The CR-Obata theorem on compact strictly pseudoconvex pseudo-Hermitian manifolds *

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Abstract. In this paper, the author proves that the CR analogue of the Obata theorem holds on a torsion-free, strictly pseudoconvex, pseudo-Hermitian CR manifold of dimension $2n + 1$ for integer $n > 1$. This result for the case $n = 1$ was proved recently by Chang and Chiu [3].

1 Introduction and main results

Let $(M, \theta)$ be a $(2n + 1)$-dimensional strongly pseudoconvex pseudo-Hermitian manifold in the sense of Webster [20]. Let $H(M)$ be the holomorphic tangent bundle over $M$. The pseudo-Hermitian form $\theta$ is a real nowhere-vanishing one-form on $M$ which annihilates $H(M) \oplus \overline{H(M)}$. Locally, one can choose $n$ complex one-forms $\theta^\alpha$ so that $(\theta^1, \theta^2, \cdots, \theta^n)$ form a basis for the holomorphic cotangent bundle $H(M)^*$ and

\begin{equation}
    d\theta = i \sum_{\alpha, \beta=1}^n h_{\alpha\beta} \theta^\alpha \wedge \theta^\beta, \quad \theta^\tau = \overline{\theta^\alpha}
\end{equation}

where $(h_{\alpha\beta})$ is an $n \times n$ positive definite matrix on $M$. It was shown by Webster [20] that there is a unique way to choose connection one form $\omega^\alpha_\beta$ and $(0, 1)$-form $\tau^\alpha$ so that

\begin{equation}
    d\theta^\alpha = \theta^\beta \wedge \omega^\alpha_\beta + \theta^\beta \wedge \tau^\alpha, \quad dh_{\alpha\beta} = h_{\gamma\beta} \omega^\alpha_\gamma + h_{\alpha\gamma} \omega^\gamma_\beta
\end{equation}

and there exist $A_{\alpha\beta} = A_{\beta\alpha}$ so that

\begin{equation}
    \tau^\alpha = \sum_{\beta=1}^n A_{\alpha\beta} \theta^\beta, \quad \text{Tor}(X, X) = 2\text{Im} \sum_{\alpha, \beta=1}^n A_{\alpha\beta} x^\alpha x^\beta, \quad X \in H(M).
\end{equation}

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Let $R_{\alpha \beta}$ be the Webster pseudo Ricci curvature and

\begin{equation}
\text{Ric}(X, X) = \sum_{\alpha, \beta=1}^{n} R_{\alpha \beta} x^{\alpha} x^{\beta}, \quad X \in H(M).
\end{equation}

Let $\Delta_{sb}$ be the sub-Laplacian on $(M, \theta)$, which is twice the real part of Kohn’s Laplacian with respect to the measure $dv = c_n \theta \wedge (d\theta)^{n-1}$ acting on functions on $M$. Let $\mu_1$ be the first positive eigenvalue of $\Delta_{sb}$ on $M$. The sharp lower bound for $\mu_1$ (analogue of Lichnerowicz theorem for the Riemannian case) was obtained by Greenleaf [8] for $n \geq 3$ and by Li and Luk [15] for $n = 2$. The results can be stated as follows:

**THEOREM 1.1** Let $(M, \theta)$ be a compact, integrable, $(2n+1)$-dimensional strictly pseudoconvex pseudo-Hermitian manifold in the sense of Webster. If

\begin{equation}
\text{Ric}_m(X, X) - \frac{n+1}{2} \text{Tor}(X, X) \geq k_0 h(X, X) \tag{1.5}
\end{equation}

for all $X \in H_m(M)$ and $m \in M$, then $\mu_1 \geq \frac{n}{n+1} k_0$. Here $k_0$ is some positive number.

For the case $n = 1$, a partial result was also obtained in [15], where the authors imposed an extra condition involving a derivative of the torsion so that Theorem 1.1 holds. In particular, Theorem 1.1 remains true when $M$ is torsion-free. An alternative extra condition was also given by Chiu in [7], where he proved that Theorem 1.1 remains true for $n = 1$ if the Paneitz operator is positive on $M$. Whether Theorem 1.1 holds for $n = 1$ without any extra condition is still open.

The following interesting and important problem about whether one has the CR version of Obata theorem for pseudo-Hermitian manifolds has been posed by Li [12] (Problem 2.3) and by Chang and Chiu [3] (Conjecture 1.1):

**Problem 1.2** Let $(M, \theta)$ be a compact, integrable, $(2n+1)$-dimensional strictly pseudoconvex pseudo-Hermitian manifold in the sense of Webster [20] so that (1.5) holds. If $\mu_1 = \frac{n}{n+1} k_0$ then $M$ is CR equivalent to the unit sphere $S^{2n+1}$ in $\mathbb{C}^{n+1}$.

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1It was pointed out by Chang and Chiu [3] that there is an error in the calculation for Bochner formula by Greenleaf [8]. The correct formula is:

\[
\frac{1}{2} \Delta |\nabla f|^2 = \|\pi_+ D^2 f\|^2 + \|\pi_- D^2 f\|^2 + \text{Re} (\nabla f, \nabla (\Delta f)) + (\text{Ric} - (n+1)/2\text{Tor})(\nabla f, \nabla f) + i(D^2 f)(X_0, (\bar{d}f)^*)
\]

The coefficient in front of Tor, which was $n/2$, is changed to the correct one $-(n+1)/2$, and the proof in [15] is still correct with a slight change coordinate.
For the case \( n = 1 \), the above problem was answered by Chang and Chiu affirmatively under the condition that \((M, \theta)\) is torsion-free. The purpose of this paper is to generalize the theorem of Chang and Chiu for \( n = 1 \) to general dimension \( n \). We will prove the following theorem:

**THEOREM 1.3** Let \((M, \theta)\) be a compact, integrable, \((2n + 1)\)-dimensional strictly pseudoconvex pseudo-Hermitian manifold in the sense of Webster [20] so that (1.5) holds. If \( \mu_1 = \frac{n}{n+1} k_0 \) and \((M, \theta)\) is torsion-free, then \( M \) is CR equivalent to the unit sphere \( S^{2n+1} \) in \( C^{n+1} \).

Here is the idea of the proof of Theorem 1.3: It follows from the theorem of Frobenius (see [10]) that any nondegenerate CR-structure on a \((2n + 1)\)-dimensional manifold \( M \) is spherical if and only if \( M \) is isomorphic to \( S^{2n+1} \). Combining this and a theorem of Pinchuk [18] and Chern and Ji in [6], which states that a simply connected manifold \( M \) with a spherical CR-structure must be CR equivalent to \( S^{2n+1} \), it suffices to prove that \((M, \theta)\) is isometric to the sphere \( S^{2n+1}(1/\lambda) \) for some \( \lambda > 0 \). To achieve this goal, the following approach was used in [3] for \( n = 1 \). We choose a local coframe \((\theta, \theta^\alpha, \theta^\overline{\alpha})\) (here \( \theta^\alpha \) may be different from the one in (1.1)) so that

\[
(1.6) \quad d\theta = i\theta^\alpha \wedge \theta^{\overline{\alpha}}
\]

For any \( \lambda \in (0, \infty) \), we let

\[
(1.7) \quad h^\lambda = \text{Re} \theta^\alpha \otimes \theta^{\overline{\alpha}} + \frac{1}{\lambda^2} \theta \otimes \theta
\]

Then \((M, h^\lambda)\) can be viewed as a compact Riemannian manifold without boundary. Let \( \Delta^\lambda \) be the Laplace-Beltrami operator on \((M, h^\lambda)\). To prove \((M, h^\lambda)\) is isomorphic to \( S^{2n+1}(1/\lambda) \) for some \( \lambda > 0 \) under the conditions of Theorem 1.3, it suffices to prove that the conditions of the Obata theorem hold on \((M, h^\lambda)\). The author will prove that this idea also works for general \( n \). The difficulty and complication lie in finding the exact formula relating the pseudo Ricci curvature \( R_{\alpha\overline{\beta}} \) \((1 \leq \alpha, \beta \leq n)\) and the Riemannian Ricci curvature \( R^\lambda_{ij} \) \((1 \leq i, j \leq 2n+1)\). Especially, this is very delicate when \( n > 1 \).

The paper is organized as follows: In Sections 2 and 3, we will derive a formula of the relations between the pseudo Ricci curvature \( R_{\alpha\overline{\beta}} \) and the Riemannian Ricci curvature \( R^\lambda_{ij} \). In Section 4, we will prove that all conditions of the Obata theorem hold on \((M, h^\lambda)\) under the condition of Theorem 1.3, and as a consequence, \((M, h^\lambda)\) is isomorphic to \( S^{2n+1}(1/\lambda) \) and \((M, \theta)\) is CR equivalent to \( S^{2n+1} \).

### 2 On the Ricci curvatures \( R_{\alpha\overline{\beta}} \) and \( R^\lambda_{ij} \)

Let \((M, \theta)\) be a \((2n + 1)\)-dimensional, strictly pseudoconvex pseudo-Hermitian CR manifold in the sense of Webster [20]. The local coframe \((\theta, \theta^\alpha, \theta^\overline{\alpha})\) is chosen
so that
\begin{equation}
(2.1) \quad d\theta = i\theta^\alpha \wedge \theta^\beta
\end{equation}
and the connection 1-forms $\omega^\beta_\beta$ are unique (in the sense of Webster [20]) and skew-Hermitian:
\begin{equation}
(2.2) \quad \omega^\beta_\beta = -\omega^{\overline{\beta}}_{\overline{\beta}}.
\end{equation}
Furthermore,
\begin{equation}
(2.3) \quad d\theta^\alpha = \sum_{\beta=1}^n \theta^\beta \wedge \omega^\alpha_\beta + \theta \wedge \tau^\alpha, \quad \tau^\alpha = \sum_{\gamma=1}^n A_{\gamma\alpha} \theta^\gamma
\end{equation}
where
\begin{equation}
(2.4) \quad \overline{A_{\alpha\beta}} = A_{\overline{\beta}\overline{\alpha}} \quad \text{and} \quad A = \left[A_{\alpha\beta}\right]
\end{equation}
is an $n \times n$ symmetric matrix whose entries are complex-valued functions.

