such that $f(a) = 0$. Moreover, since f is one-to-one, its zero is simple. Then

$$g(z) = \frac{f(z)}{\phi_a(z)}$$

is holomorphic in $D(0,1)$, $g(z) \neq 0$ in $D(0,1)$ and $|g(z)| = 1$ when $|z| = 1$. By the maximum and minimum modulus theorems, $g(z) = e^{i\theta}$. Thus,

$$f(z) = e^{i\theta}\phi_a(z), \quad z \in D(0,1).$$

The proof is complete. □

EXAMPLE 26 *Let $Aut(\mathbb{C}) = \{\phi_{a,b} : a, b \in \mathbb{C}, a \neq 0\}$ where $\phi_{a,b}(z) = az+b$.*

Proof. It is clear $\phi_{a,b} \in Aut(\mathbb{C})$. On the other hand, if $\phi \in \mathrm{Aut}(\mathbb{C})$, then $\phi : \mathbb{C} \to \mathbb{C}$ is proper. Thus, $\lim_{z\to\infty} \phi(z) = +\infty$. Thus ϕ is a polynomial. Since ϕ one-to-one, ϕ is linear and non-constant: $\phi(z) = az + b$ with $a \neq 0$. □

EXAMPLE 27 $Aut(\mathbb{C} \setminus \{0\}) = \{\phi_{a,0}(z), \phi_{a,0}(\frac{1}{z}) : a \in \mathbb{C} \setminus \{0\}\}$

Proof. For any biholomorphic map $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$,

$$f : \{0, \infty\} \to \{0, \infty\}$$

So either $f(0) = 0$ and $f(\infty) = \infty$ or $f(0) = \infty$ and $f(\infty) = 0$. This implies that $f(z) = az$ or $f(\frac{1}{z}) = az$.

EXAMPLE 28 *What is $Aut(\mathbb{C}\setminus\{0,1\})$?*

Proof. For any biholomorphic map $f:\mathbb{C}\setminus\{0,1\}\to\mathbb{C}\setminus\{0,1\}$,

$$f:\{0,1,\infty\}\to\{0,1,\infty\}.$$

Then f must satisfy one of the following six cases:

$$f(0)=0, f(1)=1,\ f(\infty)=\infty,\quad f(0)=1, f(1)=0, f(\infty)=\infty;$$

$$f(0)=\infty,\ f(\infty)=0, f(1)=1;\quad f(1)=\infty, f(\infty)=1, f(0)=0;$$

$$f(0)=\infty, f(\infty)=1, f(1)=0;\quad f(1)=\infty, f(\infty)=0, f(0)=1.$$

Since $f\in Aut(\mathbb{C}\setminus\{0,1\})$,

$$f(0)=0, f(1)=1,\ f(\infty)=\infty \iff f(z)=z$$

$$f(0)=1, f(1)=0, f(\infty)=\infty \iff f(z)=-z+1$$

$$f(0)=\infty,\ f(\infty)=0, f(1)=1 \iff f(z)=\frac{1}{z}$$

and

$$f(1)=\infty, f(\infty)=1, f(0)=0 \iff f(z)=\frac{z}{z-1}$$

$$f(0)=\infty, f(\infty)=1, f(1)=0 \iff f(z)=\frac{z-1}{z}$$

$$f(1)=\infty, f(\infty)=0, f(0)=1 \iff f(z)=\frac{1}{1-z}.$$

Therefore,

$$\text{Aut}(\mathbb{C}\setminus\{0,1\})=\Big\{z;\ \frac{1}{z};\ 1-z;\ \frac{1}{z-1};\ \frac{z}{1-z},\ \frac{z-1}{z}\Big\}$$

EXAMPLE 29 *Prove that $Aut(\overline{\mathbb{C}})=\{\frac{az+b}{cz+d}:ad-bc\neq 0\}$.*

Proof.

Case 1. If $f(\infty)=\infty$, then $f(z)=az+b$ with $a\neq 0$.

Case 2. If $f(z_0) = \infty$, let $g(w) = f(z_0 + \frac{1}{w})$ then $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is one-to-one and onto and $g(\infty) = \infty$. Thus,

$$g(w) = aw + b.$$

Thus,

$$f(z) = g(\frac{1}{z - z_0}) = \frac{a}{z - z_0} + b = \frac{a + b(z - z_0)}{z - z_0} = \frac{bz + a - bz_0}{z - z_0}$$

and

$$-bz_0 - (a - bz_0) = -a \neq 0. \quad \square$$

EXAMPLE 30 *Find* $Aut(\mathbb{R}^2_+)$.

Proposition 3.14 *Let* $f : D_1 \to D_2$ *be a biholomorphic map. Then*

$$Aut(D_1) = \{f^{-1} \circ \phi \circ f : \phi \in Aut(D_2)\}$$

Therefore,

$$\mathrm{Aut}(\mathbb{R}^2_+) = C^{-1}\mathrm{Aut}(D(0,1)) \circ C$$

where $C(z) = \frac{z-i}{z+i}$ is the Cayley transform which maps the upper halp plane to the unit disc.

3.6 Möbius transformations

Let

$$Sz = \frac{az + b}{cz + d} = \frac{a(z + b/a)}{c(z + d/c)}$$

Let $S_1 z = z + b$, $S_2 z = az$ with $a > 0$, $S_3 z = e^{i\theta} z$ and $S_4 z = \frac{1}{z}$. Then S is a combination of S_1, S_2, S_3 and S_4:

1) If $c \neq 0$ and $a = 0$, then $Sz = \frac{1}{\frac{c}{b}(z + d/c)}$;

2) If $c = 0$ and $a \neq 0$, then $Sz = a(z + b/a)$;

3) If $a, c \neq 0$ then

$$Sz = \frac{a}{c}\Big(\frac{z + b/a}{z + d/c}\Big) = \frac{a}{c}\Big(\frac{z + b/a}{z + d/c}\Big) = \frac{a}{c} + \frac{b/c - ad/c^2}{z + d/c}$$

Definition 3.15 *We say that C is a "circle" in $\mathbb{C}$ if C is a real line or real circle in $\mathbb{R}^2$.*

THEOREM 3.16 *A Möbius transformation S maps "circles" to "circles".*

Proof. A real circle can be written as

$$A|z|^2 + B(z + \overline{z}) + Ci(z - \overline{z}) + D = 0$$

where A, B, C, D are real. A straight line can be written as

$$A(z + \overline{z}) + Bi(z - \overline{z}) + C = 0.$$

It is easy to verify that S_1, S_2 and S_3 map "circles" to "circles". Let $w = S_4 z = \frac{1}{z}$. Now we verify that S_4 maps "circles" to "circles". Notice that

$$\begin{aligned} & A|z|^2 + B(z + \overline{z}) + Ci(z - \overline{z}) + D = 0 \\ \iff \ & A\frac{1}{|w|^2} + B\frac{(w + \overline{w})}{|w|^2} + Ci\frac{(\overline{w} - w)}{|w|^2} + D = 0 \\ \iff \ & A + B(w + \overline{w}) + Ci(\overline{w} - w) + D|w|^2 = 0. \end{aligned}$$

This is a "circle".

3.7 Cross ratio

We know

- Two points in $\mathbb{C}$ uniquely determine a line, and a line fixes ∞.
- A circle is uniquely determined by three points.

Question. How does one find a Möbius transformation S that maps a given "circle" to another given "circle"? Equivalently, how does one find a Möbius transformation that maps three given points in $\overline{\mathbb{C}}$ to another three given points in $\overline{\mathbb{C}}$?

To answer the above question, start with three points $\{z_1, z_2, z_3\}$ and another three points $\{w_1, w_2, w_3\}$. First, find Möbius transformations S_{z_1,z_2,z_3}

and S_{w_1,w_2,w_3} that map $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ to $\{0, \infty, 1\}$ respectively. Then

$$S = (S_{w_1,w_2,w_3})^{-1} S_{z_1,z_2,z_3}$$

maps $\{z_1, z_2, z_3\}$ to $\{w_1, w_2, w_3\}$. We construct S_{z_1,z_2,z_3} as follows:

$$S_{z_1,z_2,z_3}(z) = \frac{z - z_1}{z - z_2} \frac{z_3 - z_2}{z_1 - z_2} =: (z; z_1, z_2, z_3).$$

$(z; z_1, z_2, z_3)$ is also called the cross ratio.

THEOREM 3.17 *Möbius transformations preserve the cross ratio, i.e.* $(Sz; Sz_1, Sz_2, Sz_3) = (z; z_1, z_2, z_3)$.

Proof. It is obvious that

$$(S_j z; S_j z_1, S_j z_2, S_j z_3) = (z; z_1, z_2, z_3), \quad 1 \le j \le 3.$$

and

$$(S_4 z; S_4 z_1, S_4 z_2, S_4 z_3) = \frac{z z_2}{z z_1} \frac{z_1 z_3}{z_2 z_3} (z; z_1, z_2, z_3) = (z; z_1, z_2, z_3). \quad \square$$

3.8 Properties of the Cross Ratio

3.8.1 Symmetric points

1. We know that z and $\overline{z}$ are symmetric with respect to the real line.

2. Let L be a line. Two points z and z^* are said to be symmetric with respect to L if $[z, z^*] \perp L$ and $\text{dist}(z, L) = \text{dist}(z^*, L)$.

3. Let $C = \{z \in \mathbb{C} : |z - z_0| = R\}$ be a circle. Two points z and z^* are said to be symmetric with respect to C if

$$(i)\ \frac{z^* - z_0}{z - z_0} > 0 \text{ and (ii) } |z - z_0||z^* - z_0| = R^2.$$

Lemma 3.18 *If z and z^* are symmetric with respect to the circle $|z - z_0| = R$, then*

$$z^* = z_0 + \frac{R^2}{\overline{z} - \overline{z}_0}.$$

Proof. Notice that

$$\frac{z^* - z_0}{z - z_0} = \frac{|z^* - z_0|}{|z - z_0|} = \frac{R^2}{|z - z_0|^2}.$$

Therefore

$$z^* = z_0 + \frac{R^2}{\overline{z} - \overline{z}_0}. \quad \square$$

In particular, notice that if $z_1, z_2, z_3 \in \mathbb{R}$ and $z^* = \overline{z}$, then

$$(z^*; z_1, z_2, z_3) = (\overline{z}; z_1, z_2, z_3) = \overline{(z; z_1, z_2, z_3)}$$

Proposition 3.19 *Let C be the circle centered at z_0 with radius R. Let $z_1, z_2, z_3 \in C$ and z and z^* are symmetric with respect to C. Then*

$$(z^*; z_1, z_2, z_3) = \overline{(z; z_1, z_2, z_3)}.$$

Proof. This follows from

$$\begin{aligned}
(z^*; z_1, z_2, z_3) &= (z_0 + \frac{R^2}{\overline{z} - \overline{z}_0}; z_1, z_2, z_3) \\
&= (\frac{R^2}{\overline{z} - \overline{z}_0}; z_1 - z_0, z_2 - z_0, z_3 - z_0) \\
&= (\frac{R^2}{\overline{z} - \overline{z}_0}; \frac{R^2}{\overline{z}_1 - \overline{z}_0}, \frac{R^2}{\overline{z}_2 - \overline{z}_0}, \frac{R^2}{\overline{z}_3 - \overline{z}_0}) \\
&= (\overline{z} - \overline{z}_0; \overline{z}_1 - \overline{z}_0, \overline{z}_2 - \overline{z}_0, \overline{z}_3 - \overline{z}_0) \\
&= \overline{(z - z_0; z_1 - z_0, z_2 - z_0, z_3 - z_0)} \\
&= \overline{(z; z_1, z_2, z_3)}. \quad \square
\end{aligned}$$

