An explicit formula for the Webster pseudo Ricci curvature on real hypersurfaces and its application for characterizing balls in $\mathbb{C}^n$

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1 Introduction

Let $M$ be a $(2n + 1)$-dimensional CR manifold with CR dimension $n$. We say that $(M, \theta)$ is a strictly pseudoconvex pseudohermitian manifold in the sense of Webster [33] if there is a real one-form $\theta$ (contact form) on $M$ and at each point of $M$ there is a neighborhood with a local basis $\theta^1, \ldots, \theta^n$ for the holomorphic cotangent space $T^{1,0}$ so that

$$d\theta = i h_{\alpha \overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}},$$

where $[h_{\alpha \overline{\beta}}]$ is a positive definite $n \times n$ matrix, determined by the Levi-form $L_\theta$ on $M$. Here

$$L_\theta(w, \overline{v}) = -i d\theta(w, \overline{v}), \quad w, v \in T_{1,0}(M).$$

Let $R_{\alpha \overline{\beta}}$ be the Webster pseudo Ricci curvature and $R = h^{\alpha \overline{\beta}} R_{\alpha \overline{\beta}}$ be the pseudo scalar curvature. It is known that the contact form is neither unique nor CR invariant, but lies in a conformal class $(\theta_f = e^f \theta$ for some smooth function $f)$. There are many fundamental works done on CR manifolds by different authors. We refer to the book of Baouendi, Ebenfelt and Rothschild [1], Beals, Fefferman and Grossman [2], Folland and Stein [12], X. Huang [16], S. Webster [33] and several papers of Jerison and Lee which will be mentioned
later on. Here we will address a few major problems on Webster pseudo Ricci curvature which are related to the problems we are interested in this paper.

The CR Yamabe problem: Find a contact form $\theta_f$ so that the Webster pseudo scalar curvature $R_f$ with respect to the $\theta_f$ is a given constant. In other words, the variational equation:

$$\lambda(M) = \inf_{\theta} \lambda(\theta) = \inf_{\theta} \frac{\int_M R \theta \wedge (d\theta)^n}{(\int_M \theta \wedge (d\theta)^n)^{\frac{n}{n+1}}},$$

has a minimum. Much fundamental work has been done on the problem by D. Jerison and J. Lee in [19], [20] and [21], and N. Gama and R. Yacoub in [14] and [13]. In [19], Jerison and Lee proved that $\lambda(M) \leq \lambda(S^{2n+1})$, and solved the CR Yamabe problem for those $M$ with $\lambda(M) < \lambda(S^{2n+1}) = n(n+1)$. In particular, it was proved in [19] that if $\lambda(M) = \lambda(S^{2n+1})$ then $M$ is locally sphere. It suffices to consider $M = \partial D$ where $D$ is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^{n+1}$. In addition, if $D$ is simply connected then $D$, it was proved by Chern and Ji in [9] that locally spherical implies global spherical, or $D$ biholomorphically equivalent to the unit ball in $\mathbb{C}^{n+1}$. In this case, one can easily construct a contact form $\theta$ with constant pseudo scalar curvature (see formula in Theorem 1.1 below). It was proved by Huang and Ji in [17] that if $M = \partial D$ is locally spherical then $D$ is biholomorphically equivalent to the unit ball when $M$ is algebraic; and a counterexample was constructed by Burns and Shnieder [4] that the algebraic condition can not be replaced by real analytic. The general case, when $\lambda(M) = \lambda(S^{2n+1})$, the CR Yamabe problem was solved by N. Gama and R. Yacoub [14] and N. Gama [13]. When $M = S^{2n+1}$, the contact form $\theta$ with $\lambda(\theta) = n(n+1)$ was characterized by D. Jerison and J. Lee [20].

The existence of pseudo-hermitian Einstein metric. $(M, \theta)$ is said to be pseudo Einstein if

$$R_{\alpha\beta} = \frac{1}{n} Rh_{\alpha\beta}.$$  

This problem was solved by J. Lee in his series of excellent works [23] and [24], where he gave a few characterizations for such manifolds. In particular, if there is a non-vanishing holomorphic $(n+1)$-form on $M$ then $M$ is pseudo Einstein, which includes the boundary of strictly pseudoconvex domain in $\mathbb{C}^{n+1}$. In fact, the last result was obtained earlier by H-S Luk in [30]. He showed that the boundary of any smoothly bounded strictly pseudoconvex
domain with the contact form \( \theta = (-i/2)(\partial \rho - \bar{\partial} \rho) \) generated by the potential function \( \rho \) of Fefferman metric is pseudo Einstein.

Let \( u \in C^2(D) \), and let \( H(u) \) denote the complex hessian matrix of \( u \). For any positive function \( f(z) \in C^\infty(\overline{D}) \), it was proved that the Fefferman equation:

\[
J(\rho) = -\det \begin{bmatrix} \rho & \rho_j \\ \rho_i & \rho_{ij} \end{bmatrix} = f(z) \text{ in } D, \quad \rho = 0 \text{ on } \partial D,
\]

has a solution \( \rho \) with \( U = -\log(-\rho) \) being strictly plurisubharmonic in \( D \). Formal existence of such a solution was given by C. Fefferman in [10], existence and ‘uniqueness’ was proved by Cheng and Yau in [8] with \( \rho \in C^{n+1+3/2}(\overline{D}) \). Lee and Melrose in [25] gave an asymptotic expansion on \( \rho \), in particular, they showed \( \rho \in C^{n+3-\epsilon}(\overline{D}) \) for any \( \epsilon > 0 \).

**Eigenvalue problems for sub-Laplacian.** The best lower bound for the first positive eigenvalue of the sub-Laplace \( \Delta_{sb} = \text{Re } (\Box_b) \) where \( \Box_b \) is Kohn’s Laplacian acting on functions was given by Greenleaf [15] and Li and Luk in [29]. In the study of the eigenvalue problems, computability of the Webster pseudo Ricci curvature is very important. This leads us to one of the main purposes of the present paper: To give an explicit formula for the Webster pseudo Ricci curvatures and pseudo scalar curvature of \( \partial D \) with a major class of contact forms. In other words, we will prove the following theorem.