The pseudo curvature 2-forms $\Omega^\beta_\beta$ on $(M, \theta)$ are defined as
\begin{equation}
(2.5) \quad \Omega^\beta_\beta = d\omega^\beta_\beta - \omega^\gamma_\beta \wedge \omega^\alpha_\gamma - i\theta^\beta \wedge \tau^\alpha + i\overline{\tau}^\beta \wedge \theta^\alpha
\end{equation}
and
\begin{equation}
(2.6) \quad \overline{\Omega}^\beta_\beta = R_{\beta\alpha\rho\sigma} \theta^\rho \wedge \theta^\sigma + \lambda_{\beta\alpha} \wedge \theta^\sigma,
\end{equation}
where $\lambda_{\beta\alpha}$ are 1-forms satisfying (see page 30 in [20])
\begin{equation}
(2.7) \quad \lambda_{\beta\alpha} = W_{\beta\alpha\gamma} \theta^\gamma - W_{\beta\alpha\gamma} \theta^\overline{\gamma}, \quad W_{\beta\alpha\gamma} = W_{\gamma\beta\alpha} \quad \text{and} \quad \overline{W_{\beta\alpha\gamma}} = W_{\overline{\alpha}\overline{\beta}\overline{\gamma}},
\end{equation}
and the pseudo curvature tensor components $R_{\beta\alpha\rho\sigma}$ satisfy
\begin{equation}
(2.8) \quad R_{\beta\alpha\rho\sigma} = R_{\rho\beta\alpha\sigma} = R_{\rho\sigma\beta\alpha}.
\end{equation}
Using the fact that $dd\theta^\alpha = 0$, Webster [[(1.28) and (1.29), [20]] shows that
\begin{equation}
0 = \theta^\beta \wedge \Omega^\beta_\beta + \theta \wedge (d\tau^\alpha - \tau^\beta \wedge \omega^\alpha_\beta) = 0 \quad \text{(mod $(2,1)$-form and $(1,2)$-form)}.
\end{equation}
Combining this with (2.5)—(2.8), one has
\begin{align*}
0 &= \theta^\beta \wedge \lambda_{\beta\alpha} \wedge \theta + \theta \wedge (d\tau^\alpha - \tau^\beta \wedge \omega^\alpha_\beta) \\
&= -W_{\beta\alpha\rho} \theta^\beta \wedge \theta^\rho \wedge \theta + \theta \wedge (d\tau^\alpha - \tau^\beta \wedge \omega^\alpha_\beta).
\end{align*}
Therefore
\begin{equation}
(2.9) \quad W_{\beta\alpha\rho} \theta^\beta \wedge \theta^\rho = (d\tau^\alpha - \tau^\beta \wedge \omega^\alpha_\beta) \quad \text{(mod $\theta$)}.
\end{equation}
In particular,
\[ \lambda_{\beta\tau} = 0 \quad \text{if} \quad \tau^\alpha = 0 \quad \text{for all} \quad 1 \leq \alpha \leq n. \]

The Webster pseudo Ricci curvature \( R_{\alpha\beta} \) and pseudo scalar curvature \( R \) are defined as follows:
\[ R_{\alpha\beta} = R_{\alpha\beta\rho} = \sum_{\rho=1}^{n} R_{\alpha\beta\rho\tau}, \quad R = R_{\alpha}^\alpha = \sum_{\alpha=1}^{n} R_{\alpha\tau}. \]

By (2.5)–(2.7), one has
\[ d\omega^\alpha_{\beta} - \omega^\gamma_{\beta} \wedge \omega^\alpha_{\gamma} = R^\beta_{\alpha \rho} \wedge \theta^\rho + \Lambda^\beta_{\alpha \tau}, \]
where
\[ \Lambda^\beta_{\alpha \tau} = \lambda^\beta_{\alpha \tau} \wedge \theta + i\theta^\beta \wedge \tau^\alpha - i\tau^\beta \wedge \theta^\alpha. \]

Let
\[ w^\alpha = \text{Re}(\theta^\alpha), \quad w^{n+\alpha} = \text{Im}(\theta^\alpha), \quad w^{2n+1} = \lambda^{-1} \theta, \quad 1 \leq \alpha \leq n. \]

Then the Riemannian metric defined by (1.7) on \( M \) can be written as follows:
\[ h^\lambda = \sum_{j=1}^{2n+1} w^j \otimes w^j. \]

**Proposition 2.1** For \( 1 \leq \alpha, k \leq n \), let
\[ w^\alpha_k = w^{n+\alpha}_{n+k} = \text{Re}(\omega^\alpha_k), \quad -w^\alpha_{n+k} = w^{n+k}_\alpha = \text{Im}(\omega^\alpha_k) + \frac{1}{\lambda} \delta^\alpha_k \theta, \]
\[ -w^{2n+1}_\alpha = w^{2n+1}_{2n+1} = \lambda \text{Re}(\tau^\alpha) - \frac{1}{\lambda} w^{n+\alpha} \]
and
\[ -w^{2n+1}_{n+\alpha} = w^{2n+1}_{2n+1} = \lambda \text{Im}(\tau^\alpha) + \frac{1}{\lambda} w^\alpha, \quad w^{2n+1}_{2n+1} = 0. \]

Then the following structural equations for \( (M, h^\lambda) \) hold:
\[ dw^i = \sum_{k=1}^{2n+1} w^k \wedge w^i_k, \quad w^i_i + w^j_j = 0, \quad 1 \leq i, j \leq 2n+1. \]
Proof. By the definitions of $w^i_j$ from (2.16)—(2.18), one can easily see that $w^i_j + w^j_i = 0$ for $1 \leq i, j \leq 2n + 1$. We need only to verify that the first part of (2.19) holds. Since

$$\theta^\alpha = w^\alpha + i w^{n+\alpha} \quad \text{and} \quad d\theta^\alpha = \sum_{k=1}^{n} \theta^k \wedge \omega^\alpha_k + \theta \wedge \tau^\alpha$$

for $1 \leq \alpha \leq n$, one has

$$dw^\alpha = \sum_{k=1}^{n} \left[ w^k \wedge \text{Re} (\omega^\alpha_k) - w^{n+k} \wedge \text{Im} (\omega^\alpha_k) \right] + \theta \wedge \text{Re} (\tau^\alpha)$$

$$= \sum_{k=1}^{n} \left[ w^k \wedge \text{Re} (\omega^\alpha_k) - w^{n+k} \wedge \left( \text{Im} (\omega^\alpha_k) + \frac{\delta_{nk}}{\lambda^2} \theta \right) \right]$$

$$+ \theta \wedge \left( \text{Re} (\tau^\alpha) - \frac{w^{n+\alpha}}{\lambda^2} \right)$$

$$= \sum_{k=1}^{n} \left( w^k \wedge w^\alpha_k + w^{n+k} \wedge w^{\alpha}_{n+k} \right) + w^{2n+1} \wedge w^{\alpha}_{2n+1} \wedge$$

and

$$dw^{n+\alpha} = \sum_{k=1}^{n} \left[ w^k \wedge \text{Im} (\omega^\alpha_k) + w^{n+k} \wedge \text{Re} (\omega^\alpha_k) \right] + \theta \wedge \text{Im} (\tau^\alpha)$$

$$= \sum_{k=1}^{n} \left[ w^k \wedge \left( \text{Im} (\omega^\alpha_k) + \frac{\delta_{nk}}{\lambda^2} \theta \right) + w^{n+k} \wedge \text{Re} (\omega^\alpha_k) \right]$$

$$+ \theta \wedge \left( \text{Im} (\tau^\alpha) + \frac{w^\alpha}{\lambda^2} \right)$$

$$= \sum_{k=1}^{n} \left( w^k \wedge w^{n+\alpha}_k + w^{n+k} \wedge w^{n+\alpha}_{n+k} \right) + w^{2n+1} \wedge w^{n+\alpha}_{2n+1} \wedge$$