THEOREM 3.20 *Möbius transformations preserve symmetry; that is, if z and z^* are symmetric with respect to a "circle" C then Sz and Sz^* are symmetric with respect to $S(C)$.*

Proof. Since

$$(Sz^*; Sz_1, Sz_2, Sz_3) = (z^*; z_1, z_2, z_3) = \overline{(z; z_1, z_2, z_3)}$$

and

$$((Sz)^*, Sz_1, Sz_2, Sz_3) = \overline{(Sz; Sz_1, Sz_2, Sz_3)} = \overline{(z; z_1, z_2, z_3)}$$

Therefore, $Sz^* = (Sz)^*$. $\square$

EXAMPLE 31 *Find a conformal map which maps the upper half plane to the unit disc.*

Solution. Let

$$Sz = \frac{z-i}{z+i}.$$

This map is called the Cayley transform. Since S is a Möbius transformation with

$$S0 = -1,\ S1 = -i,\ S\infty = 1 \in \{|z| = 1\},$$

S maps the real line to the unit circle. Since additionally $Si = 0$ and $S \in Aut(\bar{\mathbb{C}})$, S maps the upper half plane to the unit disc biholomorphically (hence conformally). ∎

3.9 Construction of conformal maps

Suppose two "circles" C_1 and C_2 intersect at exactly two points z_1 and z_2. Let D be the region bounded by C_1 and C_2 and γ_1, γ_2 be the arcs from z_1 to z_2 on the circles C_1, C_2 respectively. We want to construct a conformal map S which maps D onto the unit disc. Let the angle between the two circles at z_1 be θ. Consider the map

$$f_1(z) = \frac{z - z_1}{z - z_2}$$

which sends z_1 to 0 and z_2 to ∞. Since f_1 is a Möbius transformation, $f_1(\gamma_1)$ and $f_1(\gamma_2)$ are rays emanating from the origin. Without loss of generality, suppose the angle between $f(\gamma_1)$ and $[0, \infty)$ measured counter-clockwise from the positive real axis is less than the angle between $f(\gamma_2)$ and $[0, \infty)$ measured counter-clockwise from the positive real axis. Suppose θ_1 be the angle between $f(\gamma_1)$ and $[0, \infty)$. Then we let

$$f_2(z) = e^{-i\theta_1} z$$

and

$$f_3(z) = (z)^{\pi/\theta}$$

Then $f_3 \circ f_2 \circ f_1$ maps the region D to the upper half plane. Then the Cayley transform takes it to the unit disc. (See Figure - Conformal Map 1)

EXAMPLE 32 *Find a conformal map f which map upper half disc to the unit disc. (See figure - Upper half disc to unit disc)*

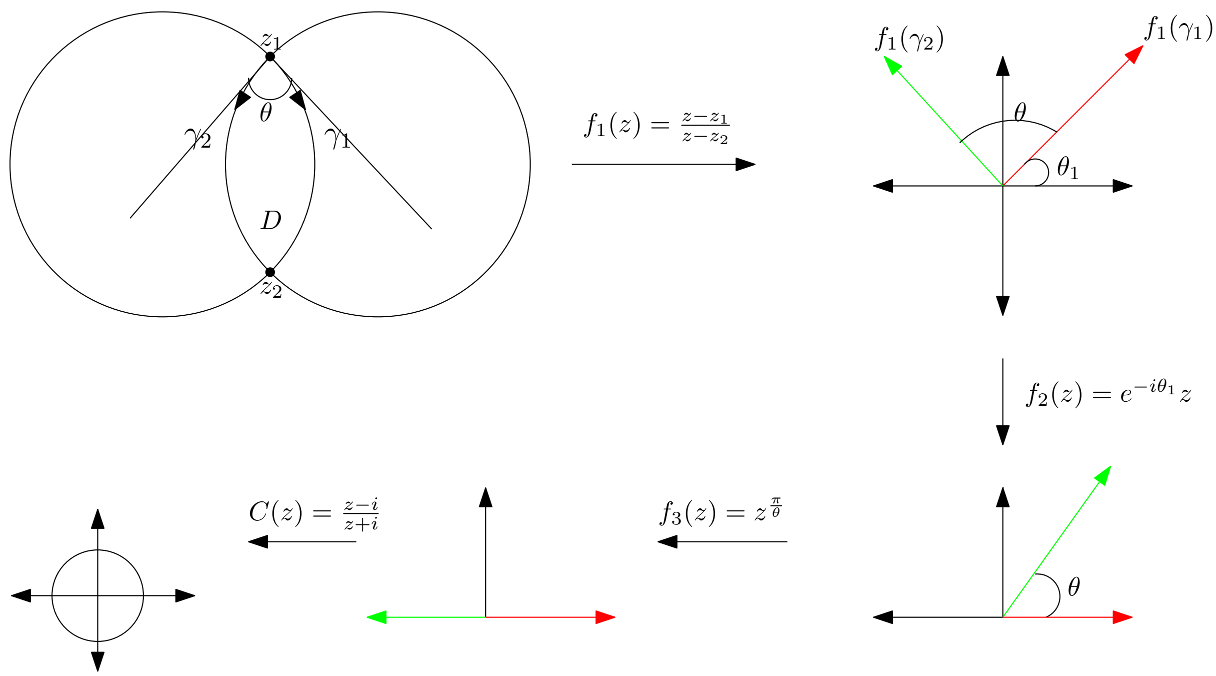

Figure 5: Conformal Map 1

Solution. Let

$$z_1 = f_1(z) = \frac{z+1}{z-1},$$

which maps the upper half disc to the first quadrant of the plane. Let

$$z_2 = f_2(z) = z_1^2,$$

which maps the first quadrant to the upper half plane. Let

$$z_3 = f_3(z) = \frac{z-i}{z+i}$$

maps the upper half plane to the unit disc. Compose the functions.

EXAMPLE 33 *Find a conformal map f which maps the region D between the two circles $|z+1| = 1$ and $|z+1/2| = 1/2$ to the unit disc. (See figure - Region between two circles to unit disc)*

Solution. Let $z_1 = \frac{1}{z}$ which maps D to the region D_1 between the two lines $x = -1$ and $x = -1/2$. Let $z_2 = i(z_1 + 1)2\pi$, which maps D_1 to the region

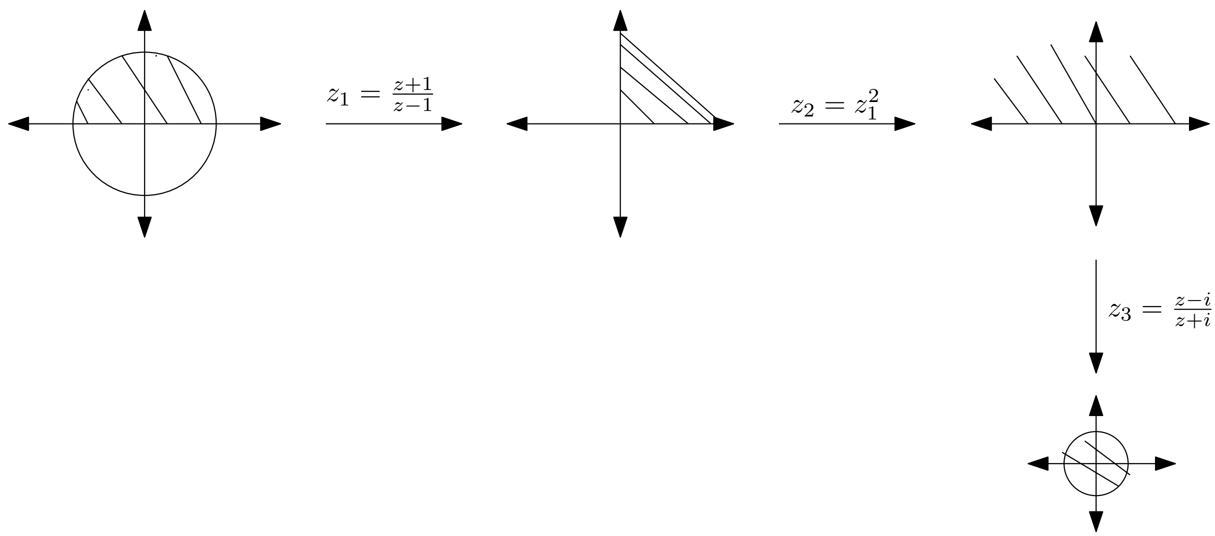

Figure 6: Upper half disc to unit disc

$D_2 = \{z = x + iy : 0 < y < \pi\}$. Let $z_3 = e^{z_2}$, which maps D_2 to the upper half plane. Let $z_4 = \frac{z_3 - i}{z_3 + i}$, which maps the upper half plane to the unit disc.

EXAMPLE 34 *Find a conformal map f which maps the region $D = \{z \in \mathbb{C} : |z| < 1, y > -\frac{\sqrt{2}}{2}\}$ to the unit disc. (See figure - conformal map 4)*

Solution. Let $p = \frac{\sqrt{2}}{2}(1-i)$ and $q = \frac{\sqrt{2}}{2}(1+i)$. The angle θ between the circle $|z| = 1$ and the line $y = -\frac{\sqrt{2}}{2}$ is

$$\theta = \pi - \arg(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = \pi - \arg(1+i) = \frac{3\pi}{4}$$