**THEOREM 1.1** Let \( M \) be a smooth strictly pseudoconvex hypersurface in \( \mathbb{C}^{n+1} \). Let \( U \) be a neighborhood of \( M \), and let \( \theta = (-i/2)(\partial \rho - \bar{\partial} \rho) \) where \( \rho \in C^3(U) \) is a defining function for \( M \) with \( J(\rho) > 0 \) in \( U \). Then for any \( w, v \in T_{1,0}(M) \), the Webster pseudo Ricci curvature is given by the formula:

\[
(1.6) \quad \text{Ric}(w, \overline{v}) = -\sum_{k,j=1}^{n+1} \frac{\partial^2 \log J(\rho)}{\partial \bar{z}^k \partial \bar{z}^j} w^k \bar{v}^j + (n+1) \frac{\det H(\rho)}{J(\rho)} \text{L}_\theta(w, \overline{v}).
\]

In particular, if \( J(\rho) \equiv 1 \) on \( D \) then

\[
(1.7) \quad \text{Ric}(w, \overline{v}) = (n+1) \det H(\rho) \text{L}_\theta(w, \overline{v}) \quad \text{and} \quad R = n(n+1) \det H(\rho).
\]

The characterization for balls in \( \mathbb{C}^{n+1} \) is always an interesting subject (see [22], [33], [9], [17] and [27]). Formula (1.7) in Theorem 1.1 and the main theorem in [27] on characterizing \( D \) to be a ball in \( \mathbb{C}^{n+1} \) lead us to the second main purpose of this paper which is to prove the following theorem.
THEOREM 1.2 Let $D$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^{n+1}$ with smooth boundary. Let $\rho \in C^3(D)$ be a defining function for $D$ with $J(\rho) \in C^2(D)$ being positive and $\log J(\rho)$ being pluriharmonic in $D$. Let $M = \partial D$ and $\theta = (-i/2)(\partial \rho - \overline{\partial} \rho)$. Let $\theta_0 = \frac{1}{2n}(\partial \rho_0 - \overline{\partial} \rho_0)$ with $\rho_0$ being the unique plurisubharmonic solution for the Monge-Ampère equation

$$
(1.8) \quad \det H(\rho_0) = J(\rho) \quad \text{in } D; \quad \rho_0(z) = |z|^2 \quad \text{on } \partial D.
$$

Then

(i) If the Webster pseudo scalar curvature $R$ satisfies

$$
(1.9) \quad R \geq n(n+1) \frac{\int_{\partial D} \theta \wedge (d\theta)^n}{\int_{\partial D} \theta_0 \wedge (d\theta_0)^n},
$$

then $D$ must be biholomorphically equivalent to the unit ball in $\mathbb{C}^{n+1}$.

(ii) If $J(\rho) \equiv 1$ on $D$ and the Webster pseudo scalar curvature $R$ satisfies (1.9) with $\theta_0 = \frac{1}{2n} \sum_{j=1}^{n+1} (z^j dz^j - z^j d\overline{z}^j)$ then $D$ must be biholomorphically equivalent to the unit ball in $\mathbb{C}^{n+1}$ having a constant Jacobian biholomorphic map.

Note: If $\phi : D \to B_{n+1}$ is a biholomorphic map with $\det \phi'(z) = c =$constant, then $R = cn(n+1)$.

The paper is organized as follows. In Section 2, we will prove Theorem 1.1. Moreover, we provide some details for computing for quantities related to the Webster pseudo Ricci curvature of a general contact form. The proof of Theorem 1.2 will be given in Section 3.

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2 Pseudo Ricci Curvatures on Hypersurfaces

Let $M$ be a smooth hypersurface (of real dimension $(2n+1)$) in $\mathbb{C}^{n+1}$. Let $\rho \in C^2(\mathbb{C}^{n+1})$ be a defining function for $M$: $M = \{\rho = 0\}$ and $\partial \rho \neq 0$ on $M$. Let $D = \{z \in \mathbb{C}^{n+1} : \rho(z) < 0\}$. Let $u$ be a $C^2$ function on $D$, we let
\[ H(u) = \left[ \frac{\partial^2 u}{\partial z \partial \overline{z}} \right]_{(n+1)\times(n+1)} \] be the complex hessian matrix of \( u \). If \( H(u) \) is invertiable then we let \( [u^\dagger] \) be the inverse of \( H(u)^t \) so that

\[ (2.1) \quad \sum_{j=1}^{n+1} u^j u^\dagger_{kj} = \sum_{k=1}^{n+1} u^k u^\dagger_{kk} = \delta_{kj}. \]

Let \( u_k = \frac{\partial u}{\partial z^k}, u^\dagger = \frac{\partial u}{\partial \overline{z}} \) and

\[ (2.2) \quad |\partial u|^2 = u^k u^\dagger_k, \quad u^j = \sum_{k=1}^{n+1} u^k u^\dagger_k \quad \text{and} \quad u^\dagger = \sum_{k=1}^{n+1} u^j u^\dagger_k. \]

**Lemma 2.1** Let \( D \) be a strictly pseudoconvex domain in \( \mathbb{C}^{n+1} \) with smooth boundary \( M \). Let \( \rho \) be a smooth defining function for \( D \) so that \( H(\rho) \) is positive definite on \( M \). Let

\[ (2.3) \quad \theta = \frac{1}{2i}[\partial \rho - \overline{\partial \rho}], \quad \theta^k = dz^k - \frac{\rho^k}{|\partial \rho|^2} \partial \rho = dz^k - h^k \partial \rho. \]

Then

\[ (2.4) \quad \theta = -i \partial \rho = i \overline{\partial \rho}, \quad d\theta = i \sum_{k, \ell=1}^{n+1} \rho_k \overline{\theta^k} \wedge \theta^\ell. \]

**Proof.** Since

\[ \rho_j h^j = \frac{\partial \rho_j}{|\partial \rho|^2} = \frac{\partial \rho_j \overline{\partial \rho}}{|\partial \rho|^2} = 1, \quad \sum_{k=1}^{n+1} \rho_k \theta^k = \partial \rho - \overline{\partial \rho} = 0 \text{ on } M. \]

Notice that since \( d\rho = 0 \) on \( \partial D \), we have \( \partial \rho = -\overline{\partial \rho} \) and \( \theta = -i \partial \rho = i \overline{\partial \rho} \) on \( M \). Therefore

\[ d\theta = i \partial \overline{\partial \rho}(z) \]

\[ = i \sum_{j,k=1}^{n+1} \rho_k \overline{z}^j \wedge dz^k \]

\[ = i \sum_{j,k=1}^{n+1} \rho_k \overline{\theta^k} \wedge dz^j + i \sum_{j,k=1}^{n+1} \rho_k h^j \partial \rho \wedge dz^j \]

\[ = i \sum_{j,k=1}^{n+1} \rho_k \overline{\theta^k} \wedge \theta^j + i \sum_{j,k=1}^{n+1} \rho_k h^j \overline{\theta^k} \wedge \overline{\partial \rho} + i \sum_{j,k=1}^{n+1} \rho_k h^j \partial \rho \wedge \overline{\theta^j}. \]
Then on \( M \)

\[
\sum_{k=1}^{n+1} \sum_{j=1}^{n} \rho_{kJ} \overline{\rho}^{\beta j} \theta^k = \frac{1}{|\partial \rho|^2} \sum_{k=1}^{n+1} \sum_{j=1}^{n+1} \sum_{l=1}^{n+1} \rho_{kJ} \overline{\rho}^{\beta} \rho_{l} \theta_k \\
= \frac{1}{|\partial \rho|^2} \sum_{k=1}^{n+1} \sum_{l=1}^{n+1} \delta_{kl} \rho_{l} \theta_k \\
= \frac{\sum_{k=1}^{n+1} \rho_{k} \theta_k}{|\partial \rho|^2} \\
= 0.
\]