Notice that $\text{Im} (A_{\alpha\gamma}) = -\text{Im} (A_{\alpha\gamma})$ and $A_{\alpha\gamma} = A_{\gamma\alpha}$, one has

$$\text{Re} (\tau^\alpha) = \sum_{\gamma=1}^{n} \text{Re} (A_{\alpha\gamma}) w^\gamma - \sum_{\gamma=1}^{n} \text{Im} (A_{\alpha\gamma}) w^{n+\gamma},$$

$$\text{Im} (\tau^\alpha) = - \sum_{\gamma=1}^{n} \text{Im} (A_{\alpha\gamma}) w^\gamma - \sum_{\gamma=1}^{n} \text{Re} (A_{\alpha\gamma}) w^{n+\gamma},$$

and

$$\sum_{\alpha=1}^{n} w^\alpha \wedge \sum_{\gamma=1}^{n} A_{\alpha\gamma} w^\gamma = \sum_{\alpha=1}^{n} w^{n+\alpha} \wedge \sum_{\gamma=1}^{n} A_{\alpha\gamma} w^{n+\gamma} = 0.$$
Therefore,

\[
dw^{2n+1} = \frac{i}{\lambda} \sum_{\alpha=1}^{n} \theta^\alpha \wedge \theta^\sigma
\]

\[
= \frac{2}{\lambda} \sum_{\alpha=1}^{n} w^\alpha \wedge w^{n+\alpha}
\]

\[
= \frac{1}{\lambda} \sum_{\alpha=1}^{n} w^\alpha \wedge w^{n+\alpha} - \frac{1}{\lambda} \sum_{\alpha=1}^{n} w^{n+\alpha} \wedge w^\alpha
\]

\[
= \sum_{\alpha=1}^{n} w^\alpha \wedge \left( \frac{1}{\lambda} w^{n+\alpha} - \lambda \sum_{\gamma=1}^{n} \text{Re} (A_{\alpha\gamma}) w^n + \lambda \sum_{\gamma=1}^{n} \text{Im} (A_{\alpha\gamma}) w^{n+\gamma} \right)
\]

\[
- \sum_{\alpha=1}^{n} w^{n+\alpha} \wedge \left( \frac{1}{\lambda} w^\alpha - \lambda \sum_{\gamma=1}^{n} \text{Re} (A_{\alpha\gamma}) w^n + \lambda \sum_{\gamma=1}^{n} \text{Im} (A_{\alpha\gamma}) w^\gamma \right)
\]

\[
= \sum_{\alpha=1}^{n} w^\alpha \wedge \left( \frac{w^{n+\alpha}}{\lambda} \right) - \sum_{\alpha=1}^{n} w^{n+\alpha} \wedge \left( \frac{w^\alpha}{\lambda} \right)
\]

\[
= \sum_{\alpha=1}^{n} \left( w^\alpha \wedge w_2^{2n+1} + w^{n+\alpha} \wedge w_n^{2n+1} \right) + w_2^{2n+1} \wedge w_n^{2n+1}.
\]

Therefore, (2.19) holds and the proof of the proposition is complete. 

On the Riemannian manifold \((M, h^\lambda)\), one has the structural equations (2.19) and the following:

\[
dw^i - \sum_{k=1}^{2n+1} w_k^i \wedge w_\ell = \frac{1}{2} \sum_{k,\ell=1}^{2n+1} R^\lambda_{ijk\ell} w^k \wedge w_\ell, \quad 1 \leq i, j \leq 2n + 1,
\]

where \(R^\lambda_{ijk\ell}\) is the Riemannian curvature tensor. Furthermore, the Riemannian Ricci curvature for \((M, h^\lambda)\) is given as follows:

\[
R^\lambda_{jk} = (h^\lambda)^{\ell\delta} R^\lambda_{j\ell k\delta} = \sum_{\ell=1}^{2n+1} R_{j\ell k\ell}^\lambda + R_{2n+1+j 2n+1}^\lambda.
\]

Next we calculate the curvatures \(R^\lambda_{ij}\) in terms of \(R_{\alpha\bar{\beta}}\). For simplicity, we introduce the following two-forms:

\[
\frac{1}{2} F_{k,\alpha}(\lambda, \tau) = \text{Re} (\tau^k \wedge (-\lambda^2 \text{Re} (\tau^\alpha) + w^{n+\alpha}) + w^{n+k} \wedge \text{Re} (\tau^\alpha) - \text{Re} (A_{k\bar{\alpha}}))
\]

\[
\frac{1}{2} F_{k,n+\alpha}(\lambda, \tau) = -\text{Re} (\tau^k \wedge (\lambda^2 \text{Im} (\tau^\alpha) + w^\alpha) + w^{n+k} \wedge \text{Im} (\tau^\alpha) - \text{Im} (A_{k\bar{\alpha}}))
\]

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and

\[
\frac{1}{2} F_{n+k,n+\alpha}(\lambda, \tau) = -\text{Im} (\tau^h) \wedge (\lambda^2 \text{Im} (\tau^\alpha) + w^\alpha) - w^k \wedge \text{Im} (\tau^\alpha) - \text{Re} (\Lambda_{k,\tau})
\]

In particular, by (2.10) and (2.13)

\[
F_{k,\alpha}(\lambda, \tau) = F_{n+k,n+\alpha}(\lambda, \tau) = F_{n+k,n+\alpha}(\lambda, \tau) = 0 \text{ if } \tau = (\tau^1, \ldots, \tau^n) = 0.
\]

Let us first build up the relation between \(R_{ijpq}^\lambda w^p \wedge w^q\) and \(R_{\alpha\beta\gamma\delta}^\theta \wedge \theta^\gamma\).

**Proposition 2.2** For \(1 \leq \alpha, k \leq n\), we have

\[
2n+1 \sum_{p,q=1}^{2n+1} R_{k\alpha pq}^\lambda w^p \wedge w^q = 2\text{Re} \left( \sum_{\rho,\sigma=1}^{n} R_{k\rho\sigma \rho}^\lambda \theta^\rho \wedge \theta^\sigma \right) + \frac{2}{\lambda^2} w^{n+k} \wedge w^{n+\alpha} - F_{k,\alpha}(\lambda, \tau),
\]

and

\[
2n+1 \sum_{p,q=1}^{2n+1} R_{n+k n+\alpha pq}^\lambda w^p \wedge w^q = 2\text{Im} \left( \sum_{\rho,\sigma=1}^{n} R_{k\rho\sigma \rho}^\lambda \theta^\rho \wedge \theta^\sigma \right) + \frac{2}{\lambda^2} w^{n+k} \wedge w^{n+\alpha} - F_{n+k,n+\alpha}(\lambda, \tau).
\]

**Proof.** By the definitions of \(w^\ell\) in (2.14) and \(w^\ell_i\) given by (2.16), and the fact that \(\text{Im} (\omega^\ell_{\alpha}) = \text{Im} (\omega^\ell_{\beta})\), one has

\[
2n \sum_{\ell=1}^{2n} w^{\ell}_{k} \wedge w^{\ell}_{\alpha} = \sum_{\ell=1}^{n} \text{Re} (\omega^\ell_{k}) \wedge \text{Re} (\omega^\ell_{\alpha}) - \sum_{\ell=1}^{n} \left( \text{Im} (\omega^\ell_{k}) + \frac{\delta_{\ell k}}{\lambda^2} \theta \right) \wedge \left( \text{Im} (\omega^\ell_{\alpha}) + \frac{\delta_{\ell \alpha}}{\lambda^2} \theta \right)
\]

\[
= \sum_{\ell=1}^{n} \text{Re} (\omega^\ell_{k}) \wedge \text{Re} (\omega^\ell_{\alpha}) - \sum_{\ell=1}^{n} \text{Im} (\omega^\ell_{k}) \wedge \text{Im} (\omega^\ell_{\alpha})
\]

\[
= \text{Re} \sum_{\ell=1}^{n} (\omega^\ell_{k} \wedge \omega^\ell_{\alpha})
\]

By (2.14), (2.16) and similar computations, one has

\[
2n \sum_{\ell=1}^{2n} w^{\ell}_{k} \wedge w^{n+\alpha}_{\ell} = \text{Im} \sum_{\ell=1}^{n} (\omega^\ell_{k} \wedge \omega^\ell_{\alpha})
\]

(2.30)
and
\[
(2.31) \quad \sum_{\ell=1}^{2n} w_{n+k}^\ell \wedge w_{\ell}^{\alpha} = \text{Re} \sum_{\ell=1}^{n} (\omega_k^\ell \wedge \omega_{\ell}^\alpha).
\]

By (2.17) and (2.25), one has
\[
(2.32) \quad w_{2n+1}^k \wedge w_{2n+1}^\alpha
= -\left( \text{Re} (\tau^k) - \frac{1}{\lambda} w_{2n+1}^k \right) \wedge \left( \frac{1}{\lambda} \text{Re} (\tau^\alpha) - \frac{1}{\lambda} w_{2n+1}^\alpha \right)
\]
\[
= - \frac{1}{\lambda^2} w_{2n+1}^k \wedge w_{2n+1}^\alpha + \text{Re} (\tau^k) \wedge \left( \lambda^2 \text{Re} (\tau^\alpha) + w_{2n+1}^\alpha \right)
\]
\[
+ w_{2n+1}^k \wedge \text{Re} (\tau^\alpha)
\]
\[
= - \frac{1}{\lambda^2} w_{2n+1}^k \wedge w_{2n+1}^\alpha + \frac{1}{2} F_{k, \alpha}(\lambda, \tau) + \text{Re} (\Lambda k \pi),
\]
by (2.17) and (2.26), one has
\[
(2.33) \quad w_{2n+1}^k \wedge w_{2n+1}^\alpha
= -\left( \lambda \text{Im} (\tau^k) - \frac{1}{\lambda} w_{2n+1}^k \right) \wedge \left( \lambda^2 \text{Im} (\tau^\alpha) + w_{2n+1}^\alpha \right)
\]
\[
= \frac{1}{\lambda^2} w_{2n+1}^k \wedge w_{2n+1}^\alpha - \text{Re} (\tau^k) \wedge \left( \lambda^2 \text{Im} (\tau^\alpha) + w_{2n+1}^\alpha \right) + w_{2n+1}^k \wedge \text{Im} (\tau^\alpha)
\]
\[
= \frac{1}{\lambda^2} w_{2n+1}^k \wedge w_{2n+1}^\alpha + \frac{1}{2} F_{k, n+\alpha}(\lambda, \tau) + \text{Im} (\Lambda k \pi)
\]
and by (2.17) and (2.27), one has
\[
(2.34) \quad w_{n+k}^\alpha \wedge w_{2n+1}^\alpha
= -\left( \lambda \text{Im} (\tau^k) + \frac{1}{\lambda} w_{n+k}^k \right) \wedge \left( \lambda \text{Im} (\tau^\alpha) + \frac{1}{\lambda} w_{n+k}^\alpha \right)
\]
\[
= - \frac{1}{\lambda^2} w_{n+k}^k \wedge w_{n+k}^\alpha - \text{Im} (\tau^k) \wedge \left( \lambda^2 \text{Im} (\tau^\alpha) + w_{n+k}^\alpha \right) - w_{n+k}^k \wedge \text{Im} (\tau^\alpha)
\]
\[
= - \frac{1}{\lambda^2} w_{n+k}^k \wedge w_{n+k}^\alpha + \frac{1}{2} F_{n+k, n+\alpha}(\lambda, \tau) + \text{Re} (\Lambda k \pi).
\]