Set

$$z_1 = \frac{z - \frac{\sqrt{2}}{2}(1-i)}{z + \frac{\sqrt{2}}{2}(1+i)}$$

Then

$$z_1(p) = 0, \quad z_1(-i\frac{\sqrt{2}}{2}) = \frac{-\sqrt{2}}{\sqrt{2}} = -1$$

and z_1 maps D to $D_1 = \{z \in \mathbb{C} : z = |z|e^{i\theta}, \frac{\pi}{4} < \theta < \pi\}$. Let

$$z_2 = e^{-\pi/4i} z_1.$$

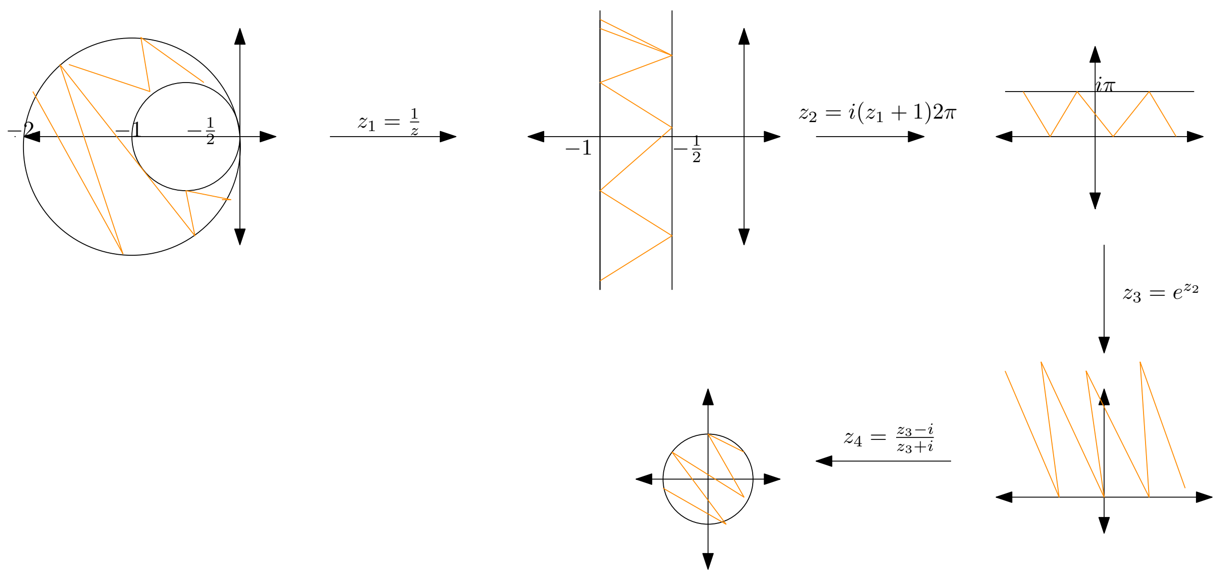

Figure 7: Region between two circles to unit disc

Then z_2 maps D_1 to $D_2 = \{z \in \mathbb{C} : z = |z|e^{i\theta}, 0 < \theta < \frac{3\pi}{4}\}$. Let

$$z_3 = z_2^{4/3}$$

Then z_3 maps D_2 to the upper half plane. Use the Cayley transform to map the upper half plane to the unit disc. □

3.10 Homework 4

- Maximum Modulus theorem and holomorphic mappings.

1. Prove or disprove that there is a sequence of holomorphic polynomials $\{p_n\}_{n=1}^{\infty}$ that converges to $\frac{1}{z}$ uniformly on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

2. Let $f(z)$ be meromorphic on $\mathbb{C}$ so that $|f(z)| = |\sin z|$ when $|z| = 1$. Find all such f.

3. Let $f(z)$ be entire and 1-1. Show that f must be linear.

4. Let D be a domain in $\mathbb{C}$. We say that $Aut(D)$ is compact if for every sequence $\{f_n\}_{n=1}^{\infty}$ in $Aut(D)$, there is a subsequence $\{f_{n_k}\}$ and an $f \in$

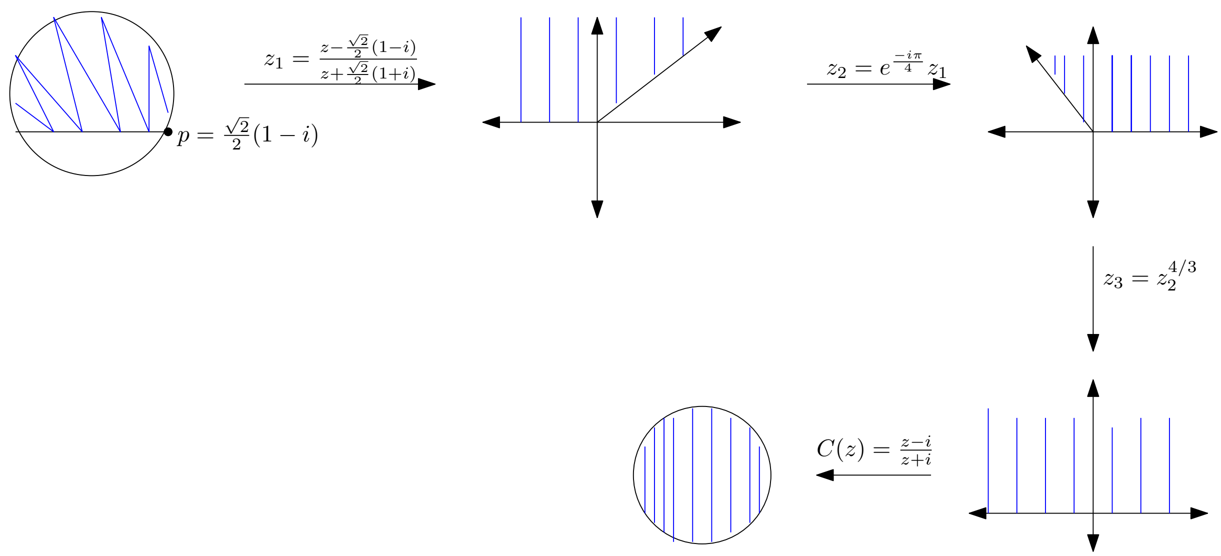

Figure 8: conformal map 4

$Aut(D)$ so that f_{n_k} converges to f uniformly on any compact subset of D. Give an example of a domain D where $Aut(D)$ is not compact.

5. Find $\text{Aut}(D)$ where $D = \mathbb{C} \setminus \{|z| \leq 1\} = \{z \in \mathbb{C} : |z| > 1\}$.

6. Let Ω be a domain in $\mathbb{C}$. Let $\phi : \Omega \to D(0,1)$ be a biholomorphic mapping and $\psi : D(0,1) \to \Omega$ be a biholomorphic map. How are ϕ and ψ related?

7. Find all proper holomorphic maps from the whole plane $\mathbb{C}$ to itself.

8. Let D be a bounded domain and let ϕ be a conformal mapping of D to itself. Let $z_0 \in D$ so that $\phi(z_0) = z_0$ and $\phi'(z_0) = 1$. Prove that $\phi(z) = z$ for all $z \in D$.

9. Find all proper holomorphic mapping from $\mathbb{C} \setminus \{0\}$ to itself.

10. Construct a conformal map (i.e. $f'(z) \neq 0$) ϕ which maps

$$\Big\{ z \in \mathbb{C} : |z| < 1, \text{Im}\, z > -\frac{1}{\sqrt{2}} \Big\}$$

to the unit disc.

11. Let T be a Möbius transformation. Let C be a circle in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $z \in \mathbb{C}$ and z^* be the symmetric point of z with respect to C. Prove that $T(z^*)$ is the symmetric point of Tz with respect to $T(C)$.

12. Construct a linear fractional transformation that sends the unit disc to the half plane that lies below the line $x + 2y = 4$.

13. Find a conformal map which maps the strip between the two lines $x + y = 1$ and $x + y = 4$ to the unit disc.

14. Find a conformal map which maps $\mathbb{C} \setminus [1, \infty)$ to the unit disc.

3.11 Midterm Review

There will be five problems on the test. The material on the test includes the following:

1) Application of Residues theorem: Evaluate some real integrals

   I. $\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta$ where $R(x, y)$ are rational function of x and y. Ex:
   $$\int_0^{2\pi} \frac{1}{a + \cos x} dx$$

   II. $\int_0^{\infty} \frac{P(x)}{Q(x)} dx$ with $\deg(Q) \geq \deg(P) + 2$ and $Q(x) \neq 0$ on $[0, \infty)$.

   III. $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{-ix} dx$ with $\deg(Q) \geq \deg(P) + 1$.
   In particular,
   $$\int_0^{\infty} \frac{\sin x}{x} dx, \quad \int_0^{\infty} \frac{1 - \cos x}{x^2} dx$$

   IV. $\int_0^{\infty} \frac{1}{1+x^3} dx$

   V. $\int_0^{\infty} \frac{P(x)}{Q(x)} x^{\alpha} dx$ with $\deg(Q) \geq \deg(P) + \alpha + 2$.

   VI. $\int_0^{\infty} \frac{P(x)}{Q(x)} \ln x dx$ with $\deg(Q) \geq \deg(P) + 2$ and $Q(x) \neq 0$ on $[0, \infty)$.
   ,

2) Argument Principle and Applications.

I. Open mapping theorem

$$\#(Z_D(f)) = \frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)-w}dz, \quad f(z) \neq w, \quad z \in \partial D.$$

If $f(z) = h(z)\prod_{j=1}^n (z - z_j)$ then

$$\frac{f'(z)}{f(z)} = \frac{h'(z)}{h(z)} + \sum_{j=1}^n \frac{1}{z - z_j}$$

II. **Example:** if f is a polynomial of deg≥ 1. If $f(z) \neq 0$ on Im$z > 1$ then $f'(z) \neq 0$ when Im$z > 1$.

3) Rouché's Theorem, Hurwitz's Theorem and their applications.

Example: Find $\#Z_{A(0;1,2)}(z^{10} - 2z^5 + 10)$;

Example: Prove that there is N such that $n \geq N$ $\sum_{k=0}^N z^k \neq 0$ on $D(0, 3/4)$.

4) Maximum Modulus Theorem/Minimum Modulus Theorem

Example: Let $f_1, \cdots, f_n$ be holomorphic in a domain D in $\mathbb{C}$. Suppose that there is a point $z_0 \in D$ such that

$$\sum_{j=1}^n |f(z)| \leq \sum_{j=1}^n |f_j(z_0)|, \quad z \in D$$

Prove that f_j must be a constant for all $1 \leq j \leq n$.

Example: Find all entire holomorphic functions such that $|f(z)| = 2$ when $|z| = 1$.

5) Schwarz lemma and Schwarz-Pick lemma

Example: If $f : D(0,1) \to D(0,1)$ such that $f(0) = f'(0) = \cdots = f^{(n)}(0) = 0$. Prove that $|f(z)| \leq |z|^{n+1}$ on D.

6) Proper, biholomorphic maps

7) Automorphism group Aut(D)

8) Möbius transformations

9) Conformal mappings.

3.12 Normal families

THEOREM 3.21 *Let D be a simply connected domain in $\mathbb{C}$ such that $D \neq \mathbb{C}$. Then for any point $z_0 \in D$ there is a unique biholomorphic map $f : D \to D(0,1)$ such that (i) $f(z_0) = 0$ and (ii) $f'(z_0) > 0$.*

Proof. First we prove that if such a map exists, then it is unique. Suppose that there are two biholomorphic maps $f_1, f_2 : D \to D(0,1)$ such that $f_j(z_0) = 0$ and $f_j'(z_0) > 0$. We will show that $f_1 = f_2$. Then $g = f_2 \circ f_1^{-1} : D(0,1) \to D(0,1)$ is a biholomorphic map with $g(0) = 0$. Moreover,

$$g'(0) = \frac{f_2(z_0)}{f_1(z_0)} > 0$$

Similarly, $g^{-1} = f_1 \circ f_2^{-1}$ also satisfies $g^{-1}(0) = 0$ and

$$(g^{-1})'(0) = \frac{f_1(z_0)}{f_2(z_0)} > 0.$$

Thus, $g'(0) = 1$ and by the Schwarz lemma, $g(z) \equiv z$. Therefore, $f_1(z) = f_2(z)$ on D.

In order to prove the existence portion of the proof, we need the theory of normal families □.

Definition 3.22 *A family $\mathcal{F}$ of holomorphic functions on a domain D is said to be a normal family if for any sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ and a holomorphic function f on D such that*

$$f_{n_k} \to f$$

uniformly on any compact subset of D. In this case, we say that f_{n_k} converges to f normally in D as $k \to \infty$.