We have proved that

\[
(2.5) \quad d\theta = i \sum_{j,k=1}^{n+1} \rho_{kJ} \theta^k \wedge \overline{\theta}^j
\]

and the proof of the lemma follows. \( \Box \)

Let \( M_1 = \{ z \in M : \rho_{n+1}(z) \neq 0 \} \). Since

\[
(2.6) \quad \sum_{j=1}^{n+1} \rho_j \theta^j = 0
\]

we can write

\[
\theta^{n+1} = - \sum_{k=1}^{n} \frac{\rho_k}{\rho_{n+1}} \theta^k
\]

Then

\[
(2.7) \quad d\theta = i \sum_{j,k=1}^{n} \left[ \rho_{kJ} \rho_{n+1} - \rho_{n+1} \overline{\rho}^{\beta j} \rho_k + \rho_{n+1} \rho_{n+1} \left| \rho_{n+1} \right|^2 \theta^k \wedge \overline{\theta}^j \right] \\
= i \sum_{\alpha, \beta=1}^{n} h_{\alpha \beta} \theta^\alpha \wedge \overline{\theta}^\beta
\]

Let

\[
(2.8) \quad Y_j = \frac{\partial}{\partial z^j} - \frac{\rho_j}{\rho_{n+1}} \frac{\partial}{\partial z^{n+1}}, \quad Y = i \sum_{j=1}^{n+1} \left( h^{j} \partial_j - h^{\overline{j}} \overline{\partial_j} \right).
\]
Then
\[
\sum_{p=1}^{n} h_{\beta p} h_{\overline{p}}
\]
\[
= \sum_{p=1}^{n} \left[ \rho_{p} \frac{\rho_{n+1} \rho_{\beta}}{\rho_{n+1}} - \rho_{n+1} \frac{\rho_{n+1} \rho_{\overline{p}}}{\rho_{n+1}^2} + \rho_{n+1} \frac{\rho_{n+1} \rho_{\beta}}{\rho_{n+1}} \right] h_{\overline{p}}
\]
\[
= \frac{\rho_{\beta}}{|\partial p|^2} - \rho_{n+1} h_{\overline{p}} h_{\overline{p}} - \rho_{n+1} h_{\overline{p}} + \rho_{n+1} h_{\overline{p}} / \rho_{n+1}
\]
\[
= -\rho_{n+1} h_{\overline{p}} + \rho_{n+1} h_{\overline{p}} h_{\overline{p}} / \rho_{n+1}
\]
\[
= -\rho_{n+1} h_{\overline{p}} + \rho_{n+1} h_{\overline{p}} / \rho_{n+1}
\]
\[
= -\rho_{n+1} h_{\overline{p}}.
\]

Moreover, for any \( f \in C^1(M_1) \) we have
\[
(2.9) \quad df = \sum_{j=1}^{n+1} f_{j} dz^{j} + f_{\overline{j}} d\overline{z}^{j}
\]
\[
= \sum_{j=1}^{n+1} f_{j} \theta^{j} + f_{\overline{j}} \theta^{j} + (i f_{j} h^{j} - i f_{\overline{j}} h^{\overline{j}}) \theta
\]
\[
= \sum_{j=1}^{n+1} (Y_{j} f \theta^{j} + Y_{\overline{j}} f \theta^{j}) + \sum_{j=1}^{n+1} \left( \frac{f_{n+1}}{\rho_{n+1}} \theta^{j} + \frac{\rho_{n+1}}{f_{n+1}} \theta^{\overline{j}} \right) + Y(f) \theta
\]
\[
= \sum_{j=1}^{n+1} (Y_{j} f \theta^{j} + Y_{\overline{j}} f \theta^{j}) + Y(f) \theta
\]
\[
= \sum_{j=1}^{n} (Y_{j} f \theta^{j} + Y_{\overline{j}} f \theta^{j}) + Y(f) \theta.
\]

Notice that since \( \theta^{\alpha} = dz^{\alpha} - i h^{\alpha} \theta \), we have
\[
(2.10) \quad d\theta^{\alpha} = -i d h^{\alpha} \wedge \theta - i h^{\alpha} d \theta
\]
\[
= -i d h^{\alpha} \wedge \theta + h^{\alpha} h_{\gamma \beta} \theta^{\gamma} \wedge \theta^{\overline{\beta}}
\]
\[= \sum_{\gamma=1}^{n} \theta^\gamma \wedge \left( h^\alpha \sum_{\beta=1}^{n} h_{\gamma \beta} \theta^\beta - i Y_{\gamma} h^\alpha \theta \right) + \theta \wedge \left( \sum_{\beta=1}^{n} i Y_\beta h^\alpha \theta^\beta \right) \]
\[
= \sum_{\gamma=1}^{n} \theta^\gamma \wedge \omega_{\gamma}^{\alpha} + \theta \wedge \tau^\alpha
\]

where

\[
(2.11) \quad \omega_{\gamma}^{\alpha} = h^\alpha \sum_{\beta=1}^{n} h_{\gamma \beta} \theta^\beta - i Y_{\gamma} h^\alpha \theta, \quad \tau^\alpha = i \sum_{\beta=1}^{n} Y_\beta h^\alpha \theta^\beta
\]

and

\[
(2.12) \quad \tau_{\alpha} = h_{\alpha \gamma} \tau^\gamma = -i \sum_{\beta=1}^{n} \sum_{\gamma=1}^{n} h_{\alpha \gamma} Y_\beta h^\gamma \theta^\beta = \sum_{\beta=1}^{n} A_{\alpha \beta} \theta^\beta
\]

with

\[
(2.13) \quad A_{\alpha \beta} = (-i) \sum_{\gamma=1}^{n} h_{\alpha \gamma} Y_\beta h^\gamma
\]

The torsion of \( M \) with respect to \( \theta \) is defined as follows:

\[
(2.14) \quad \text{Tor}(z^\alpha Y_\alpha, w^\beta Y_\beta) = i (A_{\alpha \beta} z^\alpha w^\beta - A_{\beta \alpha} z^\alpha w^\beta).
\]