For $1 \leq k, \alpha \leq n$, by (2.23), (2.29), (2.32) and (2.12), one has
\[
(2.35) \quad \sum_{p,q=1}^{2n+1} R_{k, \alpha pq}^\lambda w^p \wedge w^q
= 2 \left( dw_{k}^\alpha - \sum_{\ell=1}^{2n+1} w_{k}^\ell \wedge w_{\ell}^\alpha \right)
\]

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\[
\begin{align*}
&= 2\text{Re} \left( d\omega_k^\alpha - \sum_{\ell=1}^n \omega_k^\ell \wedge \omega_k^\alpha \right) + \frac{2}{\lambda^2} w^{n+k} \wedge w^{n+\alpha} - F_{k,\alpha}(\lambda, \tau) - 2\text{Re} (\Lambda_{k\alpha}) \\
&= 2\text{Re} \left( \sum_{\rho,\sigma=1}^n R_{k\alpha\rho\sigma} \theta^\rho \wedge \theta^\sigma \right) + \frac{2}{\lambda^2} w^{n+k} \wedge w^{n+\alpha} - F_{k,\alpha}(\lambda, \tau)
\end{align*}
\]
and by (2.23), (2.16), (2.30), (2.33), (2.12) and (2.13),

\[
(2.36) \quad \sum_{p,q=1}^{2n+1} R_{k,n+\alpha}^{\lambda p q} w^p \wedge w^q
\]

\[
= 2 \left( dw^{n+\alpha} - \sum_{p=1}^{2n+1} w_k^p \wedge w_p^{n+\alpha} \right)
\]

\[
= 2\text{Im} \left( d\omega_k^\alpha - \sum_{\ell=1}^n \omega_k^\ell \wedge \omega_k^\alpha \right)
\]

\[
+ \frac{2}{\lambda^2} \delta_{k\alpha} d\theta - \frac{2}{\lambda^2} w^{n+k} \wedge w^{\alpha} - F_{k,n+\alpha}(\lambda, \tau) - 2\text{Im} (\Lambda_{k\alpha})
\]

\[
= 2\text{Im} \left( \sum_{\rho,\sigma=1}^n R_{k\alpha\rho\sigma} \theta^\rho \wedge \theta^\sigma \right) + \frac{2}{\lambda^2} \delta_{k\alpha} d\theta - \frac{2}{\lambda^2} w^{n+k} \wedge w^{\alpha} - F_{k,n+\alpha}(\lambda, \tau).
\]

Finally, for \(1 \leq k, \alpha \leq n\), by (2.23), (2.31), (2.34), (2.12) and (2.13), we have

\[
(2.37) \quad \sum_{p,q=1}^{2n+1} R_{n+k,n+\alpha}^{\lambda p q} w^p \wedge w^q
\]

\[
= 2 \left( dw^{n+\alpha} - \sum_{p=1}^{2n+1} w_k^p \wedge w_p^{n+\alpha} \right)
\]

\[
= 2\text{Re} \left( d\omega_k^\alpha - \sum_{\ell=1}^n \omega_k^\ell \wedge \omega_k^\alpha \right) + \frac{2}{\lambda^2} w^k \wedge w^{\alpha} - F_{n+k,n+\alpha}(\lambda, \tau) - 2\text{Re} (\Lambda_{k\alpha})
\]

\[
= 2\text{Re} \left( \sum_{\rho,\sigma=1}^n R_{k\alpha\rho\sigma} \theta^\rho \wedge \theta^\sigma \right) + \frac{2}{\lambda^2} w^k \wedge w^{\alpha} - F_{n+k,n+\alpha}(\lambda, \tau).
\]

Therefore, by (3.35)–(3.37), the proof of the proposition is complete. \(\Box\)

In order to compare \(R_{ijk}^\lambda\) with \(R_{\alpha\beta\gamma\delta}\), we write \(R_{k\alpha\rho\sigma} \theta^\rho \wedge \theta^\sigma\) in terms of \(w^j\). Notice that

\[
(2.38) \quad \sum_{\rho,\sigma=1}^n R_{k\alpha\rho\sigma} \theta^\rho \wedge \theta^\sigma
\]

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\[
\sum_{\rho, \sigma = 1}^n R_{k\pi \rho \sigma} \left[ (w^\rho \wedge w^\sigma + w^{n+\rho} \wedge w^{n+\sigma}) + i(-w^\rho \wedge w^{n+\sigma} + w^{n+\rho} \wedge w^\sigma) \right]
\]

\[
= \sum_{\rho, \sigma = 1}^n \left[ \text{Re} (R_{k\pi \rho \sigma})(w^\rho \wedge w^\sigma + w^{n+\rho} \wedge w^{n+\sigma}) - \text{Im} (R_{k\pi \rho \sigma})(-w^\rho \wedge w^{n+\sigma} + w^{n+\rho} \wedge w^\sigma) \right] + i \sum_{\rho, \sigma = 1}^n \left[ \text{Im} (R_{k\pi \rho \sigma})(w^\rho \wedge w^\sigma + w^{n+\rho} \wedge w^{n+\sigma}) + \text{Re} (R_{k\pi \rho \sigma})(-w^\rho \wedge w^{n+\sigma} + w^{n+\rho} \wedge w^\sigma) \right]
\]

\[
= \sum_{\rho, \sigma = 1}^n \frac{1}{2} \left( \text{Re} (R_{k\pi \rho \sigma}) - \text{Re} (R_{k\pi \sigma \rho}) \right) (w^\rho \wedge w^\sigma + w^{n+\rho} \wedge w^{n+\sigma}) + \sum_{\rho, \sigma = 1}^n \left( \text{Im} (R_{k\pi \rho \sigma}) + \text{Im} (R_{k\pi \sigma \rho}) \right) w^\rho \wedge w^{n+\sigma} + \frac{i}{2} \sum_{\rho, \sigma = 1}^n \left( \text{Im} (R_{k\pi \rho \sigma}) - \text{Im} (R_{k\pi \sigma \rho}) \right) (w^\rho \wedge w^\sigma + w^{n+\rho} \wedge w^{n+\sigma}) - i \sum_{\rho, \sigma = 1}^n \left( \text{Re} (R_{k\pi \rho \sigma}) + \text{Re} (R_{k\pi \sigma \rho}) \right) w^\rho \wedge w^{n+\sigma}.
\]

For \(1 \leq i, j, p, q \leq 2n + 1\), we write

\[
(2.39) \quad F_{i,j}(\lambda, \tau) := \sum_{p, q = 1}^{2n+1} E_{ijpq}(\lambda) w^p \wedge w^q \quad \text{with} \quad E_{ijpq} = -E_{jipq} = E_{jiqp}.
\]

Then by Proposition 2.2 with (2.35), (2.38) and (2.39), for \(1 \leq k, \alpha \leq n\), we have

\[
(2.40) \quad \sum_{p, q = 1}^{2n+1} \frac{\lambda^k}{\lambda^2} w^p \wedge w^q = \sum_{\rho, \sigma = 1}^n \left[ \text{Re} (R_{k\pi \rho \sigma}) - \text{Re} (R_{k\pi \sigma \rho}) \right] (w^\rho \wedge w^\sigma + w^{n+\rho} \wedge w^{n+\sigma}) + \sum_{\rho, \sigma = 1}^n \left( \text{Im} (R_{k\pi \rho \sigma}) + \text{Im} (R_{k\pi \sigma \rho}) \right) w^\rho \wedge w^{n+\sigma} + \frac{2}{\lambda^2} w^{n+k} \wedge w^{n+\alpha} - F_{k,\alpha}(\lambda, \tau).
\]
For $1 \leq k, \alpha, p, q \leq n$, (2.40) gives

(2.40.1) \[ R^\lambda_{k, p, q} = \text{Re} \left( R_{k, p, q} - R_{k, q, p} \right) + E_{k, p, q}, \]

(2.40.2) \[ R^\lambda_{k, p, n+q} = \text{Im} \left( R_{k, p, q} + R_{k, q, p} \right) + E_{k, p, n+q} \]

and

(2.40.3) \[ R^\lambda_{k, \alpha, n+p, n+q} = \text{Re} \left( R_{k, \alpha, q} - R_{k, \alpha, p} \right) + \frac{\delta_{pk} \delta_{qa} - \delta_{pa} \delta_{kq}}{\lambda^2} + E_{k, \alpha, n+p, n+q}. \]