EXAMPLE 35 *$\mathcal{F} = \{f_n(z) =: nz : n \in N\}$ is not a normal family on $D(0,1)$.*

Proof. Note that $f_n(0) = 0$ for any $n \in N$ and $f_n(\frac{1}{2}) = n/2 \to \infty$ as $n \to \infty$. $\square$

Definition 3.23 *Let $\mathcal{F}$ be a family of continuous functions on a domain D. We say that*

(i) $\mathcal{F}$ is locally bounded if for any compact subset K of D there is a constant $C_K > 0$ such that

$$|f(z)| \leq C_K, \quad z \in K, f \in \mathcal{F}$$

(ii) $\mathcal{F}$ is bounded pointwise if for each point $z \in D$ there is a constant C_z such that

$$|f(z)| \leq C_z, \quad f \in \mathcal{F}$$

(iii) $\mathcal{F}$ is equicontinuous on D if for any $\epsilon > 0$ there is a $\delta > 0$ such that if $z_1, z_2 \in D$ and $|z_1 - z_2| < \delta$ then $|f(z_1) - f(z_2)| < \epsilon$ for any $f \in \mathcal{F}$.

THEOREM 3.24 *(Arzela-Ascoli Theorem) Let $\mathcal{F}$ be a family of continuous functions on a compact set K. Then $\mathcal{F}$ is a normal family on K if and only if $\mathcal{F}$ is both bounded and equicontinuous on K.*

THEOREM 3.25 *(Montel's Theorem) Let $\mathcal{F}$ be a family of holomorphic functions on a domain D. Then $\mathcal{F}$ is a normal family on D if and only if $\mathcal{F}$ is locally bounded on D.*

Proof. If $\mathcal{F}$ is a normal family, then by the Arzela-Ascoli theorem, one has that $\mathcal{F}$ is locally bounded. Conversely, assume that $\mathcal{F}$ is locally bounded on D; we will show that $\mathcal{F}$ is a normal family. Let $\{D_n^1\}_{n=1}^\infty$ and $\{D_n\}_{n=1}^\infty$ be two sequence of domains in D such that

1. $\overline{D}_n^1 \subset D_n$, D_n bounded and $D_n^1 \to D$ increasingly as $n \to \infty$
2. $\partial D_n \subset D$ is piecewise C^1.
3. Let M_n be the constant such that

$$|f(z)| \leq M_n, \quad z \in \overline{D}_n, f \in \mathcal{F}.$$

For any $f \in \mathcal{F}$ and $z_1, z_2 \in D_n^1$,

$$\begin{aligned} |f(z_1) - f(z_2)| &= \Big|\frac{1}{2\pi i}\int_{\partial D_n} \frac{f(w)}{(w-z_1)(w-z_2)}(z_2 - z_1)dw\Big| \\ &\leq \frac{M_n \cdot length(\partial D_n)}{2\pi d(\partial D_n^1, \partial D_n)^2}|z_2 - z_1| \end{aligned}$$

It is clear that $\mathcal{F}$ is equicontinuous on $\overline{D}_n^1$, $n = 1, 2, 3, \cdots$.

Since $\mathcal{F}$ is bounded on $\overline{D}_n^1$, by Arzela-Ascoli, for any sequence $\{f_n\}_{n=1}^\infty \subset \mathcal{F}$ there is a subsequence $\{f_{1,n}\}_{n=1}^\infty$ which converges uniformly on $\overline{D}_1^1$ and for each k, there is a subsequence $\{f_{k,n}\}_{n=1}^\infty$ of $\{f_{k-1,n}\}_{n=1}^\infty$ which converges uniformly on $\overline{D}_k^1$. Let $\{g_n\}_{n=1}^\infty = \{f_{n,n}\}_{n=1}^\infty$. Then for all k, $\{g_n\}_{n=1}^\infty$ converges uniformly on $\overline{D}_k^1$ to a holomorphic function g_k. Let

$$f(z) = g_k(z), \quad z \in D_k^1$$

Since D_k^1 is increasing, by the uniqueness theorem, f is well-defined and holomorphic in D. Since any compact subset K of D is a subset of some $\overline{D}_k^1$, the subsequence $\{g_n\}_{n=1}^\infty$ converges uniformly to f on any compact subset K of D as desired. □

3.12.1 Examples for normal families

The Bergman space $A^2(D)$ consists of all holomorphic functions on D such that

$$\|f\|_{A^2}^2 = \int_{D(0,1)} |f(z)|^2 dA(z)$$

EXAMPLE 36 *The unit ball in the Bergman space $A^2(D(0,1))$ is a normal family.*

Proof. For any $f \in A^2(D(0,1))$,

$$|f(z)| \leq \frac{\|f\|_{A^2}}{\sqrt{\pi} d(z, \partial D)}, \quad z \in D$$

Therefore, the unit ball $\mathcal{F}$ of $A^2(D)$ is locally bounded on D. By Montel's theorem, $\mathcal{F}$ is a normal family. □

The Hardy space $H^2(D)$ on the unit disc $D(0,1)$ consists of all holomorphic functions f on $D(0,1)$ such that

$$\|f\|_{H^2} = \sup_{0<r<1} \Big(\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta\Big)^{1/2} < \infty.$$

EXAMPLE 37 *The unit ball in the Hardy space $H^2(D(0,1))$ is a normal family.*

Proof. For any $f \in H^2(D(0,1))$, by the Cauchy integral formula, we have

$$|f(z)^2| = \lim_{r\to 1^-} \Big|\frac{1}{2\pi i}\int_{\partial D(0,r)} \frac{f(w)^2}{w-z} dw\Big| \le \frac{1}{2\pi} \lim_{r\to 1^-} \int_0^{2\pi} \frac{|f(re^{i\theta})|^2}{r-|z|} r d\theta \le \frac{\|f\|_{H^2}^2}{2\pi(1-|z|)}.$$

Thus,

$$|f(z)| \le \frac{\|f\|_{H^2}}{\sqrt{2\pi}(1-|z|)^{1/2}}, \quad z \in D$$

Therefore, the unit ball $\mathcal{F}$ of $H^2(D)$ is locally bounded on D. By Montel's theorem, $\mathcal{F}$ is a normal family. $\square$

The Dirichlet space $\mathcal{D}(D(0,1))$ consists of all holomorphic functions on $D(0,1)$ such that

$$\|f\|_{\mathcal{D}}^2 = \int_{D(0,1)} |f'(z)|^2 dA(z) < \infty.$$

EXAMPLE 38 *The unit ball in the Dirichlet space $\mathcal{D}(D(0,1))$ is a normal family.*

The Bloch space $\overline{B}(D) =: \{f : f \text{ is holomorphic in } D, \|f\|_{\mathcal{B}} < \infty\}$, where

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup\{|f'(z)|\text{dist}(z, \partial D) : z \in D\}$$

EXAMPLE 39 *The unit ball in the Bloch space $\mathcal{B}(D(0,1))$ is a normal family.*

3.13 The Riemann Mapping Theorem

THEOREM 3.26 ***(Riemann Mapping Theorem)*** *Let D be a simply connected domain in $\mathbb{C}$ such that $D \neq \mathbb{C}$. Then for any point $z_0 \in D$ there is a unique biholomorphic map $f : D \to D(0,1)$ such that (i) $f(z_0) = 0$ and (ii) $f'(z_0) > 0$.*

The uniqueness portion was shown in Theorem 3.21. We need the next several lemmas to prove the existence portion.

Remark: The map given in the Riemann Mapping Theorem is often called the Riemann map.

3.13.1 Existence of the Riemann map

Lemma 3.27 *Let D be a simply connected domain in $\mathbb{C}$ and $w \in \mathbb{C} \setminus D$. Then there is a one-to-one holomorphic function h on D such that*

(i) $h(z)^2 = z - w$, $z \in D$

(ii) *if* $b \in h(D) \setminus \{0\}$ *then* $-b \notin \overline{h(D)}$.

Proof. Since $z - w \neq 0$ on D and D is simply connected, there is a holomorphic function h on D such that $h(z)^2 = z - w$.

Let $b \in h(D)$. It is clear that $b \neq 0$. We claim that $-b \notin h(D)$. Otherwise, there are $z_1 \in D$ and $z_2 \in \overline{D}$ such that $h(z_1) = b$ and $h(z_2) = -b$. Thus,

$$b^2 = h(z_1)^2 = z_1 - w, \quad b^2 = (-b)^2 = h(z_2)^2 = z_2 - w.$$

Thus, $z_1 = z_2$, and $b = -b$. This is a contradicts that $b \neq 0$. We now claim that $-b \notin \overline{h(D)}$. By the open mapping theorem, for sufficiently small δ,

$$D(b, \delta) \subset h(D) \setminus \{0\}.$$

Our proof thus far show that $D(-b, \delta) \cap h(D) = \emptyset$. Thus, $-b \notin \overline{h(D)}$. $\square$

Lemma 3.28 *Let D be a simply connected domain in $\mathbb{C}$ and $D \neq \mathbb{C}$. Then for any $z_0 \in D$, there is a one-to-one holomorphic function $f : D \to D(0,1)$ such that*

1. $f(z_0) = 0$;

2. $f'(z_0) > 0$

Proof. Let $w \in \mathbb{C} \setminus D$. Then there is a one-to-one holomorphic function h on D such that $h(z)^2 = z - w$ on D. Choose $b \in h(D) \setminus \{0\}$. Then $-b \notin \overline{h(D)}$. Let

$$r = \operatorname{dist}(-b, \overline{h(D)}) > 0$$

and

$$f(z) =: \frac{re^{i\theta}}{2}\Big(\frac{1}{h(z)+b} - \frac{1}{h(z_0)+b}\Big).$$

where θ is to be determined. Then $f(z)$ is holomorphic on D, it is clearly one-to-one and $f(z_0) = 0$. $f(D) \subset D(0,1)$ because

$$|f(z)| < \frac{r}{2}(\frac{1}{r} + \frac{1}{r}) = 1.$$

Since $h^2 = z - w$,

$$h'(z_0) = \frac{1}{2h(z_0)};$$

hence,

$$f'(z_0) = \frac{re^{i\theta}}{2}\Big(\frac{-h'(z_0)}{(h(z_0)+b)^2}\Big) = \frac{re^{i\theta}}{2}\Big(\frac{-1}{2h(z_0)(h(z_0)+b)^2}\Big),$$

which can be made positive by choosing θ appropriately. ∎

Let $\mathcal{F}(D; z_0)$ be the set of all one-to-one holomorphic functions from D to $D(0,1)$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.