Using the fact \( d\theta = i h_{\alpha \beta} \theta^\alpha \wedge \theta^\beta \) on \( M_1 \), one has

\[
0 = dd\theta = dh_{\alpha \beta} \wedge \theta^\alpha \wedge \theta^\beta + h_{\alpha \beta} d\theta^\alpha \wedge \theta^\beta - h_{\alpha \beta} \theta^\alpha \wedge d\theta^\beta
\]
\[
= dh_{\alpha \beta} \wedge \theta^\alpha \wedge \theta^\beta + h_{\alpha \beta} \left[ (\theta^\gamma \wedge \omega_{\gamma}^{\alpha} + \theta^\gamma \theta^\alpha) \wedge \theta^\beta - \theta^\alpha \wedge (\theta^\gamma \wedge \omega_{\gamma}^{\beta} + \theta \wedge \tau^\beta) \right]
\]
\[
= (dh_{\alpha \beta} - h_{\gamma \beta} \omega^\gamma_{\alpha} - h_{\alpha \gamma} \omega^\gamma_{\beta}) \theta^\alpha \wedge \theta^\beta + \theta \wedge \tau^\gamma \wedge \theta^\beta - \theta \wedge \tau_{\alpha} \wedge \theta^\alpha.
\]

This implies that

\[
(2.15) \quad \tau_{\alpha} \wedge \theta^\alpha = 0 \quad \text{or} \quad A_{\alpha \beta} = A_{\beta \alpha}
\]

and

\[
(2.16) \quad dh_{\alpha \beta} - \omega^\gamma_{\alpha \beta} - \omega^\gamma_{\beta \alpha} = dh_{\alpha \beta} - h_{\gamma \beta} \omega^\gamma_{\alpha} - h_{\alpha \gamma} \omega^\gamma_{\beta} = A_{\alpha \beta \gamma} \theta^\gamma + B_{\alpha \beta} \theta^\gamma,
\]

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where

$$A_{\alpha \beta \gamma} = A_{\gamma \beta \alpha}, \quad B_{\alpha \beta \gamma} = A_{\beta \alpha \gamma}$$

Then

$$\omega'_{\alpha \beta} = h_{\gamma \beta} \omega'_{\alpha}$$

$$= h_{\gamma \beta} \left( h^\gamma \sum_{k=1}^{n} h_{\alpha k} \theta^k - iY_{\alpha} h^\gamma \theta \right)$$

$$= h_{\gamma \beta} \sum_{k=1}^{n} h_{\alpha k} \theta^k - ih_{\gamma \beta} Y_{\alpha} h^\gamma \theta$$

and

$$A_{\alpha \beta \gamma} = Y_{\gamma} h_{\alpha \beta} - h_{\alpha} h_{\gamma \beta}.$$

If we let

$$\omega_{\alpha \beta} = \omega'_{\alpha \beta} + (Y_{\gamma} h_{\alpha \beta} - h_{\alpha} h_{\gamma \beta}) \theta^\gamma$$

$$= (Y_{\gamma} h_{\alpha \beta} - h_{\alpha} h_{\gamma \beta}) \theta^\gamma + h_{\gamma \beta} h_{\alpha \gamma} \theta^\gamma - ih_{\gamma \beta} Y_{\alpha} h^\gamma \theta$$

and

$$\omega_{\beta \alpha} = (Y_{\gamma} h_{\alpha \beta} - h_{\alpha} h_{\gamma \beta}) \theta^\gamma + h_{\alpha \beta} h_{\gamma \alpha} \theta^\gamma + ih_{\alpha \beta} Y_{\gamma} h^\gamma \theta,$$

then

$$dh_{\alpha \beta} - h_{\gamma \beta} \omega'_{\alpha} - h_{\alpha} \omega'_{\beta} = 0.$$

Moreover,

$$Y_{\gamma} h_{\alpha \beta} - h_{\alpha} h_{\gamma \beta} = Y_{\alpha} h_{\gamma \beta} - h_{\gamma} h_{\alpha \beta}.$$

Let

$$\Omega^\alpha_{\beta} = d\omega^\alpha_{\beta} - \omega^\alpha_{\gamma} \wedge \omega^\gamma_{\beta} - i\theta^\alpha \wedge \theta^\gamma + i\tau^\alpha \wedge \theta^\gamma.$$

Then

$$\Omega_{\alpha \beta} = h_{\gamma \beta} \Omega^\gamma_{\alpha}$$

$$= h_{\gamma \beta} \left[d\omega^\alpha_{\gamma} - \omega^k_{\alpha} \wedge \omega^\gamma_{k} - i\theta^\alpha \wedge \theta^\gamma + i\tau^\alpha \wedge \theta^\gamma \right]$$

$$= -dh_{\gamma \beta} \wedge \omega^\alpha_{\gamma} + d\omega^\alpha_{\beta} - \omega^k_{\alpha} \wedge \omega_{\gamma k} - i\theta^\alpha \wedge \tau^\gamma + i\tau^\alpha \wedge \theta^\gamma$$

$$= -[h_{\gamma \beta} \omega^k_{\gamma} + h_{\gamma \beta} \omega^k_{\gamma} \wedge \omega^\gamma_{k} + d\omega^\alpha_{\gamma} - \omega^k_{\alpha} \wedge \omega_{\gamma k} - i\theta^\alpha \wedge \tau^\gamma + i\tau^\alpha \wedge \theta^\gamma]$$

$$= -[\omega^\alpha_{\gamma} + \omega^\gamma_{\alpha} \wedge \omega^\alpha_{\gamma} + d\omega^\alpha_{\beta} - \omega^k_{\alpha} \wedge \omega_{\gamma k} - i\theta^\alpha \wedge \tau^\gamma + i\tau^\alpha \wedge \theta^\gamma]$$