By (2.36), (2.38) and (2.39), one has

(2.41) \[ \sum_{p, q=1}^{2n+1} R^\lambda_{k, p, q} w^p \wedge w^q = \sum_{p, q=1}^{n} \left( \text{Re} \left( R_{k, p, q} \right) - \text{Re} \left( R_{k, q, p} \right) \right) \left( w^p \wedge w^q + w^{n+p} \wedge w^{n+q} \right) + 2 \sum_{\rho, \sigma=1}^{n} \left( \text{Im} \left( R_{k, \rho, \sigma} \right) + \text{Im} \left( R_{k, \sigma, \rho} \right) \right) w^\rho \wedge w^\sigma + 2 \frac{\delta_{pk} \delta_{qa} - \delta_{pa} \delta_{kq}}{\lambda^2} + E_{k, p, q}. \]

Thus, for $1 \leq k, \alpha, p, q \leq n$, one has

(2.41.1) \[ R^\lambda_{k, n+\alpha, p, q} = \text{Im} \left( R_{k, p, q} - R_{k, q, p} \right) + E_{k, n+\alpha, p, q}, \]

(2.41.2) \[ R^\lambda_{k, n+\alpha, n+p, n+q} = \text{Im} \left( R_{k, p, q} - R_{k, q, p} \right) + E_{k, n+\alpha, n+p, n+q}, \]

(2.41.3) \[ R^\lambda_{k, n+\alpha, n+p, n+q} = -\text{Re} \left( R_{k, p, q} + R_{k, q, p} \right) + 2 \delta_{k\alpha} \frac{\delta_{pq}}{\lambda^2} + \frac{\delta_{p\alpha}}{\lambda^2} \delta_{kq} + E_{k, n+\alpha, n+p, n+q}, \]

and

(2.41.4) \[ R^\lambda_{k, n+\alpha, n+p, n+q} = \text{Re} \left( R_{k, p, q} + R_{k, q, p} \right) - 2 \delta_{k\alpha} \frac{\delta_{pq}}{\lambda^2} - \frac{\delta_{p\alpha}}{\lambda^2} \delta_{kq} + E_{k, n+\alpha, n+p, n+q}. \]

Furthermore, by (2.37) and (2.38)

(2.42) \[ \sum_{p, q=1}^{2n+1} R^\lambda_{k, n+k, p, q} w^p \wedge w^q = \sum_{p, q=1}^{n} \left( \text{Re} \left( R_{k, p, q} \right) - \text{Re} \left( R_{k, q, p} \right) \right) \left( w^p \wedge w^q + w^{n+p} \wedge w^{n+q} \right) + 2 \sum_{\rho, \sigma=1}^{n} \left( \text{Im} \left( R_{k, \rho, \sigma} \right) + \text{Im} \left( R_{k, \sigma, \rho} \right) \right) w^\rho \wedge w^\sigma + \frac{2}{\lambda^2} w^k \wedge w^{\alpha} - F_{n+k, n+\alpha}(\lambda, \tau). \]
For $1 \leq p, q \leq n$, (2.42) and (2.39) imply that

\[
R_{n+k+n+α,n+p+q}^\lambda = \text{Re} \left( (R_{k+q} - R_{k+p}) + E_{n+k+n+α,n+p+q} \right)
\]

and

\[
R_{n+k+n+α,p+q}^\lambda = \text{Re} \left( (R_{k+q} - R_{k+p}) + \frac{δ_{pk}δ_{aq} - δ_{po}δ_{kq}}{λ^2} + E_{n+k+n+α,p+q} \right)
\]

and

\[
R_{n+k+n+α,p+q}^\lambda = \text{Im} \left( (R_{k+q} + R_{k+p}) + E_{n+k+n+α,p+q} \right).
\]

Next we will derive a formula for $R_{2n+1\alpha,pq}^\lambda$ in terms of $R_{k+q}^\lambda$ and $τ^α$. By (2.17), one has

\[
dw_{2n+1}^α = \frac{1}{λ} \text{Im} dθ^α + λ \text{Re} dτ^α
\]

Using the fact that $\text{Im}(ω^α_β) = \text{Im}(ω^α_β)$ and (2.16)–(2.18), one has

\[
\sum_{ℓ=1}^{2n} w_{2n+1}^α \wedge w_ℓ^α = \sum_{ℓ=1}^n w_{2n+1}^α \wedge w_ℓ^α + \sum_{ℓ=1}^n w_{2n+1}^α \wedge w_ℓ^α
\]

\[
= \sum_{ℓ=1}^n \left[ λ \text{Re}(τ^β) - λ^{-1} w^{n+β} \wedge (\text{Im}(ω_β^α) - λ^{-2} δ_{nα,δ_{nα}}θ) \right] \wedge (\text{Im}(ω_β^α) - λ^{-2} δ_{nα,δ_{nα}}θ)
\]

\[
- \frac{1}{λ} \sum_{ℓ=1}^n w^{n+β} \wedge (\text{Im}(ω_β^α) - λ^{-2} δ_{nα,δ_{nα}}θ) \wedge (\text{Im}(ω_β^α) - λ^{-2} δ_{nα,δ_{nα}}θ)
\]

Combining the above two identities and (2.23), one has

\[
R_{2n+1\alpha,pq}^\lambda w^p \wedge w^q = 2 \left( dw_{2n+1}^α - \sum_{ℓ=1}^{2n} w_{2n+1}^α \wedge w_ℓ^α \right)
\]
\[ E_{2n+1, \alpha p q} = -E_{\alpha 2n+1 pq} = E_{\alpha 2n+1 qp} \text{ are chosen so that} \]

\[
\sum_{p,q=1}^{2n+1} E_{2n+1, \alpha p q} w^p \wedge w^q,
\]

where \( E_{2n+1, \alpha p q} = -E_{\alpha 2n+1 pq} = E_{\alpha 2n+1 qp} \) are chosen so that

\[
(2.44) \quad \sum_{p,q=1}^{2n+1} E_{2n+1, \alpha p q} w^p \wedge w^q = 2 \lambda \text{Re} d\tau^\alpha - 2 \lambda \sum_{\ell=1}^{n} \text{Re} (\tau^\ell) \wedge \text{Re} (\omega^\alpha_{\ell}) + 2 \lambda \sum_{\ell=1}^{n} \text{Im} (\tau^\ell) \wedge \text{Im} (\omega^\alpha_{\ell})
\]

\[
+ 4 \text{Im} (\tau^\alpha) \wedge w^{2n+1}
\]

\[
= 2 \lambda \text{Re} d\tau^\alpha - 2 \lambda \sum_{\ell=1}^{n} \text{Re} (\tau^\ell \wedge \omega^\alpha_{\ell}) + 4 \text{Im} (\tau^\alpha) \wedge w^{2n+1}.
\]

Therefore, with the suitable choice of \( E_{2n+1, \alpha p q} \) and \( 1 \leq \alpha \leq n \),

\[
(2.45) \quad R^\lambda_{2n+1, \alpha p q} = \frac{\delta_{\alpha p} \delta_{q 2n+1}}{\lambda^2} + E_{2n+1, \alpha p q} = -R^\lambda_{2n+1, \alpha q p}, \quad 1 \leq p, q \leq 2n+1.
\]

Similarly, by (2.16)–(2.18)

\[
(2.46) \quad R^\lambda_{2n+1, n+\alpha p q} w^p \wedge w^q
\]

\[
= 2 \left( dw^{n+\alpha}_{2n+1} - \sum_{\ell=1}^{2n} w^\ell_{2n+1} \wedge w^n_{\ell+\alpha} \right)
\]

\[
= \frac{2}{\lambda} \text{Re} (d\theta^\alpha) + 2 \lambda \text{Im} (\tau^\alpha)
\]

\[
+ 2 \sum_{\ell=1}^{n} \left( \frac{1}{\lambda} w^{n+\ell} - \lambda \text{Re} (\tau^\ell) \right) \wedge \left( \text{Im} (\omega^\alpha_{\ell}) + \frac{1}{\lambda^2} \delta_{\alpha \ell} \theta \right)
\]

\[
- 2 \sum_{\ell=1}^{n} \left( \frac{1}{\lambda} w^\ell + \lambda \text{Im} (\tau^\ell) \right) \wedge \text{Re} (\omega^\alpha_{\ell})
\]

\[
= \frac{2}{\lambda} \sum_{\ell=1}^{n} \left( w^\ell \wedge \text{Re} (\omega^\alpha_{\ell}) - w^{n+\ell} \wedge \text{Im} (\omega^\alpha_{\ell}) \right) - 2 \text{Re} (\tau^\alpha) \wedge w^{2n+1}
\]

\[
+ 2 \lambda \text{Im} (d\tau^\alpha) + 2 \frac{\lambda^2}{\lambda^2} \sum_{\ell=1}^{n} \delta_{\alpha \ell} w^{n+\ell} \wedge w^{2n+1}
\]

\[
+ \frac{2}{\lambda} \sum_{\ell=1}^{n} w^{n+\ell} \wedge \text{Im} (\omega^\alpha_{\ell}) - 2 \lambda \sum_{\ell=1}^{n} \text{Re} (\tau^\ell) \wedge \text{Im} (\omega^\alpha_{\ell}) - 2 \text{Re} (\tau^\alpha) \wedge w^{2n+1}
\]

\[
- \frac{2}{\lambda} \sum_{\ell=1}^{n} w^\ell \wedge \text{Re} (\omega^\alpha_{\ell}) - 2 \lambda \sum_{\ell=1}^{n} \text{Im} (\tau^\ell) \wedge \text{Re} (\omega^\alpha_{\ell})
\]