Lemma 3.29 *Let D be a simply connected domain in $\mathbb{C}$ with $D \neq \mathbb{C}$. Let $z_0 \in D$ and $f \in \mathcal{F}(D; z_0)$. If $f : D \to D(0,1)$ is not onto, then there is a $g \in \mathcal{F}(D; z_0)$ such that $g'(z_0) > f'(z_0)$.*

Proof. Let $w \notin D(0,1) \setminus f(D)$ and let

$$\psi(z) = \frac{f(z) - w}{1 - \overline{w} f(z)} \neq 0.$$

Then $\psi : D \to D(0,1) \setminus \{0\}$ is one-to-one, holomorphic, and

$$\psi(z_0) = -w, \quad \psi'(z_0) = f'(z_0) + \overline{w} f'(z_0)(-w) = f'(z_0)(1 - |w|^2).$$

Since D is simply connected, there is a one-to-one holomorphic function h on D such that $h(z)^2 = \psi(z)$ on D. Let

$$g(z) = \frac{h(z_0)}{|h(z_0)|} \frac{h(z) - h(z_0)}{1 - \overline{h(z_0)}h(z)} : D \to D(0,1)$$

Then g is one-to-one, holomorphic and $g(z_0) = 0$. We will prove that $g'(z_0) > f(z_0)$. Notice that $|h(z_0)| = \sqrt{|w|}$ and

$$\begin{aligned}
g'(z_0) &= \frac{h(z_0)}{|h(z_0)|} \frac{h'(z_0)(1 - |h(z_0)|^2)}{(1 - \overline{h(z_0)}h(z_0))^2} \\
&= \frac{h(z_0)}{|h(z_0)|} \frac{\psi'(z_0)}{2h(z_0)(1 - |\psi(z_0)|)} \\
&= \frac{1}{|h(z_0)|} \frac{f'(z_0)(1 - |w|^2)}{2(1 - |\psi(z_0)|)} \\
&= \frac{1}{\sqrt{|w|}} \frac{f'(z_0)(1 - |w|^2)}{2(1 - |w|)} \\
&= \frac{f'(z_0)(1 + |w|)}{2\sqrt{|w|}} \\
&> f'(z_0).
\end{aligned}$$

The proof is complete. $\square$

3.13.2 Proof of Riemann mapping theorem

Proof. Let $\mathcal{F}(D; z_0)$ be the set of all one-to-one holomorphic functions from D to $D(0,1)$ such that $f(z_0) = 0$ and $f'(z_0) > 0$. Since D is simply connected and $D \neq \mathbb{C}$, by lemma 3.28, $\mathcal{F}(D; z_0) \neq \emptyset$. Let

$$R_{D:z_0} := R = \sup\{f'(z_0) : f \in \mathcal{F}(D; z_0)\}$$

We claim that there is $F \in \mathcal{F}(D; z_0)$ such that $F'(z_0) = R$. By the last lemma, F is the Riemann map.

Choose $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}(D; z_0)$ such that

$$R_{D;z_0} = \lim_{n\to\infty} f_n'(z_0)$$

Since $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence on D, by Montel's theorem, there is a subsequence $\{f_{n_k}\}$ and a holomorphic function f on D such that $f_{n_k} \to f$

uniformly on any compact subset of D. We claim that $f \in \mathcal{F}(D; z_0)$. It is clear that

$$f(z_0) = 0, \quad f'(z_0) = R$$

and $|f(z)| \le 1$ on D. By the maximum modulus theorem, $|f(z)| < 1$ on D and f is one-to-one on D by Hurwitz's Theorem. Therefore, $f \in \mathcal{F}(D; z_0)$. By the previous lemma and definition of R, we have $f : D \to D(0,1)$ is onto and therefore the Riemann map. ∎

We will call $R_{D;z_0}$ is the radius of Riemann map from D to $D(0,1)$ at z_0.

EXAMPLE 40 $R_{D(0,1);z_0} = \frac{1}{1-|z_0|^2}$

Proof. The Riemann map from $D(0,1) \to D(0,1)$ with $f(z_0) = 0$ and $f'(z_0) > 0$ is

$$f(z) = \frac{z - z_0}{1 - \overline{z}_0 z}$$

and

$$R_{D(0,1);z_0} = f'(z_0) = \frac{1}{1 - |z_0|^2}$$

The proof is complete. ∎

3.14 Homework 5

• Conformal holomorphic mappings and normal family.

1. Find a conformal map which maps $D(0,1)$ onto $D(0,1) \setminus \{0\}$.

2. (Hard) Find a conformal map which maps $D(0,1) \setminus \{0\}$ onto $D(0,1)$.

3. Let $\mathcal{F}$ be a family of holomorphic functions on the unit disc such that for any $f \in \mathcal{F}$,

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta \le 1, \text{ for all } 0 < r < 1.$$

   Prove that $\mathcal{F}$ is a normal family.

4. Let $\mathcal{F}$ be a family of holomorphic functions on the unit disc such that for any $f \in \mathcal{F}$,

$$|f(0)|^2 + \int_{D(0,1)} |f'(z)|^2 dA(z) \leq 1.$$

Prove that $\mathcal{F}$ is a normal family.

5. Let $\mathcal{F}$ be a family of holomorphic functions on the unit disc such that for any $f \in \mathcal{F}$,

$$|f(0)| + (1 - |z|)|f'(z)| \leq 1.$$

Prove that $\mathcal{F}$ is a normal family.

6. Let D be a bounded domain in $\mathbb{C}$ and let ϕ be a conformal self-map of D. Let $z_0 \in D$ so that

$$\phi(z) = z_0 + (z - z_0) + O((z - z_0)^2)$$

Prove that ϕ must be the identity (i.e., $\phi(z) = z$).

7. Let $a \in D(0,1)$ and let

$$\phi_a(z) = \frac{z - a}{1 - \bar{a}z}.$$

Define $\phi_a^1(z) = \phi_a(z)$, and for $j \geq 1$, $\phi_a^{j+1}(z) = \phi_a \circ \phi_a^j(z)$. Prove that $\{\phi_a^k\}_{k=1}^{\infty}$ converges normally to a holomorphic function f on $D(0,1)$, and find the function f.

8. Let G be a simply connected domain in $\mathbb{C}$ so that $D \neq \mathbb{C}$. Let $\mathcal{F}$ be a family of holomorphic maps $f : D(0,1) \to G$. Prove or disprove that $\mathcal{F}$ is a normal family.

9. Let $\mathcal{F}(D(0,1))$ be a family of holomorphic functions on the unit disk $D(0,1)$ so that for $f \in \mathcal{F}(D(0,1))$, one has

$$(\mathrm{Re}\, f(z))^2 \neq (\mathrm{Im}\,(f(z))^2, \quad z \in D(0,1).$$

Prove that $\mathcal{F}(D(0,1))$ is a normal family in the sense that every sequence has a normally convergent subsequence or a subsequence that conveges to ∞ normally.

3.15 The Reflection Principle

Let D be a domain in $\mathbb{C}$. Define the reflection domain of D with respect to the real line $\mathbb{R}$

$$D^* = D^*_{\mathbb{R}} = \{\overline{z} : z \in D\}$$

THEOREM 3.30 *Let D be a domain in $\mathbb{R}^2_+$ such that $(a,b) \subset \overline{D}$. Let f be holomorphic in D, continuous on $D \cup (1,b)$ and $f : (a,b) \to \mathbb{R}$. Then there is a holomorphic function F on $D \cup (a,b) \cup D^*$ such that $F = f$ on D.*

Proof. Let

$$F(z) = \begin{cases} f(z), & \text{if } z \in D \cup (a,b); \\ \overline{f(\overline{z})}, & \text{if } z \in D^*. \end{cases}$$

We will prove F is holomorphic in $D \cup (a,b) \cup D^*$.

First we show $F(z)$ is holomorphic in D^*. Fix $z_0 \in D^*$, and let $w_0 = \overline{z}_0 \in D$. Since F is holomorphic in D

$$f(z) = \sum_{n=0}^{\infty} a_n (z - w_0)^n, \quad |z - w_0| < \delta$$

for some $\delta > 0$. Then

$$\overline{f(\overline{z})} = \overline{\sum_{n=0}^{\infty} a_n (\overline{z} - w_0)^n} = \sum_{n=0}^{\infty} \overline{a}_n \overline{(\overline{z} - w_0)}^n = \sum_{n=0}^{\infty} \overline{a}_n (z - z_0)^n,$$

which is holomorphic in $|z - z_0| < \delta$. Therefore, F is holomorphic in D^*. Since $f(a,b) \subset \mathbb{R}$. It is easy to see that F is continuous on $D_e =: D \cup (a,b) \cup D^*$. One may apply Morera's theorem or Cauchy's theorem to prove that F is holomorphic in D_e (see fall quarter, homework 6, problem 1.)

EXAMPLE 41 *Let $f(z)$ be holomorphic in $\mathbb{R}^2_+$ and is continuous on $\mathbb{R}^2_+ \cup (0,1)$. Let $f(x) = \cos x + i \sin x$ when $x \in (0,1)$ Find all such f.*

Solution. Let

$$g(z) = f(z) - (\cos z + i \sin z), \quad z \in \mathbb{R}^2_+ \cup (0,1)$$

Then g is holomorphic in $\mathbb{R}^2_+$ and continuous on $\mathbb{R}^2_+ \cup (0,1)$ and $g(x) = 0$ when $x \in (0,1)$. By the reflection principle, g can be extended holomorphically to $D_e =: \mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty))$ and $g(x) = 0$ when $x \in (0,1)$. D_e is a domain which includes $(0,1)$. By the uniqueness theorem of holomorphic functions, $g(z) \equiv 0$ on D_e. Therefore, $f(z) = \cos z + i \sin z$ on $\mathbb{R}^2_+ \cup (0,1)$. ∎

Let D be a domain in $D(z_0, R)$, we define the reflection domain of D with respect to the real line $\partial D(z_0, R)$

$$D^* = D^*_{\partial D(z_0,R)} = \{z_0 + \frac{R^2}{\overline{z} - \overline{z}_0} : z \in D(z_0, R)\}$$

THEOREM 3.31 *Let D be a domain in $D(z_0, R)$ and Γ a portion of the circle $|z - z_0| = R$ such that $\Gamma \subset \overline{D}$. If f is holomorphic in D, continuous on $D \cup \Gamma$, and $f|_\Gamma : \Gamma \to \mathbb{R}$, then there is a holomorphic function F on $D \cup (a, b) \cup D^*$ such that $F = f$ on D.*

Proof. Let S be a Möbius transformation from $\mathbb{R}^2_+$ onto $D(z_0, R)$. Let $g(z) = f(S(z))$ and $(a, b) = S^{-1}(\Gamma)$. Then g can be extended to be holomorphic in $D_e = \mathbb{R}^+_2 \cup (a, b) \cup \mathbb{R}^-_2$. Thus

$$F(z) = g(S^{-1}(z))$$

is holomorphic in $S(D_e) = D \cup \Gamma \cup D^*$. ∎

3.16 Homework 6

- Problems related to holomorphic extensions

1. Let $f(z)$ be holomorphic on $D(0, 1)$ and continuous on $\overline{D(0, 1)}$ such that $f(e^{i\theta}) = 2$ for all $\theta \in [0, \pi/4)$. Prove that f is a constant.

2. Let f be holomorphic on $\mathbb{R}^2_+$ and continuous on $\mathbb{R}^2_+ \cup (0, 1)$. Assume that
$$f(x) = i \sin(x), \quad x \in (0, 1).$$
What is f?

3. Let $f(z)$ be entire such that $f(x + ix) \in \mathbb{R}$ for all $x \in \mathbb{R}$. Assume that $f(i) = 0$. Prove that $f(1) = 0$.