$$= dh_{\alpha \beta} - \omega^\alpha_{\gamma} \wedge \omega^\alpha_{\gamma} - i\theta^\alpha \wedge \tau^\gamma + i\tau^\alpha \wedge \theta^\gamma.$$
We compute
\[
d\omega_{\alpha \beta} \quad (\mod \theta, \theta^p \wedge \theta^q, \theta^\alpha \wedge \theta^\beta)
\]
\[
= \left[ -Y_\ell Y_\gamma h_{\alpha \beta} + Y_\ell (h_\alpha h_{\gamma \beta}) + Y_\gamma (h_\beta h_{\alpha \gamma}) \right] \theta^\gamma \wedge \theta^\beta + (Y_\gamma h_{\alpha \beta} - h_\alpha h_{\gamma \beta}) d\theta^\gamma
+ h_\gamma h_{\alpha \gamma}d\theta^\alpha + h_\beta Y_\alpha h^j h_\gamma \theta^\gamma \wedge \theta^\beta
\]
\[
= \left[ -Y_\ell Y_\gamma h_{\alpha \beta} + Y_\ell (h_\alpha h_{\gamma \beta}) + Y_\gamma (h_\beta h_{\alpha \gamma}) + h_\gamma h_{j \beta} Y_\alpha h^j \right] \theta^\gamma \wedge \theta^\beta
+ (Y_\gamma h_{\alpha \beta} - h_\alpha h_{\gamma \beta}) h^j \sum_{k=1}^n h_{\beta k} \theta^j \wedge \theta^k + h_\gamma h_{\alpha \gamma} h^j \sum_{k=1}^n h_{k j} \theta^\gamma \wedge \theta^k
\]
\[
= \left[ -Y_\ell Y_\gamma h_{\alpha \beta} + Y_\ell (h_\alpha h_{\gamma \beta}) + Y_\gamma (h_\beta h_{\alpha \gamma}) + h_\gamma h_{j \beta} Y_\alpha h^j + (Y_\gamma h_{\alpha \beta} - h_\alpha h_{\gamma \beta}) h^j h_\gamma - h_\gamma h_{\alpha \gamma} h^j h_\gamma \right] \theta^\gamma \wedge \theta^\beta
\]
\[
= \left[ -Y_\ell Y_\gamma h_{\alpha \beta} + Y_\ell (h_\alpha h_{\gamma \beta}) + Y_\gamma (h_\beta h_{\alpha \gamma}) + h_\gamma h_{j \beta} Y_\alpha h^j + (Y_\gamma h_{\alpha \beta} - h_\alpha h_{\gamma \beta}) h^j h_\gamma - 2h_\gamma h_\alpha h_\gamma \right] \theta^\gamma \wedge \theta^\beta
\]
and
\[
\omega^\gamma_{\alpha} = h^\gamma_{\alpha \beta} \omega_{\alpha \beta}
\]
\[
= h^\gamma_{\alpha \beta} (Y_\beta h_{\alpha \gamma} - h_\alpha h_{\beta \gamma}) \theta^\beta + h^\gamma_{\alpha \beta} h^j_{\beta \gamma} \theta^j \mod \theta
\]
\[
= (h^\gamma_{\alpha \beta} Y_\beta h_{\alpha \gamma} - h_\alpha h_{\beta \gamma}) \theta^\beta + h^\gamma_{\alpha \beta} h^j_{\beta \gamma} \theta^j \mod \theta.
\]
By (2.20), we have
\[
(2.25) \quad \omega_{\beta \gamma} = (Y_{\beta \gamma} h_{\alpha \beta} - h_{\beta \gamma} h_{\alpha \gamma}) \theta^\beta \wedge \theta^\gamma + h_{\gamma \alpha \beta} \theta^\beta \mod \theta
\]
Thus
\[
\omega_{\beta \gamma} \wedge \omega^\gamma_{\alpha} \quad (\mod \theta, \theta^p \wedge \theta^q, \theta^\alpha \wedge \theta^\beta)
\]
\[
= [(Y_{\beta \gamma} h_{\alpha \beta} - h_{\beta \gamma} h_{\alpha \gamma}) \theta^\beta \wedge \theta^\gamma + h_{\gamma \alpha \beta} \theta^\beta \wedge \theta^\gamma] \wedge [(h^\gamma_{\alpha \beta} Y_\beta h_{\alpha \gamma} - h_\alpha h_{\beta \gamma}) \theta^\beta + h^\gamma_{\alpha \beta} h^j_{\beta \gamma} \theta^j] \wedge \theta^\gamma
\]
\[
= h_\beta h^\gamma h_{\alpha \beta} \theta^\beta \wedge \theta^\gamma \wedge \theta^\gamma - (Y_{\beta \gamma} h_{\alpha \beta} - h_{\beta \gamma} h_{\alpha \gamma}) (h^\gamma_{\alpha \beta} Y_\beta h_{\alpha \gamma} - h_\alpha h_{\beta \gamma}) \theta^\gamma \wedge \theta^\gamma
\]
\[
= h_\beta h^\gamma h_{\alpha \beta} \theta^\beta \wedge \theta^\gamma \wedge \theta^\gamma - (Y_{\beta \gamma} h_{\alpha \beta} - h_{\beta \gamma} h_{\alpha \gamma}) (h^\gamma_{\alpha \beta} Y_\beta h_{\alpha \gamma} - h_\alpha h_{\beta \gamma}) \theta^\gamma \wedge \theta^\gamma
\]
\[
= h_\beta h^\gamma h_{\alpha \beta} \theta^\beta \wedge \theta^\gamma \wedge \theta^\gamma - (h^\gamma_{\alpha \beta} Y_\beta h_{\alpha \gamma} - h_\alpha h_{\beta \gamma}) \theta^\gamma \wedge \theta^\gamma
\]
\[ + h^\gamma_\gamma h_\gamma^\gamma Y_\gamma^\gamma h_\alpha \alpha^\gamma - h_\gamma h_\alpha h_\gamma^\gamma \delta_{\gamma\gamma} \right] \theta^\gamma \wedge \theta^\gamma \]

\[ = [h_\gamma h_\gamma^\gamma Y_\gamma^\gamma h_\alpha \alpha^\gamma - h_\gamma h_\alpha h_\gamma^\gamma \delta_{\gamma\gamma} \right] \theta^\gamma \wedge \theta^\gamma \]

\[ = [h_\gamma h_\gamma^\gamma Y_\gamma^\gamma h_\alpha \alpha^\gamma - h_\gamma h_\alpha h_\gamma^\gamma \delta_{\gamma\gamma} \right] \theta^\gamma \wedge \theta^\gamma \]

Therefore

\[ (2.26) \quad \Omega_{\alpha\beta} = R_{\alpha\beta\gamma\delta} \theta^\gamma \wedge \theta^\delta + \theta \wedge \lambda_{\alpha\beta} \]

where

\[ (2.27) \quad R_{\alpha\beta\gamma\delta} = \]

\[ - Y_\gamma^\gamma Y_\alpha^\beta + Y_\eta^\gamma Y_\alpha^\gamma (h_\gamma h_\eta^\gamma) + Y_\gamma (h_\gamma^\gamma h_\alpha^\gamma) \]

\[ + h_\gamma^\gamma h_\beta^\gamma Y_\eta^\gamma h_\alpha^\gamma + h_\gamma^\gamma h_\alpha^\gamma Y_\eta^\gamma h_\beta^\gamma - 2h_\gamma^\gamma h_\alpha^\gamma h_\gamma^\gamma \]

\[ + h_\gamma^\gamma Y_\eta^\gamma h_\alpha^\gamma h_\gamma^\gamma - h_\alpha^\gamma Y_\eta^\gamma h_\gamma^\gamma + h_\gamma^\gamma h_\gamma^\gamma \]