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\[ \begin{align*}
= & -4\text{Re}(\tau^\alpha) \wedge w^{2n+1} + 2\lambda \text{Im}(d\tau^\alpha) - 2\lambda \sum_{\ell=1}^{n} \text{Im}(\tau^\ell \wedge \omega_\ell^\alpha) \\
+ & \frac{2}{\lambda^2} \sum_{\ell=1}^{n} \delta_\alpha \xi w^{n+\ell} \wedge w^{2n+1} \\
= & \sum_{p,q=1}^{2n+1} E_{2n+1+\alpha \, p \, q} w^p \wedge w^q + \frac{2}{\lambda^2} \sum_{\ell=1}^{n} \delta_\alpha \xi w^{n+\ell} \wedge w^{2n+1},
\end{align*} \]

where \( E_{2n+1+\alpha \, p \, q} = -E_{n+\alpha \, 2n+1 \, p \, q} = E_{\alpha \, 2n+1 \, q \, p} \) are chosen so that

\[ \sum_{p,q=1}^{2n+1} E_{2n+1+\alpha \, p \, q} w^p \wedge w^q = 2\lambda \text{Im}(d\tau^\alpha) - 4\text{Re}(\tau^\alpha) \wedge w^{2n+1} - 2\lambda \sum_{\ell=1}^{n} \text{Im}(\tau^\ell \wedge \omega_\ell^\alpha). \]

Therefore, for \( 1 \leq \alpha \leq n \) and \( 1 \leq p, q \leq 2n + 1 \), one has

\[ \sum_{p,q=1}^{2n+1} E_{2n+1+\alpha \, p \, q} w^p \wedge w^q = 2\lambda \text{Im}(d\tau^\alpha) - 4\text{Re}(\tau^\alpha) \wedge w^{2n+1} - 2\lambda \sum_{\ell=1}^{n} \text{Im}(\tau^\ell \wedge \omega_\ell^\alpha). \]

As a summary of (2.40), (2.40.1)–(2.40.3), (2.41), (2.41.1)–(2.41.4), (2.42), (2.42.1)–(2.42.3), (2.43), (2.45), one can write them as the following proposition:

**Proposition 2.3** If \((M^{2n+1}, \theta)\) is a strictly pseudoconvex pseudo-Hermitian manifold in the sense of Webster [20], then for \( 1 \leq k, \alpha, m, \ell \leq n \) and for \( 1 \leq p, q \leq 2n + 1 \)

\[ \begin{align*}
(2.49.1) & \quad R^\lambda_{k \alpha m \ell} = \text{Re}(R_{k \overline{\alpha m} \ell} - R_{k \overline{\alpha m} \ell}) + E_{k \alpha m \ell}, \\
(2.49.2) & \quad R^\lambda_{k \alpha m n + \ell} = \text{Im}(R_{k \overline{\alpha m} \ell} + R_{k \overline{\alpha m} \ell}) + E_{k \alpha m n + \ell}, \\
(2.49.3) & \quad R^\lambda_{k \alpha n \alpha m \ell} = \text{Re}(R_{k \overline{\alpha \alpha m} \ell}) + \frac{1}{\lambda^2} (\delta_{m \alpha} \delta_{\ell \alpha} - \delta_{m \alpha} \delta_{\ell k}) + E_{k \alpha n + \alpha m \ell}, \\
(2.49.4) & \quad R^\lambda_{k n + \alpha m \ell} = \text{Im}(R_{k \overline{\alpha \alpha m} \ell}) + E_{k n + \alpha m \ell}, \\
(2.49.5) & \quad R^\lambda_{k n + \alpha m + m \ell} = \text{Re}(R_{k \overline{\alpha \alpha m} \ell} + R_{k \overline{\alpha \alpha m} \ell}) + \frac{1}{\lambda^2} (\delta_{m \alpha} \delta_{\ell k} + 2\delta_{k \alpha} \delta_{m \ell}) + E_{k n + \alpha m + m \ell}, \\
(2.49.6) & \quad R^\lambda_{k n + \alpha m + n \ell} = -\text{Re}(R_{k \overline{\alpha m} \ell} + R_{k \overline{\alpha m} \ell}) + \frac{1}{\lambda^2} (\delta_{m \alpha} \delta_{\ell k} + 2\delta_{k \alpha} \delta_{m \ell}) + E_{k n + \alpha m + n \ell},
\end{align*} \]
(2.49.7) \[ R_{n+k+n+\alpha m \ell} = \text{Re}(R_{k\pi m}) - \text{Re}(R_{k\pi m^\alpha}) \]
\[ + \frac{1}{\lambda^2}(\delta_{km} \delta_{\ell\alpha} - \delta_{m\alpha} \delta_{k\ell}) + E_{n+k+n+\alpha m \ell}, \]

(2.49.8) \[ R_{n+k+n+\alpha n+\ell} = \text{Im}(R_{k\pi m}) + \text{Im}(R_{k\pi m^\alpha}) + E_{n+k+n+\alpha n+\ell}, \]

(2.49.9) \[ R_{n+k+n+m+n+\ell} = \text{Re}(R_{k\pi m}) - \text{Re}(R_{k\pi m^\alpha}) + E_{n+k+n+m+n+\ell}, \]

(2.49.10) \[ R_{2n+1 \alpha pq} = R_{2n+1 \alpha n+p q} = \frac{1}{\lambda^2} \delta_{p\alpha} \delta_{q 2n+1} + E_{2n+1 \alpha p q}, \]

(2.49.11) \[ R_{n+k+n+2n+1 p} = E_{n+k+n+2n+1 p}, \quad R_{k\alpha 2n+1 p} = E_{k\alpha 2n+1 p}, \]

and

(2.49.12) \[ R_{k\alpha 2n+1 p} = E_{k\alpha 2n+1 p}. \]

Let

(2.50.1) \[ E_{k\ell} = \sum_{p=1}^{2n+1} E_{p k\ell p}, \quad 1 \leq k, \ell \leq 2n + 1. \]

Then by (2.10), (2.13), (2.28) and (2.39),

(2.50.2) \[ E_{k\ell} = 0 \quad \text{if} \quad \tau = (\tau^1, \cdots, \tau^n) = 0, \quad 1 \leq k, \ell \leq 2n + 1. \]

In general, one has the following proposition.

**Proposition 2.4** *With the notation above, for \( 1 \leq \alpha, \ell \leq n \), one has*

(2.51.1) \[ R_{\alpha \ell} = 2\text{Re}(R_{\alpha \ell}) - \frac{2}{\lambda^2} \delta_{\alpha \ell} + E_{\alpha \ell}, \quad R_{n+\alpha \ell} = 2\text{Im}(R_{\pi \alpha \ell}) + E_{n+\alpha \ell}, \]

(2.51.2) \[ R_{2n+1 \ell} = E_{2n+1 \ell}, \quad R_{2n+1 n+\ell} = E_{2n+1 n+\ell}, \]

(2.51.3) \[ R_{n+\alpha n+\ell} = 2 \sum_{k=1}^{n} \text{Re}(R_{k\pi m}) - \frac{2}{\lambda^2} \delta_{\alpha \ell} + E_{n+\alpha n+\ell}, \]

and

(2.51.4) \[ R_{2n+1 2n+1} = \sum_{m=1}^{2n} R_{mn 2n+1} = \frac{2n}{\lambda^2} + E_{2n+1 2n+1}. \]
Proof. By (2.40.1), (2.41.3), (2.45) and (2.8), for $1 \leq k, \ell \leq n$, we have

$$R^\lambda_{\alpha \ell} = \sum_{k=1}^{2n} R^\lambda_{k \alpha k \ell k} + R^\lambda_{2n+1 \alpha 2n+1 \ell}$$

$$= \sum_{k=1}^{n} R^\lambda_{k \alpha k \ell k} - \sum_{k=1}^{n} R^\lambda_{\alpha n+k \ell n+k} + R^\lambda_{2n+1 \alpha 2n+1 \ell}$$

$$= \text{Re} \sum_{k=1}^{n} (R^\lambda_{k \alpha k \ell k} - R^\lambda_{\alpha k \ell k}) + \text{Re} \sum_{k=1}^{n} (R^\lambda_{\alpha k \ell k} + R^\lambda_{\alpha \ell k k})$$

$$- \frac{1}{\lambda^2} \sum_{k=1}^{n} (2\delta_{\alpha k} \delta_{k \ell} + \delta_{k \ell} \delta_{k \alpha}) + \frac{1}{\lambda^2} \delta_{\alpha \ell} + E_{\alpha \ell}$$

$$= 2\text{Re} (R^\lambda_{\alpha \ell}) - \frac{2}{\lambda^2} \delta_{\alpha \ell} + E_{\alpha \ell}.$$

By (2.41.1), (2.42.3), (2.48) and (2.8), one has

$$R^\lambda_{n+\alpha \ell} = \sum_{k=1}^{n} R^\lambda_{k n+\alpha \ell k} + \sum_{k=1}^{n} R^\lambda_{n+k n+\alpha \ell n+k} + R^\lambda_{2n+1 n+\alpha \ell 2n+1}$$

$$= \sum_{k=1}^{n} \left( \text{Im} (R^\lambda_{k \alpha k \ell k}) - \text{Im} (R^\lambda_{k \alpha k \ell}) \right) + \sum_{k=1}^{n} \left( \text{Im} (R^\lambda_{\alpha k \ell k}) + \text{Im} (R^\lambda_{\alpha k \ell}) \right)$$

$$+ 0 + E_{n+\alpha \ell}$$

$$= \sum_{k=1}^{n} \text{Im} (R^\lambda_{k \alpha k \ell k} + R^\lambda_{k \alpha k \ell}) + E_{n+\alpha \ell}$$

$$= 2\text{Im} (R^\lambda_{\alpha \ell}) + E_{n+\alpha \ell}.$$

Therefore, (2.51.1) is proved.