4. Let $L = \{x + iy : y = ax + b,\ x \in \mathbb{R}\}$ be a line in $\mathbb{C}$. Let $\mathbb{R}^2_L = \{z = x + iy \in \mathbb{C} : y > ax + b,\ x \in \mathbb{R}\}$. Suppose f is holomorphic in $\mathbb{R}^2_L$ and continuous on $\overline{\mathbb{R}^2_L}$ so that $f(L) \subset L$. Prove that there is an entire function F so that $f(z) = F(z)$ for $z \in \mathbb{R}^2_L$.

5. Let
$$f(z) = \sum_{n=1}^{\infty} \frac{z^{2n}}{n}$$
be holomorphic in $D(0,1)$. Find all singular points on $\partial D(0,1)$ and justify your answer.

6. Prove that every boundary point of
$$f(z) = \sum_{n=1}^{\infty} \frac{z^{n!}}{n^{100}}$$
is a singular point.

7. Prove that every boundary point of
$$f(z) = \sum_{n=1}^{\infty} \frac{z^{2^n}}{2^n}$$
is a singular point.

8. Prove or disprove that there is a nonconstant holomorphic function f on the domain
$$D = \{z \in \mathbb{C} : |z+1| > 1 \text{ and } |z+2| < 2\}$$
and continuous on $\overline{D}$ with $f(z) = 0$ when z is in $S = \{z \in \mathbb{C} : |z+2| = 2 \text{ and } \operatorname{Re} z \in (-2,-1)\}$.

3.17 Singular Points and Regular Points

Definition 3.32 *Let D be a domain in $\mathbb{C}$, f a holomorphic function on D; a point $z_0 \in \partial D$ is said to be a regular point for f on D if there is a holomorphic function F in $D(z_0, \epsilon)$ such that $F = f$ on $D \cap D(z_0, \epsilon)$ for some $\epsilon > 0$. If z_0 is not a regular point, then we say that z_0 is a singular point for f on D.*

EXAMPLE 42 *Consider*
$$f(z) = \sum_{n=0}^{\infty} z^n, \quad z \in D(0,1).$$

We know that

$$f(z) = \frac{1}{1-z}, \quad z \in D(0,1).$$

It is easy see that $z = 1$ *is a singular point for* f *on* $D(0,1)$*, and every other point in* $\partial D(0,1)$ *is a regular point.*

EXAMPLE 43 *Consider*

$$f(z) = \sum_{n=0}^{\infty} z^{n!}, \quad z \in D(0,1).$$

Prove that every point in $\partial D(0,1)$ *is a singular point for* f *on* $D(0,1)$.

Proof. Consider

$$z_0 = e^{i2\pi p/q}, \quad p, q \in \mathbb{N},\ gcd(p,q) = 1.$$

Then

$$f(rz_0) = \sum_{n=0}^{q-1} (rz_0)^{n!} + \sum_{n=q}^{\infty} (rz_0)^{n!} = \sum_{n=0}^{q-1} (rz_0)^{n!} + \sum_{n=q}^{\infty} r^{n!}$$

It is easy to see that

$$\lim_{r \to 1^-} f(rz_0) = +\infty.$$

So z_0 is a singular point. Since $\{e^{i2\pi p/q} : p, q \in \mathbb{N}\}$ is dense in $\partial D(0,1)$, every point $z_0 \in \partial D(0,1)$ is a singular point. □

EXAMPLE 44 *Consider*

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{n!}}{2^n}, \quad z \in D(0,1)$$

Prove every point in $\partial D(0,1)$ *is a singular point for* f *on* $D(0,1)$.

Hint: Consider $f'(z)$ instead of $f(z)$.

4 Infinite Products

Let f be holomorphic in a domain D. If $\{z_1, \cdots, z_m\}$ are zeros of f counting multiplicity, then

$$g(z) = \frac{f(z)}{\prod_{j=1}^{m} (z - z_j)}$$

is holomorphic in D and has no zeros in D.

Question. If $m = \infty$, what does $\prod_{j=1}^{\infty} (z - z_j)$ mean?

4.1 Basic properties of infinite products

Definition 4.1 *Let* $\{z_n\}_{n=1}^\infty$ *be a sequence of non-zero complex numbers. We say that* $\prod_{j=1}^\infty z_n$ *defines a complex number* z *if*

$$\lim_{n\to\infty}\prod_{j=1}^{n} z_j = z$$

When $z \neq 0$, then

$$z_n = \frac{\prod_{j=1}^{n} z_j}{\prod_{j=1}^{n-1} z_j} \to \frac{z}{z} = 1 \quad \text{as } n \to \infty.$$

THEOREM 4.2 *Let* $\{z_n\}_{n=1}^\infty$ *be a sequence of non-zero complex numbers. Then* $\prod_{j=1}^\infty z_n$ *converges to a non-zero complex number if and only if* $\sum_{n=1}^\infty \ln z_n$ *converges.*

Proof. If $\sum_{n=1}^\infty \ln z_n$ converges to $z \in \mathbb{C}$, then

$$\prod_{j=1}^{\infty} z_j = e^{\sum_{j=1}^\infty \ln z_j} = e^z.$$

Conversely, if $w_n = \prod_{j=1}^n z_j$ converges to $z \neq 0$, then

$$\ln w_n = \sum_{j=1}^{n} \ln z_j + 2k_n \pi i$$

where k_n is some integer. Then

$$\ln w_{n+1} - \ln w_n = 2(k_{n+1} - k_n)\pi i \to 0 \quad \text{as } n \to \infty.$$

Therefore, there is an N such that if $n \geq N$, one has $k_n = k_N$. Therefore,

$$\ln w_n = \sum_{j=1}^{n} \ln z_j + k_N 2\pi i, \quad n \geq N$$

Therefore,

$$\sum_{j=1}^{\infty} \ln z_j = \ln z + 2k_N \pi i.$$

The proof is complete. □

Definition 4.3 *Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of non-zero complex numbers. We say that $\prod_{j=1}^{\infty} z_n$ converges to a non-zero complex number z absolutely if*

$$\sum_{j=1}^{\infty} |\ln z_j| < \infty.$$

Therefore, we have

1) $\prod_{n=1}^{\infty}(1+z_n)$ converges to non-zero complex number if and only if $\sum_{j=1}^{\infty} \ln(1+z_j)$ converges.

2) $\prod_{n=1}^{\infty}(1+z_n)$ converges to non-zero complex number absolutely if and only if $\sum_{j=1}^{\infty} |\ln(1+z_j)|$ converges.

Notice that

$$\frac{|z|}{2} \le \ln(1+|z|) \le 2|z|, \quad |z| \le 1/2.$$

We have

THEOREM 4.4 $\prod_{n=1}^{\infty}(1+z_n)$ *converges to a non-zero complex number absolutely if and only if $\sum_{n=1}^{\infty} |z_n|$ converges and for all n, $z_n \neq -1$.*

Proof. Without loss of generality, we may assume that $|z_n| << 1$. Then

$$|\ln(1+z_n)| = \sqrt{(\ln|1+z_n|)^2 + \theta_n^2}, \quad \theta_n = \tan^{-1}\frac{y_n}{1+x_n}.$$

Assume that $|x_n| \le \frac{1}{2}$. Then

$$\frac{|y_n|}{2} \le |\theta_n| \le \frac{|y_n|}{1+x_n} \le 2|y_n|$$

and

$$-2|z_n| \le \ln(1-|z_n|) \le \ln|1+z_n| \le \ln(1+|z_n|) \le 2|z_n|.$$

Notice that

$$\ln|1+z_n| = \frac{1}{2}\ln[(1+x_n)^2 + y_n^2].$$

If $x_n \ge 0$ and $|x_n| \le |y_n|$, then $|x_n| \le 2|\theta_n|$. Otherwise, if $-\frac{1}{2} \le x_n < 0$ and $|x_n| > |y_n|$, then $x_n + x_n^2 + y_n^2 \le 0$. Therefore,

$$\ln|1+z_n| = \frac{1}{2}\ln[(1+x_n)^2 + y_n^2] \ge \ln(1+x_n) \ge \frac{x_n}{2}, \quad x_n \le 1/2$$

If $x_n < 0$ then

$$\ln|1+z_n| = \frac{1}{2}\ln[1+2x_n+x_n^2+y_n^2] \leq \frac{1}{2}\ln(1+x_n) \leq \frac{x_n}{2}, \quad x_n \leq 1/2$$

Therefore

$$|z_n| \leq 2(|\ln|1+z_n|| + |\theta_n|) \leq 4|\ln(1+z_n)|, \quad \text{if } |z_n| \leq 1/2.$$

In summary, we have

$$\sum_{n=1}^{\infty} |\ln(1+z_n)| \text{ converges if and only if } \sum_{n=1}^{\infty} |z_n| \text{ converges .} \quad \square$$

4.2 Examples

EXAMPLE 45 *Determine if each of the following infinite products converges:*

(i) $\prod_{j=1}^{\infty}(1-\frac{i}{j^2})$

(ii) $\prod_{n=2}^{\infty}(1-\frac{1}{n})$.

Additionally, show that the following infinite product converges and find its value:

(iii) $\prod_{j=2}^{\infty}(1-\frac{1}{j^2})$.

Solution.

(i) Since $|\frac{i}{j^2}| = \frac{1}{j^2}$ and $\sum_{j=1}^{\infty}\frac{1}{j^2}$ converges. By the previous theorem, we have $\sum_{j=1}^{\infty}(1-\frac{i}{j^2})$ converges absolutely.

(ii) Observe that

$$\prod_{n=2}^{m}(1-\frac{1}{n}) = \prod_{n=2}^{m}\frac{n-1}{n} = \frac{1}{m} \to 0 \quad \text{as } m \to \infty.$$

But $\prod_{n=2}^{\infty}(1-\frac{1}{n})$ does not converge absolutely.

(iii) Notice that

$$\prod_{j=2}^{m}(1-\frac{1}{j^2})=\prod_{j=2}^{m}\frac{j+1}{j}\frac{j-1}{j}=\frac{m+1}{2}\frac{1}{m}\to\frac{1}{2}$$

as $m\to\infty$. Therefore, $\prod_{n=2}^{\infty}(1-\frac{1}{j^2})=\frac{1}{2}$ and it converges absolutely. ∎

Definition 4.5 *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on a domain D. We say that $\prod_{j=1}^{\infty}(1+f_n)$ converges absolutely and uniformly on a compact set $K\subset D$ there is a positive integer $N=N(K)$ if*

$$\sum_{j=N}^{\infty}|\ln(1+f_n(z))|$$

converges uniformly on K.

THEOREM 4.6 *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on a domain D. If $\prod_{j=1}^{\infty}(1+f_n)$ converges absolutely and uniformly on any compact set $K\subset D$ then*

$$\prod_{n=1}^{\infty}(1+f_n(z))$$

defines a holomorphic function on D.