Let

\[ (2.28) \quad g(z) = \log \det (h_{\alpha\beta}), \quad z \in M_1. \]

Then for \(1 \leq \gamma, \ell \leq n\), we have

\[ R_{\gamma\ell} \]

\[ = h^\alpha^\beta R_{\alpha\beta\gamma\ell} \]

\[ = - h^\alpha^\beta Y_\gamma^\gamma Y_\alpha^\beta + Y_\eta^\gamma (h_\gamma) + h_\gamma^\gamma Y_\eta^\gamma (h_\alpha^\gamma) + Y_\gamma (h_\eta^\gamma) + h_\gamma^\gamma Y_\eta^\gamma (h_\alpha^\gamma) + h_\gamma^\gamma Y_\gamma^\gamma (h_\alpha^\gamma) + h_\gamma^\gamma Y_\gamma^\gamma (h_\alpha^\gamma) \]

\[ - 2h_\alpha^\gamma h_\gamma^\gamma - h_\gamma^\gamma h_\gamma^\gamma \]

\[ + h_\gamma^\gamma Y_\eta^\gamma h_\alpha^\gamma \]

\[ - h_\gamma^\gamma Y_\eta^\gamma h_\alpha^\gamma \]

\[ = - h_\gamma^\gamma Y_\eta^\gamma Y_\alpha^\beta + h_\gamma^\gamma h_\beta^\gamma Y_\eta^\gamma Y_\alpha^\gamma + Y_\gamma (h_\eta^\gamma) + Y_\gamma (h_\eta^\gamma) + [h_\eta^\gamma h_\alpha^\gamma + h_\gamma^\gamma h_\eta^\gamma] \]

\[ = - Y_\gamma^\gamma g(z) + Y_\eta^\gamma Y_\gamma^\gamma (\log \rho_{\eta\gamma}) - h_\gamma^\gamma Y_\gamma^\gamma (\log \rho_{\eta\gamma}) + [h_\gamma^\gamma h_\alpha^\gamma + h_\gamma^\gamma h_\eta^\gamma] \]

\[ = - Y_\gamma^\gamma g(z) - Y_\gamma^\gamma (\log \rho_{\eta\gamma}) + Y_\gamma^\gamma (\log \rho_{\eta\gamma}) + [h_\gamma^\gamma h_\alpha^\gamma + h_\gamma^\gamma h_\eta^\gamma] h_\gamma^\gamma \]

\[ = - Y_\gamma^\gamma g(z) - Y_\gamma^\gamma (\log \rho_{\eta\gamma}) + Y_\gamma^\gamma (\log \rho_{\eta\gamma}) + [h_\gamma^\gamma h_\alpha^\gamma + h_\gamma^\gamma h_\eta^\gamma] h_\gamma^\gamma \]

Since

\[ (2.29) \quad [Y_\ell^\gamma, Y_\eta^\gamma] = - Y_\gamma^\gamma \left( \frac{\rho_{\gamma\ell}}{\rho_{\eta\gamma}} \right) \partial_{\eta\gamma} + Y_\gamma^\gamma \left( \frac{\rho_{\gamma\eta}}{\rho_{\eta\gamma}} \right) \partial_{\eta\gamma} = h_\gamma^\gamma \left[ \frac{\partial_{n+1}}{\rho_{n+1} - \rho_{n+1}} \right] \]

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we get

\begin{equation}
[\bar{Y}_\tilde{\tau}, Y_\gamma] (\log \rho_{n+1}) = \left( \frac{\rho_{n+1} \log \rho_{n+1}}{|\rho_{n+1}|^2} - \frac{\rho_{n+1} \log \rho_{n+1}}{|\rho_{n+1}|^2} \right) h_{\gamma \tilde{\tau}}.
\end{equation}

Since

\begin{equation}
g(z) + \log |\rho_{n+1}|^2 = \log [J(\rho) + \rho \det H(\rho)]
\end{equation}

and \( Y_{\tilde{\tau}}, Y_\gamma \) are tangential to \( M_1 \), combining all the above, we have, on \( M \)

\begin{equation}
R_{\gamma \tilde{\tau}} = -Y_{\tilde{\tau}} Y_\gamma (g + \log |\rho_{n+1}|^2)
+ \left( Y_\alpha (h^\alpha) + h^\alpha Y_\alpha (g) - 2h^\alpha h_\alpha + \frac{\rho_{n+1} \log \rho_{n+1}}{|\rho_{n+1}|^2} \right) h_{\gamma \tilde{\tau}}
= -Y_{\tilde{\tau}} Y_\gamma (\log J(\rho))
+ \left( Y_\alpha (h^\alpha) + h^\alpha Y_\alpha (g) - 2h^\alpha h_\alpha + \frac{\rho_{n+1} \log \rho_{n+1}}{|\rho_{n+1}|^2} \right) h_{\gamma \tilde{\tau}}
\end{equation}

where \( \alpha \) is summing from 1 to \( n \).

The main result for this section is the following proof of Theorem 1.1.

**The proof of Theorem 1.1.**

**Proof.** We first assume that \( H(\rho) \) is positive definite on \( M \). Since

\begin{equation}
\sum_{a=1}^{n} \sum_{k=1}^{n+1} \rho^a \bar{\rho}_{\tilde{\tau}} Y_\alpha (\rho_{\tilde{\tau}}) = \sum_{a=1}^{n} \sum_{k=1}^{n+1} \rho^a \bar{\rho}_{\tilde{\tau}} (\rho_{a \tilde{\tau}} - \frac{\rho_a}{\rho_{n+1}} \rho_{a+n \tilde{\tau}}) = n
\end{equation}

and

\begin{equation}
Y_\gamma (\rho^a) = -\sum_{\rho, q=1}^{n+1} \rho^{a q} \rho^q \bar{\rho} Y_\gamma (\rho_{\tilde{\tau}})
\end{equation}

we have, by \( h^\alpha = \rho^a / |\partial \rho|^2 \),

\begin{equation}
|\partial \rho|^2 \sum_{a=1}^{n} Y_\alpha (h^\alpha) = \rho^a \bar{\rho}_{\tilde{\tau}} Y_\alpha (\rho_{\tilde{\tau}}) + \rho_{\tilde{\tau}} Y_\alpha (\rho^a) - h^\alpha Y_\alpha (|\partial \rho|^2)
= n - \rho^a \rho^q \rho^q \bar{\rho} Y_\alpha (\rho_{\tilde{\tau}}) - h^\alpha Y_\alpha (|\partial \rho|^2)
= n - |\partial \rho|^2 \rho^a \bar{\rho} Y_\alpha (\rho_{\tilde{\tau}}) - h^\alpha Y_\alpha (|\partial \rho|^2).
\end{equation}
Since $J(\rho) = \det H(\rho)(|\partial \rho|^2 - \rho)$,