By (2.45) and (2.48), one has

$$R^\lambda_{2n+1 \ell} = \sum_{k=1}^{n} R^\lambda_{k 2n+1 \ell k} + \sum_{k=1}^{n} R^\lambda_{n+k 2n+1 \ell n+k}$$

$$= - \sum_{k=1}^{n} R^\lambda_{2n+1 \ell k k} - \sum_{k=1}^{n} R^\lambda_{2n+1 n+k \ell n+k}$$

$$= 0 + 0 + E_{2n+1 \ell}$$

and

$$R^\lambda_{2n+1 n+\ell} = \sum_{k=1}^{n} R^\lambda_{k 2n+1 n+\ell k} + \sum_{k=1}^{n} R^\lambda_{n+k 2n+1 n+\ell n+k}$$

$$= \sum_{k=1}^{n} R^\lambda_{2n+1 k k n+\ell} - \sum_{k=1}^{n} R^\lambda_{2n+1 n+k n+\ell n+k}$$

$$= 0 + 0 + E_{2n+1 n+\ell}$$

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and the proof of (2.51.2) is complete.

Next we prove (2.51.3). By (2.41.4), (2.42.1), (2.48) and (2.8), one has

\[
R_{\lambda}^n + \alpha n + \ell = \sum_{k=1}^{n} R_{\lambda}^k n + \alpha n + \ell + R_{2n+1}^\lambda n + \alpha n + \ell 2n+1 = \sum_{k=1}^{n} \left[ \Re(R_{k}^\alpha \delta_{\ell k}) - \frac{3}{\lambda^2} \delta_{k \alpha} \delta_{\ell k} + \Re(R_{k}^\alpha \delta_{\ell k}) \right] + \frac{1}{\lambda^2} \delta_{n + \alpha n + \ell} + E_{n + \alpha n + \ell} = 2 \sum_{k=1}^{n} \Re(R_{k}^\alpha \delta_{\ell k}) - \frac{2}{\lambda^2} \delta_{\ell n} + E_{n + \alpha n + \ell}
\]

and (2.51.3) follows. Finally, by (2.45) and (2.48), one has

\[
R_{2n+1}^\lambda = \sum_{k=1}^{n} R_{2n+1}^\lambda k + \sum_{k=1}^{n} R_{n + k 2n+1}^\lambda n + k = \frac{2n}{\lambda^2} + E_{2n+1} 2n+1
\]

and (2.51.4) is proved. Therefore, the proof of the proposition is complete. 

**Remark 1** If \((M, \theta)\) is torsion-free, then \(E_{i,j} = 0\) for all \(1 \leq i, j \leq 2n + 1\).

### 3 Proof of the main theorem

In this section, we prove Theorem 1.3. First, we will study the relation between the eigenvalues of the sub-Laplacian \(\Delta_{sb}\) and the eigenvalues of \(\Delta^\lambda\).

With the notations in the previous sections, we define the following covariant derivatives:

\[
f_j = X_j f, \quad f_{\alpha j} = X_j f_{\alpha} - \Gamma_{\alpha j}^\gamma f_{\gamma}, \quad f_{\bar{\alpha} j} = X_j f_{\bar{\alpha}} - \Gamma_{\bar{\alpha} j}^\gamma f_{\gamma},
\]

and

\[
f_{\beta \alpha j} = X_j f_{\beta \alpha} - \Gamma_{\beta j}^\gamma f_{\gamma \alpha} - \Gamma_{\alpha j}^\gamma f_{\beta \gamma}, \quad \text{for } \alpha, \beta \in I \cup \bar{T}, \; j \in J,
\]

where \(I = \{1, 2, \ldots, n\}, \bar{T} = \{1, \bar{2}, \ldots, \bar{n}\}\) and \(J = \{0\} \cup I \cup \bar{T}\).

The relations on the order of covariant derivatives were given by Greenleaf in [8]:

\[
f_{\beta \alpha} = f_{\alpha \beta} + i \delta_{\alpha \bar{\beta}} f_{0}, \quad \beta \in I, \; \alpha \in I \cup \bar{T},
\]

\[18\]
(3.4) \[ X^\alpha_\alpha = -X_\alpha + \sum_{\beta} \Gamma^\alpha_{\beta\beta}. \]

and

(3.5) \[ \Delta_{sb} f = -(X^\alpha_{\alpha}X_\alpha + X^*_\alpha X_\alpha)f = 2\text{Re}(\text{tr}(\pi D^2 f)) = \sum_{\alpha=1}^n \left( f_{\alpha\alpha} + f_{\alpha\alpha}^* \right). \]

Let \( \mu_1 \) be the first positive eigenvalue of \( \Delta_{sb} \) with real eigenfunction \( f \) on \( M \). Then \( \Delta_{sb} f = -\mu_1 f \) on \( M \). The following theorems were proved by Greenleaf [8] and Li and Luk [15]:

(a) If the assumption (1.5) holds, then \( \mu_1 \geq \frac{n}{n+1} k_0 \);

(b) If \( \mu_1 = \frac{n}{n+1} k_0 \) and (1.5) holds, then

(3.6) \[ \int_M \left[ \text{Ric}_m(\tilde{\nabla} f, \tilde{\nabla} f) - \frac{n+1}{2} \text{Tor}(\tilde{\nabla} f, \tilde{\nabla} f) \right] dv = k_0 \int_M |\tilde{\nabla} f|^2 dv \]

and

(3.7) \[ f_{\alpha\beta} = 0, \quad \text{on} \quad M \quad \text{for all} \quad 1 \leq \alpha, \beta \leq n. \]

A straightforward calculation shows that the following proposition holds.

**Proposition 3.1** Let \( \Delta^\lambda \) be the Laplace-Beltrami operator for \( (M, h^\lambda) \). Then

(3.8) \[ \Delta^\lambda = 2\Delta_{sb} + \lambda^2 X^2_0. \]

Let \( f \) be a real-valued non-constant function on \( M \) such that

(3.9) \[ \Delta_{sb} f(z) = -\mu_1 f(z), \quad \text{on} \quad M. \]

Then

(3.10) \[ (-\Delta^\lambda f, f) = 2\mu_1 (f, f) + \lambda^2 (X_0 f, X_0 f). \]

It was proved by Greenleaf in [8], on page 209, equation (6.2), that

\[
\int_M \left[ \left( 1 + \frac{2c}{n} \right) \|\pi_+ D^2 f\|^2 + \frac{n-2c}{n} \|\pi_- D^2 f\|^2 - \frac{4(1-c)}{n} |\text{tr}(\pi_+ D^2 f)|^2 - \frac{n-2+c}{2n} (\tilde{\Delta} f)^2 \right] dv \\
= -\int_M \left( \frac{n-2c}{n} \text{Ric} + \frac{2-2c-n}{2} \text{Tor} \right) (\tilde{\nabla} f, \tilde{\nabla} f) dv.
\]

Here \( \tilde{\Delta} = \Delta_{sb} \).
Notice that

\[(3.11) \quad \|\pi_+ D^2 f\|^2 \geq \frac{1}{n} |\text{tr}(\pi_+ D^2 f)|^2.\]

Assuming that \(2c > 0,\) one has

\[
\left(1 + \frac{2c}{n}\right)\|\pi_+ D^2 f\|^2 - \frac{4(1 - c)}{n} |\text{tr}(\pi_+ D^2 f)|^2 \\
\geq \frac{1}{n}\left(1 + \frac{2c}{n}\right) - \frac{4(1 - c)}{n} \frac{n|m|}{n} |\text{tr}(\pi_+(D^2 f))|^2 \\
= -\frac{3n + 2c(1 + 2n)}{n^2} |\text{tr}(\pi_+(D^2 f))|^2.
\]

Since

\[(3.12) \quad f_{\alpha\alpha} - f_{\alpha\alpha} = i f_0, \quad 1 \leq \alpha \leq n,
\]

we have that

\[
|\text{tr}(\pi_+ D^2 f)|^2 = |\text{Re}(\text{tr}(\pi_+ D^2 f))|^2 + |\text{Im}(\text{tr}(\pi_+ D^2 f))|^2 = \frac{1}{4}(\tilde{\Delta} f)^2 + \frac{n^2}{4}|f_0|^2
\]

and

\[
-\frac{3n + 2c(1 + 2n)}{n^2} |\text{tr}(\pi_+(D^2 f))|^2 - \frac{n - 2 + 2c}{2n} (\tilde{\Delta} f)^2 \\
= -\frac{3n + 2c(1 + 2n)}{4}|f_0|^2 - \frac{n - 2 + 2c}{4n^2} (\tilde{\Delta} f)^2 + \frac{2c - 2n^2 + n}{4n^2} (\tilde{\Delta} f)^2.
\]