Proof. This is straight forward. ∎

4.3 Infinite Products and Factorization Factors

For each $p\geq 0$, we let

$$E_p(z)=\begin{cases}(1-z)e^{z+z^2/2+\cdots+z^p/p}; & p\geq 1,\\ 1-z; & p=0\end{cases}$$

Lemma 4.7 *For any non-negative integer p, $E_p(z)$ satisfies*

(i) $E_p(0)=1$;

(ii) $|E_p(z)-1|\leq|z|^{p+1}$ *if* $|z|\leq 1$.

Proof. It is obvious that $E_p(0) = 1$. Now, we write

$$E_p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n.$$

Then

$$E_p'(z) = \sum_{n=1}^{\infty} n b_n z^{n-1}.$$

If $p \geq 1$ then

$$\begin{aligned} E_p'(z) &= e^{z+z^2/2+\cdots+z^p/p}[-1 + (1-z)(1+z+\cdots+z^{p-1}] \\ &= -z^p e^{z+z^2/2+\cdots+z^p/p} \end{aligned}$$

This implies that

(a) $b_n \leq 0$ for $n \geq 1$

(b) $b_k = 0$ if $1 \leq k \leq p$

(c) $0 = E_p(1) = 1 + \sum_{n=1}^{\infty} b_n$.

Therefore,

$$|E_p(z) - 1| = |\sum_{n=1}^{\infty} b_n z^n| \leq |z|^{p+1} |\sum_{n=1}^{\infty} b_n| = |z|^{p+1}. \quad \square$$

THEOREM 4.8 *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $\lim_{n\to\infty} a_n = \infty$. Then there is a holomorphic function f on $\mathbb{C}$ such that $Z(f) = \{a_1, a_2, \cdots\}$.*

Proof. We define

$$f(z) = \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right)$$

It is clear that $E_{n-1}(\frac{z}{a_n})$ is holomorphic in $\mathbb{C}$ and $E_{n-1}(\frac{z}{a_n}) = 0$ if only if $z = a_n$, and it is a simple zero. But the previous lemma, we have

$$|E_{n-1}(\frac{z}{a_n}) - 1| \leq \frac{|z|^n}{|a_n|^n}, \quad \text{if } |z| \leq |a_n|.$$

Therefore, the infinite product $\prod_{n=1}^{\infty} E_{n-1}(z/a_n)$ converges uniformly on any compact subset of $\mathbb{C}$. Therefore, f is holomorphic in $\mathbb{C}$ and $Z(f) = \{a_1, a_2, a_3, \cdots\}$. $\square$

THEOREM 4.9 *Let $D \neq \mathbb{C}$ be a domain in $\mathbb{C}$ and let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of points in D without any accumulation points in D. Then there is a holomorphic function f on D such that $Z(f) = \{a_1, a_2, \cdots\}$.*

Proof. Since $D \neq \mathbb{C}$, there is a $a_n^* \in \partial D$ such that $|a_n - a_n^*| = \text{dist}(a_n, \partial D)$. Let

$$f(z) = \prod_{n=1}^{\infty} E_{n-1}\left(\frac{a_n - a_n^*}{z - a_n^*}\right)$$

It is clear that $E_{n-1}(\frac{a_n - a_n^*}{z - a_n^*})$ is holomorphic in D and $E_{n-1}(\frac{a_n - a_n^*}{z - a_n^*}) = 0$ if only if $z = a_n$, and it is a simple zero. By the previous lemma,

$$|E_{n-1}(\frac{a_n - a_n^*}{z - a_n^*}) - 1| \leq \frac{|a_n - a_n^*|^n}{|z - a_n^*|^n}, \quad \text{if } |a_n - a_n^*| \leq |z - a_n^*|.$$

For any compact subset K of D, since $\lim_{n\to\infty} |a_n - a_n^*| = 0$, there is an N such that if $n \geq N$ then

$$|z - a_n^*| \geq 2|a_n - a_n^*| \quad \text{for all } z \in K.$$

This implies that f is well-defined and holomorphic in D. Moreover, $Z(f) = \{a_1, a_2, a_3, \cdots\}$. ∎

4.4 Weierstrass Factorization Theorem

THEOREM 4.10 *Let D be a domain in $\mathbb{C}$ and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of points in D without any accumulation points in D. Then there is a holomorphic function f on $\mathbb{C}$ such that $Z(f) = \{a_1, a_2, \cdots\}$.*

Proof. Case 1. If $D = \mathbb{C}$, then $\lim_{n\to\infty} a_n = \infty$. Theorem 4.8 shows the existence of such holomorphic function f.

Case 2. $D \neq \mathbb{C}$. Since $\{a_n : n \in \mathbb{N}\}$ has no accumulation point in D, there is a $z_0 \in D$ and $r > 0$ such that $|a_n - z_0| \geq r$. Let $\tilde{D} = \{1/(z - z_0) : z \in D\}$ and $\tilde{a}_n = 1/(a_n - z_0)$. Then $\{\tilde{a}_n : n \in \mathbb{N}\}$ is a bounded sequence in $\tilde{D}$. By the previous theorem, there is a the sequence $\{\tilde{a}_n^*\}_{n=1}^{\infty} \subset \partial\tilde{D}$ with $|\tilde{a}_n - \tilde{a}_n^*| = \text{dist}(\tilde{a}_n, \partial\tilde{D})$ and the holomorphic function

$$g(z) = \prod_{n=1}^{\infty} E_{n-1}(\frac{\tilde{a}_n - \tilde{a}_n^*}{z - \tilde{a}_n^*}), z \in \tilde{D}$$

such that

$$Z(g) = \{\tilde{a}_n : n \in \mathbb{N}\}.$$

Since $\{\tilde{a}_n\}_{n=1}^\infty$ and $\{\tilde{a}_n^*\}_{n=1}^\infty$ are bounded, one has when z is very large,

$$\Big|E_{n-1}(\frac{\tilde{a}_n - \tilde{a}_n^*}{z - \tilde{a}^*}) - 1\Big| \leq \Big|\frac{\tilde{a}_n - \tilde{a}_n^*}{z - \tilde{a}^*}\Big|^n \leq \frac{1}{2^n}$$

Therefore, since $E_p(0) = 1$, one can see that

$$\lim_{z \to \infty} g(z) = b \in \mathbb{C} \setminus \{0\}.$$

Let

$$f(z) = g(\frac{1}{z - z_0}), \quad z \in D \setminus \{z_0\}, \quad f(z_0) = b.$$

Then f is holomorphic in D and $Z(f) = \{a_n : n \in \mathbb{N}\}$. ▯

The following the Weierstrass' factorization theorem for a meromorphic function.

THEOREM 4.11 *Let D be a domain in $\mathbb{C}$ and let f be a meromorphic function on D then there are three holomorphic functions g, f_1, f_2 on D such that*

1. $Z(f) = Z(f_1)$;
2. $P(f) = Z(f_2)$;
3. $f(z) = \frac{f_1(z)}{f_2(z)}$.

where $P(f)$ is the set of poles of f counting orders.

Proof. Let $P(f) = \{w_1, w_2, \cdots, w_m, \cdots\}$ be the poles of f in D counting orders. Then, by the previous theorem, there is a holomorphic function f_2 on D such that

$$f_2(z) = \prod_{n=1}^\infty E_{n-1}\left(\frac{z}{w_n}\right)$$

Let

$$f_1(z) = f(z) f_2(z)$$

Then $f_1(z)$ is holomorphic in D and $Z(f) = Z(f_1)$. Moreover,

$$f(z) = \frac{f_1(z)}{f_2(z)}.$$

THEOREM 4.12 *Let f be a meromorphic function on $\mathbb{C}$. Let $Z(f) = \{z_n : n \in \mathbb{N}\}$ be zero set of f counting multiplicity and $P(f) = \{w_n : n \in \mathbb{N}\}$ denote the set of poles counting order. Then there is a holomorphic function g on $\mathbb{C}$ such that*

$$f(z) = e^{g(z)} \frac{f_1(z)}{f_2(z)}$$

with

$$f_1(z) = \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{z_n}\right) \quad \text{and} \quad f_2(z) = \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{w_n}\right).$$

Proof. By the definitions of f_1 and f_2, one has

$$h(z) := \frac{f(z) f_2(z)}{f_1(z)}$$

is holomorphic in $\mathbb{C}$ and has no zeros in $\mathbb{C}$. Since $\mathbb{C}$ is simply connected, there is a holomorphic function g on $\mathbb{C}$ such that $h(z) = e^{g(z)}$. This implies that

$$f(z) = e^{g(z)} \frac{f_1(z)}{f_2(z)}.$$

4.5 Application to Singular Points

THEOREM 4.13 *Let D be a domain in $\mathbb{C}$. Then there is a holomorphic f on D such that every point $z_0 \in \partial D$ is a singular point for f on D.*

Proof. Choose a sequence of bounded domains D_n in D such that $\partial D_n \subset D$ and $D_n \to D$ increasingly and

$$\frac{1}{2n} \le \text{dist}(\partial D_n, \partial D) < \frac{1}{n} \quad \text{if } m \ne n.$$

Choose k_n many points $z_{n,k}$ in ∂D_n such that $|z_{n,k} - z_{n,j}| > 2^{-n+10}$ and $\partial D_n \subset \cup_{k=1}^{k_n} D(z_{n,k}, 2^{-n})$. Then $Z = \{z_{n,k} : k = 1, \cdots, k_n, n \in \mathbb{N}\}$ is a set of points in D having no accumulation points in D. Every point in ∂D is an accumulation point of Z. There is a non-constant holomorphic function f on D such that $Z(f) = Z$. Then every point in ∂D is a singular point for f in D. Otherwise, there is $z_0 \in \partial D$ such that f can be extended to be holomorphic in $D(z_0, \delta_0)$ for $\delta > 0$, but z_0 is an accumulation point of $Z(f)$. The Uniqueness theorem of holomorphic function implies $f \equiv 0$. This is a contradiction. ▯

4.6 Mittag-Leffler's Theorem

Definition 4.14 *A singular part about point z_0 is a function*

$$S_{z_0}(z) = \sum_{k=-m}^{-1} a_k (z - z_0)^k, \quad z \in \mathbb{C} \setminus \{z_0\}.$$

Existence of the prescribing singular parts theorem

THEOREM 4.15 *(Mittag-Leffler's theorem) Let D a domain in $\mathbb{C}$, and let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points in D with no accumulation point in D. Let*

$$S_n(z) = \sum_{k=-m_n}^{-1} a_{n,k} (z - z_n)^{-k}$$

be a singular part about z_n. Then there is a meromorphic function f on D and holomorphic in $D \setminus \{z_n : n \in \mathbb{N}\}$ such that $f(z) - S_n(z)$ is holomorphic in a neighborhood of z_n.

Proof. First, we assume that D is bounded. Then there is a sequence $\{w_n \in \partial D\}$ such that

$$\text{dist}(z_n, \partial D) = |z_n - w_n|, \quad n \in \mathbb{N}.$$

Assume that there is a holomorphic function $T_n(z)$ in D such that

$$|T_n(z) - S_n(z)| < 2^{-n}, \quad |z - w_n| > 2|z_n - w_n|$$

Then we define

$$f(z) = \sum_{n=1}^{\infty} (T_n(z) - S_n(z)), \quad z \in D \setminus \{z_n : n \in \mathbb{N}\}$$

Then f is holomorphic in $D \setminus \{z_n : n \in \mathbb{N}\}$. In fact, for any compact subset $K \subset D \setminus \{z_n : n \in \mathbb{N}\}$, there is an N such that

$$|z - w_n| > 2|z_n - w_n|, \quad n \geq N, z \in K.$$

Then $\sum_{n=1}^{\infty} (T_n(z) - S_n(z))$ converges uniformly on K. This implies that f is holomorphic in $D \setminus \{z_n : n \in \mathbb{N}\}$. Since T_n is holomorphic, f is meromorphic in D and has the singular part S_n at z_n.