\[
\frac{\rho^\alpha h^p Y_\alpha(\rho)}{p} = \frac{\rho^\alpha h^p \rho \rho \rho}{\rho n + 1} - \frac{\rho^\alpha h^p}{\rho n + 1} \rho n + 1 \rho
\]

\[
= h^p \partial_\rho \log \det H(\rho) - \rho^{\alpha+1} h^p \rho n + 1 \rho - \frac{\rho^\alpha h^p}{\rho n + 1} \rho n + 1 \rho
\]

\[
= h^p \partial_\rho \log \det H(\rho) - h^p \rho n + 1 \rho \rho n + 1 \rho
\]

\[
= h^p \partial_\rho \log J(\rho) - h^p \partial_\rho \log (|\partial \rho|^2 - \rho) - h^p \rho n + 1 \rho
\]

where we sum $p$ and $q$ from 1 to $n + 1$, and $\alpha$ from 1 to $n$. Thus, since $\sum_{k=1}^{n+1} \rho_k h^k = 1$, $Y_\alpha \log |\partial \rho|^2 = Y_\alpha \log (|\partial \rho|^2 - \rho)$ and

\[
\frac{1}{\rho n + 1} \partial_\rho \log (|\partial \rho|^2 - \rho) = \frac{1}{\rho n + 1} \partial_\rho \log (|\partial \rho|^2 - \rho)
\]

we have

\[
\sum_{\alpha=1}^{n} Y_\alpha(h^n)
\]

\[
= \frac{n}{|\partial \rho|^2} - h^p \partial_\rho \log J(\rho) + h^p \partial_\rho \log (|\partial \rho|^2 - \rho) + \frac{h^p \rho n + 1 \rho}{\rho n + 1} - h^p Y_\alpha \log (|\partial \rho|^2)
\]

\[
= \frac{n}{|\partial \rho|^2} - h^p \partial_\rho \log J(\rho) + \frac{\rho n + 1 \rho}{\rho n + 1} \partial_\rho \log (|\partial \rho|^2 - \rho) + \frac{h^p \rho n + 1 \rho}{\rho n + 1}
\]

\[
= \frac{n}{|\partial \rho|^2} - h^p \partial_\rho \log J(\rho) + \frac{\rho n + 1 \rho}{\rho n + 1} \partial_\rho \log (|\partial \rho|^2 - \rho)
\]

\[
= \frac{n}{|\partial \rho|^2} - h^p \partial_\rho \log J(\rho) + \frac{\rho n + 1 \rho}{\rho n + 1} \partial_\rho \log (|\partial \rho|^2 - \rho)
\]

\[
= \frac{n - 1}{|\partial \rho|^2} - h^p \partial_\rho \log J(\rho) + \frac{2}{\rho n + 1} \partial_\rho \log (|\partial \rho|^2)
\]

\[
- \frac{\rho n + 1 \rho}{|\partial \rho|^2 \rho n + 1}
\]

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\[ \begin{align*}
&= \frac{n-1}{|\partial \rho|^2} - h^p \partial_p \log J(\rho) + \frac{2}{\rho_{n+1}} \partial_{n+1} \log (|\partial \rho|^2) \\
&\quad - \frac{\partial_{n+1}(\rho_p \rho_p) - \rho_p \rho_{n+1}}{|\partial \rho|^2 \rho_{n+1}} \partial_{n+1}(\rho_{\overline{\pi} \overline{\pi}}) - \rho_{\overline{\pi} \overline{\pi}} \rho_{n+1} \overline{\gamma} \\
&= \frac{n-1}{|\partial \rho|^2} - h^p \partial_p \log J(\rho) + \frac{2}{\rho_{n+1}} \partial_{n+1} \log (|\partial \rho|^2) \\
&\quad - 2 \frac{\partial_{n+1} \log |\partial \rho|^2}{\rho_{n+1}} + \frac{\rho^p \rho_{n+1} + \rho_{\overline{\pi} \overline{\pi}} \rho_{n+1} \overline{\gamma}}{|\partial \rho|^2 \rho_{n+1}} \\
&= \frac{n-1}{|\partial \rho|^2} - h^p \partial_p \log J(\rho) + \frac{h^p \partial_{n+1}(\rho_p) + h^\pi \partial_{n+1}(\rho_{\overline{\pi}})}{\rho_{n+1}}.
\end{align*} \]

Since

\[ h^\alpha Y_a g = h^\alpha Y_a (- \log |\rho_{n+1}|^2 + \log J(\rho)) \]

\[ = -h^\alpha \frac{\rho_{an+1}}{\rho_{n+1}} - h^\alpha \frac{\rho_{an+1}}{\rho_{n+1}} + h^\alpha \frac{\rho_a (\rho_{n+1}a + \rho_{n+1}a)}{\rho_{n+1}a + \rho_{n+1}a} + h^\alpha Y_a \log J(\rho) \]

and

\[ (2.36) \quad -h^p \partial_p \log J(\rho) + \sum_{\alpha=1}^{n} h^\alpha Y_a \log J(\rho) = -\frac{1}{\rho_{n+1}} \partial_{n+1} \log J(\rho), \]

we have

\[ \begin{align*}
\sum_{\alpha=1}^{n} Y_a(h^\alpha) + h^\alpha Y_a g(z) + \frac{\rho_{n+1}a}{|\rho_{n+1}|^2} - \frac{\rho_{n+1}a}{\rho_{n+1}a} \\
&= \frac{n-1}{|\partial \rho|^2} - \frac{1}{\rho_{n+1}} \partial_{n+1} \log J(\rho) + \frac{h^p \partial_{n+1}(\rho_p) + h^\pi \partial_{n+1}(\rho_{\overline{\pi}})}{\rho_{n+1}} \\
&\quad - h^\alpha \frac{\rho_{an+1}}{\rho_{n+1}} - h^\alpha \frac{\rho_{an+1}}{\rho_{n+1}} + h^\alpha \frac{\rho_a (\rho_{n+1}a + \rho_{n+1}a)}{\rho_{n+1}a + \rho_{n+1}a} + \frac{\rho_{n+1}a}{|\rho_{n+1}|^2} - \frac{\rho_{n+1}a}{\rho_{n+1}a} \\
&= \frac{n-1}{|\partial \rho|^2} - \frac{1}{\rho_{n+1}} \partial_{n+1} \log J(\rho) + \frac{h_{n+1}^p \rho_{n+1}a + h_{n+1}^\pi \rho_{n+1}a}{\rho_{n+1}} \\
&\quad + \left( - h^\alpha \frac{\rho_{an+1}}{\rho_{n+1}a} - h^\pi \rho_{an+1} a \right) + h^\alpha \frac{\rho_a (\rho_{n+1}a + \rho_{n+1}a)}{\rho_{n+1}a + \rho_{n+1}a} + \frac{\rho_{n+1}a}{|\rho_{n+1}|^2} - \frac{\rho_{n+1}a}{\rho_{n+1}a} \\
&= \frac{n-1}{|\partial \rho|^2} - \frac{1}{\rho_{n+1}} \partial_{n+1} \log J(\rho) + \frac{\rho_{n+1}a}{\rho_{n+1}^2} + \frac{\rho_{n+1}a h_{n+1}^p \rho_{n+1}a}{|\rho_{n+1}|^2}.
\end{align*} \]
\[ \begin{aligned}
&\left(-\frac{h^\alpha}{\rho_{n+1}}\rho_{\alpha n+1} + \frac{h^\alpha}{\rho_{n+1}}\rho_{n+1\alpha}\right) + \frac{h^\alpha}{\rho_{n+1}}\rho_{\alpha n+1} \frac{\rho_{n+1} + 1}{\rho_{n+1}} + \frac{\rho_{n+1} + 1}{\rho_{n+1}^2} - \frac{\rho_{n+1} + 1}{\rho_{n+1}^2} \\
&= \frac{n-1}{|\partial \rho|^2} - \frac{1}{\rho_{n+1}} \partial_{n+1} \log J(\rho) + 2\frac{\rho_{n+1} + 1}{|\rho_{n+1}|^2} + E(\rho)
\end{aligned} \]