Therefore, with the assumption \((f, f) = 1\) and \(\mu_1 = nk/(n + 1),\)

\[
-\frac{3n + 2c(1 + 2n)}{4} \int_M |X_0 f|^2 dv \\
\leq \frac{2n^2 - n - 2c}{4n^2} \int_M (\tilde{\Delta} f)^2 dv \\
- \frac{n - 2c}{n} \int_M \left(\text{Ric} - \frac{(n + 2c - 2)n}{2(n - 2c)} \text{Tor}\right)(\tilde{\nabla} f, \tilde{\nabla} f) dv \\
= \frac{2n^2 - n - 2c}{4n^2} \int_M (\tilde{\Delta} f)^2 dv - \frac{n - 2c}{n} \int_M \left(\text{Ric} - \frac{n + 1}{2} \text{Tor}\right)(\tilde{\nabla} f, \tilde{\nabla} f) dv \\
+ \frac{-3n + 2c(2n + 1)}{2n} \int_M \text{Tor}(\tilde{\nabla} f, \tilde{\nabla} f) dv
\]
\[
\begin{align*}
&= \frac{2n^2 - n - 2c}{4n^2} \mu_1^2 - \frac{n - 2c}{n} k \mu_1 \frac{k}{2} + \frac{-3n + 2c(n + 1)}{2n} \int_M \text{Tor}(\tilde{\nabla} f, \tilde{\nabla} f) \, dv \\
&= \mu_1^2 \left[ \frac{2n^2 - n - 2c}{4n^2} - \frac{n - 2c (n + 1)}{2n} \right] \\
&\quad + \frac{-3n + 2c(2n + 1)}{2n} \int_M \text{Tor}(\tilde{\nabla} f, \tilde{\nabla} f) \, dv \\
&= \frac{\mu_1^2}{4n^2} \left[ 2n^2 - n - 2c + (-2n + 4c)(n + 1) \right] \\
&\quad + \frac{-3n + 2c(2n + 1)}{2n} \int_M \text{Tor}(\tilde{\nabla} f, \tilde{\nabla} f) \, dv \\
&= \frac{\mu_1^2}{4n^2} \left[ -3n + 2c(2n + 1) \right] + \frac{-3n + 2c(2n + 1)}{2n} \int_M \text{Tor}(\tilde{\nabla} f, \tilde{\nabla} f) \, dv.
\end{align*}
\]

Thus, if \(2c > \frac{3n}{2n + 1}\) then
\[
\int_M |X_0 f|^2 \, dv \leq \frac{\mu_1^2}{n^2} + \frac{2}{n} \int_M \text{Tor}(\tilde{\nabla} f, \tilde{\nabla} f) \, dv.
\]

Taking \(0 < 2c < \frac{3n}{2n + 1}\), we get the reverse of the above inequality. Thus,
\[
(3.12) \quad \int_M |X_0 f|^2 \, dv = \frac{\mu_1^2}{n^2} + \frac{2}{n} \int_M \text{Tor}(\tilde{\nabla} f, \tilde{\nabla} f) \, dv.
\]

Assume that
\[
(3.13) \quad \text{Ric}(X, X) - \frac{n + 1}{2} \text{Tor}(X, X) \geq k|X|^2 \text{ with } k = \frac{n + 1}{\lambda^2}
\]

for all \(X \in H(M)\). Thus, if \(\mu_1 = \frac{n}{\lambda^2}\) then
\[
(3.14) \quad 2\mu_1 + \frac{\lambda^2}{n^2} \mu_1^2 = \frac{2n}{\lambda^2} + \frac{\lambda^2 n^2}{n^2 \lambda^2} = \frac{2n + 1}{\lambda^2}.
\]

If we let \(\mu_{\lambda, 1}\) be the first positive eigenvalue of \(-\Delta^\lambda\), then by (3.10), (3.13), (3.14) and the assumption that \((f, f) = 1\), we have
\[
(3.15) \quad \mu_{\lambda, 1} \leq 2\mu_1 + \frac{\lambda^2}{n^2} \mu_1^2 + \frac{2\lambda^2}{n} \int_M \text{Tor}(\tilde{\nabla} f, \tilde{\nabla} f) \, dv
\]
\[
= \frac{(2n + 1)}{\lambda^2} + \frac{2\lambda^2}{n} \int_M \text{Tor}(\tilde{\nabla} f, \tilde{\nabla} f) \, dv.
\]

Next, we will use the results in the previous two sections to get a lower bound for the matrix \([R^\lambda_{jk}]\) and then for \(\mu_{\lambda, 1}\). By (1.5), we have
\[
(3.16) \quad \text{Ric}(z, z) - \frac{n + 1}{2} \text{Tor}(z, z) \geq k|z|^2,
\]

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for all $z \in H(M)$. Replacing $z$ by $e^{i\theta}z$, by a suitable choice of $\theta \in [0, 2\pi)$, one has that

$$\text{(3.17)} \quad \operatorname{Ric}(z, z) - \frac{n+1}{2} |\operatorname{Tor}(z, z)| \geq k|z|^2.$$  

Assume that $[\mathcal{R}_{\alpha\beta}]$ is an $n \times n$ positive definite Hermitian matrix. Write

$$\text{(3.18)} \quad C = \begin{bmatrix} \mathcal{R}_{\alpha\beta} \end{bmatrix} = A + iB$$

where $A$ and $B$ are real matrices. Thus, $A$ is symmetric positive definite, and $B$ is skew symmetric. Let $z = x + iy \in \mathbb{C}^n$ and let $s \in \mathbb{R}$. Then

$$\text{(3.19)} \quad (Cz, z) = (Ax - By + iAy + iBx, x + iy) = (Ax, x) - (By, x) + (Ay, y) + (Bx, y).$$

Let

$$\text{(3.20)} \quad G = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}, \quad \tilde{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad X = \begin{bmatrix} x \\ y \\ s \end{bmatrix}.$$  

Then $G$ is symmetric and

$$\text{(4.21)} \quad (G\tilde{X}, \tilde{X}) = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x \\ y \\ s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

By Proposition 2.4, one has

$$\text{(3.22)} \quad \left[ R^{\lambda}_{jk} \right] = 2 \begin{bmatrix} \text{Re} \left[ \mathcal{R}_{\alpha\beta} \right] - \frac{1}{\lambda^2} I_n & -\text{Im} \left[ \mathcal{R}_{\alpha\beta} \right] & 0 \\ \text{Im} \left[ \mathcal{R}_{\alpha\beta} \right] & \text{Re} \left[ \mathcal{R}_{\alpha\beta} \right] - \frac{1}{\lambda^2} I_n & 0 \\ 0 & 0 & \frac{n}{\lambda^2} \end{bmatrix} + E.$$  

By (3.16)–(3.22) and (3.49), one has

$$\text{(3.23)} \quad \left[ \left[ R^{\lambda}_{jk} \right] X, X \right] = 2 \left( \left[ \mathcal{R}_{\alpha\beta} \right] z, z \right) - \frac{2}{\lambda^2} |z|^2 + \frac{2n}{\lambda^2}s^2 + (EX, X)$$

$$= 2 \left( \left[ \mathcal{R}_{\alpha\beta} \right] z, z \right) - n\operatorname{Tor}(z, z) - \frac{2}{\lambda^2} |z|^2 + \frac{2n}{\lambda^2}s^2.$$  

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\[ +4\lambda \text{Re} \left[ \sum_{\alpha,k=1}^{n} \left( X_{k} A_{\alpha k} - \sum_{\ell=1}^{n} A_{\ell k} \Gamma_{\alpha k}^\ell - \sum_{\ell=1}^{n} A_{\ell \alpha} \Gamma_{k k}^\ell \right) z_{\alpha} \right] s \]
\[ -2\lambda^2 \text{tr}(A^*A)s^2 - \lambda^2 \text{Re} (A_0 z, \bar{z}) \]
\[ \geq 2 \left( k - \frac{1}{\lambda^2} \right) |z|^2 + |\text{Tor}(z, z)| + \frac{2n}{\lambda^2} s^2 \]
\[ + 4s \lambda \text{Re} \left[ \sum_{\alpha,k=1}^{n} \left( X_{k} A_{\alpha k} - \sum_{\ell=1}^{n} A_{\ell k} \Gamma_{\alpha k}^\ell - \sum_{\ell=1}^{n} A_{\ell \alpha} \Gamma_{k k}^\ell \right) z_{\alpha} \right] \]
\[ - \lambda^2 \text{Re} (A_0 z, \bar{z}) - 2\text{tr}(A^*A)\lambda^2 s^2. \]

Assume that \((M, \theta)\) is torsion-free. Then \(A_{\alpha \beta} = 0\). Thus,

\[(3.24) \quad (R^\lambda X, X) \geq \frac{2n}{\lambda^2} |X|^2, \quad k = \frac{n + 1}{\lambda^2}. \]

By the Lichnerowicz’s theorem for \(\Delta_{h^\lambda}\) on \((M, h^\lambda)\), the first positive eigenvalue \(\mu_{\lambda;1}\) for \(\Delta_{h^\lambda}\) satisfies the estimate:

\[(3.25) \quad \mu_{\lambda;1} \geq \frac{2n + 1}{\lambda^2}. \]

By \((3.15)\) with \((M, \theta)\) being torsion-free and \((3.25)\), we have

\[(3.26) \quad \mu_{\lambda;1} = \frac{2n + 1}{\lambda^2}. \]

By \((3.24)\) and \((3.26)\), one has that all conditions of Obata’s theorem hold for the Riemannian manifold \((M, h^\lambda)\). By Obata’s theorem, one has that \((M, h^\lambda)\) is isometric to the sphere \(S^{2n+1}(\lambda)\). Therefore, \((M, \theta)\) is CR equivalent to the unit sphere \(S^{2n+1}\), and the proof of the theorem is complete. \(\square\)

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