Next, we shall prove the existence of $T_n(z)$. Consider

$$\frac{1}{z-z_n} = \frac{1}{z-w_n-(z_n-w_n)} = \frac{1}{z-w_n}\sum_{k=0}^{\infty}\Big(\frac{z_n-w_n}{z-w_n}\Big)^k$$

Choose $K_n \geq n$ such that

$$\sum_{k=K_n}^{\infty}(1/2)^k \leq 2^{-n}\frac{\min\{1,|w_n-z_n|^{m_n+1}\}}{(\sum_{k=-m_n}^{-1}|a_{n,k}||k|+1)}.$$

Let

$$t_n(z) = \frac{1}{z-w_n}\sum_{k=0}^{K_n}(\frac{z_n-w_n}{z-w_n})^k.$$

When $|z-w_n| > 2|z_n-z_n|$ we have

$$\Big|\frac{1}{z-z_n}-t_n(z)\Big| \leq 2^{-n}\frac{\min\{|w_n-z_n|^{m_n},1\}}{(\sum_{k=-m_n}^{-1}|a_{n,k}||k|+1)}.$$

Let

$$T_n(z) = \sum_{k=1}^{m_n} a_{n,-k}t_n(z)^k.$$

$T_n(z)$ is a polynomial.

When $|z-w_n| \geq 2|z_n-w_n|$, one has

$$|z-z_n| \geq |z-w_n|-|z_n-w_n| \geq |z_n-w_n|$$

and

$$|\frac{1}{z-z_n}| \leq \frac{1}{|z_n-w_n|}, \quad |t_n(z)| \leq \frac{1}{|z_n-w_n|}$$

and

$$\begin{aligned}
|S_n - T_n(z)| &\leq \sum_{k=1}^{m_n}|a_{n,-k}|\Big|\Big(\frac{1}{z-z_n}\Big)^k - t_n(z)^k\Big| \\
&\leq \sum_{k=1}^{m_n}|a_{n,-k}|\sum_{\ell=0}^{k-1}\frac{|t_n(z)|^\ell}{|z-z_n|^{k-1-\ell}}\Big|\frac{1}{z-z_n}-t_n(z)\Big| \\
&\leq \sum_{k=1}^{m_n}\frac{|a_{n,-k}|k}{|z_n-w_n|^{k-1}}\Big|\frac{1}{z-z_n}-t_n(z)\Big| < 2^{-n}.
\end{aligned}$$

The claim is proved and the theorem is proved. ∎

Second, we assume $D = \mathbb{C}$.
Since $\lim_{n\to\infty} z_n = \infty$. When $|z| < 2|z_n$, one has

$$\frac{1}{z - z_n} = -\frac{1}{z_n}\frac{1}{1 - \frac{z}{z_n}} = -\frac{1}{z_n}\sum_{k=0}^{\infty}\Big(\frac{z}{z_n}\Big)^k$$

Following the argument above, one can find k_n such that

$$T_n(z) = \sum_{k=1}^{m_n} a_{n,-k}\Big(-\sum_{\ell=0}^{k_n}\frac{z^\ell}{z_n^{\ell+1}}\Big)^k$$

such that

$$|S_n(z) - T_n(z)| < 2^{-n}, \quad \text{when } |z| < 2|z_n|$$

With the same argument as the first case, one can complete the proof of the theorem.

Third, When $D \neq \mathbb{C}$ and is unbounded. One may follow the proof the Weierstrass Factorization theorem to deal with the case. Omit it here.

4.7 Homework 7

1. Determine whether $\prod_{n=2}^{\infty}\Big(1 + (-1)^n\frac{1}{n}\Big)$ converges. Do the same for $\prod_{n=2}^{\infty}\Big(1 + (-1)^n\frac{1}{\sqrt{n}}\Big)$.

2. For which z does $\prod_{n=1}^{\infty}(1 + z^{3^n})$ converge?

3. If $|z| < R$, then prove that

$$\prod_{n=0}^{\infty}\Big(\frac{R^{2^n} + z^{2^n}}{R^{2^n}}\Big) = \frac{R}{R - z}.$$

4. Let f be entire and have a first-order (simple) zero at each of the nonpositive integers. Prove that

$$f(z) = ze^{g(z)}\prod_{j=1}^{\infty}\Big[(1 + \frac{z}{j})e^{-z/j}\Big]$$

for some entire function g.

5. Suppose that $\sum_{n=1}^{\infty}|\alpha_n - \beta_n| < \infty$. Determine the set of z for which $\prod_{n=1}^{\infty}\frac{z-\alpha_n}{z-\beta_n}$ converges normally.

6. Prove that
$$\frac{\sin z}{\sin(\pi z)} = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{n \sin n}{z^2 - n^2}.$$

7. Let $\{a_n\}_{n=1}^{\infty} \subset D(0,1)$ be such that $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$. Discuss the convergence of
$$\prod_{n=1}^{\infty} B_{a_n}(z), \quad B_{a_n} = \frac{\overline{a}_n}{|a_n|}\frac{a_n - z}{1 - \overline{a}_n z}.$$

8. Use Mittag–Leffler's theorem to prove the following: Let $\{a_j\}_{j=1}^{\infty}$ be a squence of ponits in a simply connected domain D without any accummulation points in D. Then there is a holomorphic function f so that $Z(f) = \{a_j\}_{j=1}^{\infty}$ counting multiplicity.

9. Suppose that g_1, g_2 are entire functions with no common zeros. Prove that there are entire functions f_1 and f_2 such that
$$f_1 g_1 + f_2 g_2 \equiv 1.$$

10. Construct a meromorphic function f on $\mathbb{C}$ so that $\mathbb{N} = \{1, 2, 3, \cdots\}$ is its set of poles and the singular part at $z = n$ is
$$S_n(z) = \sum_{k=-2^n}^{-1} 3^k (z - n)^k, \quad n = 1, 2, 3, \cdots.$$

Hint for Problem 6:

Consider
$$g(z) = \frac{\sin z}{\sin(\pi z)} - \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{n \sin n}{z^2 - n^2}.$$

Prove that

(i) g is holomorphic in $\mathbb{C}$.

(ii) g is bounded on $z = \pm(n + \frac{1}{2}) + iy$ and $z = x + \pm(n + 1/2)$.

Conclude that g is a constant and $g(0) = 0$.

Hint for Problem 3: Prove that

$$f(z) = \prod_{n=0}^{\infty} (1 + z^{2^n}) = \frac{1}{1-z}.$$

Notice that

$$f(z) = (1+z) \prod_{n=1}^{\infty} (1 + z^{2^n}) = (1+z) \prod_{n=0}^{\infty} (1 + (z^2)^{2^n}) = (1+z) f(z^2).$$

Write

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n.$$

Since

$$a_{2n} = a_n, \quad a_{2n+1} = a_n, \quad a_0 = 1,$$

$a_n = a_0 = 1$ for all n. So,

$$f(z) = \frac{1}{1-z}. \quad \square$$

Final Review

Final Review = Midterm Review + the following materials

1. Proper, biholomorphic maps

EXAMPLE 46 *(i) Find all proper holomorphic maps f on $D(0,1)$ with $f \in C(\overline{D}(0,1))$.*

(ii) Prove or disprove that there is a proper holomorphic map $f : \mathbb{C} \setminus \{0,1\} \to D(0,1) \setminus \{0\}$.

2. Automorphism group Aut(D)

EXAMPLE 47 *Find the following:*

(i) Aut$(D(0,1))$

(ii) Aut$(\mathbb{C} \setminus \{0\})$

(iii) Aut$(\mathbb{C} \setminus \overline{D}(0,1))$

3. Möbius transformations, Cross ratio

EXAMPLE 48 *(i) Find a Möbius transformation that maps $2i$,$i+1$ and -3 to the points $1+i$, $-1-i$ and $2+2i$;*

(ii) Prove that the cross ratio has the following property: Let L be the ‘circle” determined by three points z_1, z_2 and z_3. Let $z \in \mathbb{C} \setminus L$ and z^* be the symmetric point of z with respect to L. Then

$$(z^*; z_1, z_2, z_2) = \overline{(z; z_1, z_2, z_3)}.$$

4. Conformal mappings

EXAMPLE 49 *(i) Find a conformal map which maps $D = D(2,2) \setminus \overline{D}(1,1)$ to the unit disc;*

(ii) Find a conformal map which maps the unit disc to $D(0,1) \setminus \{0\}$.

5. Riemann mapping theorem

EXAMPLE 50 *(i) State the Riemann mapping theorem*

(ii) Prove the uniqueness portion of the Riemann mapping theorem.

(iii) Let D be a simply connected domain in $\mathbb{C}$, $D \neq \mathbb{C}$ and $z_0 \in D$. Prove that $\mathcal{F} = \{f : D \to D(0,1)$ is one-to-one , $f(z_0) = 0\}$ is not empty.

6. Normal families

EXAMPLE 51 *(i) Let $\mathcal{F}$ be a family of holomorphic functions on $D(0,1)$ such that $f(0) = i$ and Im $f(z) \neq \operatorname{Re} f(z)$ for all $z \in D(0,1)$. Prove that $\mathcal{F}$ is a normal family;*

(ii) Prove that the unit ball in Hardy space is a normal family.

7. Reflection Principle

EXAMPLE 52 *Let f be holomorphic in $D = \{z = x + iy : x > y\}$ and it is continuous on $D \cup \{(x + ix : x \in [0,1]\}$ and*

$$f(x + ix) = \sin((1 - i)x), \quad x \in (0,1)$$

Find all such f.

8. Infinite products

EXAMPLE 53 *(i) State Weierstrass Factorization theorem;*

(ii) Prove that $\prod_{j=1}^{\infty} \cos \frac{1}{j}$ converges absolutely;

(iii) Prove that $|E_p(z) - 1| \leq |z|^{p+1}$ if $|z| \leq 1$;

(iv) Find an entire holomorphic function f on $\mathbb{C}$ such that zero set of f is $\{2^n : n \in \mathbb{N}\}$.

9. Problems associated to singular points

EXAMPLE 54 *(i) Prove that every point* $z_0 \in \partial D(0,1) \setminus \{1\}$ *is regular point for* f *in* $D(0,1)$ *and* $z = 1$ *is a singular point for* f *in* $D(0,1)$*, where*

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

(ii) Prove that every point $z_0 \in \partial D(0,1)$ *is a singular point for* f *in* $D(0,1)$ *where*

$$f(z) = \sum_{n=1}^{\infty} \frac{z^{n!}}{2^n}$$

(iii) Let D *be a domain in* $\mathbb{C}$*. Construct a holomorphic function* f *in* D *such that every point* $z_0 \in \partial D$ *is a singular point of* f*.*