where

\[(2.37) \quad E(\rho) = \sum_{\alpha=1}^n \left(-\frac{h^\alpha}{\rho_{n+1}}\rho_{\alpha n+1} + \frac{h^\alpha}{\rho_{n+1}}\rho_{n+1\alpha}\right) + (h^\alpha \rho_{\alpha} - h^\alpha \rho_{\alpha}) \frac{\rho_{n+1} + 1}{|\rho_{n+1}|^2} \]

is a pure imaginary number.

For 1 ≤ p, q ≤ n + 1, since

\[ B_{p\bar{q}} = \sum_{\alpha, \beta=1}^n h_{\alpha\beta}^\alpha \rho_{\alpha n+1} \rho_{\beta n+1} \]

\[ = \sum_{\alpha, \beta=1}^n \left(\rho_{\alpha n+1\beta} - \frac{\rho_{\alpha n+1}\rho_{n+1\beta}}{\rho_{n+1}}\right) - \frac{\rho_{\alpha n+1\beta}(\rho_{\alpha n+1} + 1)}{|\rho_{n+1}|^2} \rho_{n+1} \rho_{n+1\beta} \]

\[ = \sum_{\beta=1}^n \rho_{\alpha n+1\beta} (\delta_{\alpha n+1\beta} - \rho_{n+1\beta}) - \sum_{\alpha=1}^n \rho_{\alpha n+1\beta} \left(\delta_{\alpha n+1\beta} - \rho_{n+1\beta} \rho_{n+1\beta}\right) \rho_{\alpha n+1\beta} \]

\[ - \sum_{\beta=1}^n \rho_{\alpha n+1\beta} (\delta_{n+1\beta} - \rho_{n+1\beta} \rho_{n+1\beta}) \rho_{\alpha n+1\beta} + \frac{\rho_{n+1\beta}(\rho_{n+1\beta} + 1)}{|\rho_{n+1}|^2} \rho_{n+1} \rho_{n+1\beta} \rho_{\alpha n+1\beta} \]

we have

\[ \sum_{\alpha=1}^n h^\alpha h_{\alpha n+1} |\partial \rho|^4 = \sum_{\alpha, \beta=1}^n h_{\alpha\beta}^\alpha h_{\beta n+1} |\partial \rho|^4 \]

\[ = \sum_{p, q=1}^{n+1} B_{p\bar{q}} \rho_{p\bar{q}} \]

\[ = \sum_{p, q=1}^{n+1} \rho_{p\bar{q}} \sum_{\beta=1}^n \rho_{p\bar{q}} (\delta_{\beta n+1\beta} - \rho_{n+1\beta} \rho_{n+1\beta}) \]

\[ - \sum_{p, q=1}^{n+1} \rho_{p\bar{q}} \sum_{\alpha=1}^n \rho_{p\alpha} \left(\delta_{p n+1\alpha} - \rho_{n+1}\rho_{n+1\alpha}\right) \rho_{p\alpha} \]

\[ - \sum_{p, q=1}^{n+1} \rho_{p\bar{q}} \sum_{\beta=1}^n \frac{\rho_{p n+1\beta}}{\rho_{n+1}} (\delta_{n+1\beta} - \rho_{n+1\beta}) \rho_{p n+1\beta} \]

\[ 15 \]
\[ + \sum_{\rho, \eta = 1}^{n} \rho_{\rho} \psi_{\eta} \frac{\rho_{n+1 \eta}^{\eta+1}}{\rho_{n+1}^{\eta+1}} - \rho_{n+1}^{\eta+1} \rho_{n+1}^{\eta+1} + \rho_{n+1}^{\eta+1} \rho_{n+1}^{\eta+1} \]

\[ = \sum_{\beta = 1}^{n} \rho_{\beta}^{\beta} \rho_{\beta}^{\beta} - \rho_{n+1}^{\eta+1} \rho_{n+1}^{\eta+1} \]

\[ - \sum_{\alpha = 1}^{n} \rho_{\alpha}^{\alpha} \rho_{\alpha}^{\alpha} + \sum_{\alpha = 1}^{n} \rho_{\beta}^{\beta} \rho_{\beta}^{\beta} \frac{\rho_{n+1}^{\eta+1}}{\rho_{n+1}^{\eta+1}} \]

\[ = - |\partial \rho|^2 + \frac{\rho_{n+1}^{\eta+1}}{\rho_{n+1}^{\eta+1}} \rho_{n+1}^{\eta+1} \rho_{n+1}^{\eta+1} \]

\[ + \left[ \sum_{\alpha = 1}^{n} \rho_{\alpha}^{\alpha} \rho_{n+1}^{\eta+1} \rho_{n+1}^{\eta+1} \sum_{\beta = 1}^{n} \rho_{\beta}^{\beta} \rho_{n+1}^{\eta+1} \rho_{n+1}^{\eta+1} \right] \frac{\rho_{n+1}^{\eta+1}}{\rho_{n+1}^{\eta+1}} \frac{\rho_{n+1}^{\eta+1}}{\rho_{n+1}^{\eta+1}} \]

\[ = - |\partial \rho|^2 + |\partial \rho|^4 \frac{\rho_{n+1}^{\eta+1}}{\rho_{n+1}^{\eta+1}} \frac{\rho_{n+1}^{\eta+1}}{\rho_{n+1}^{\eta+1}} \]

Therefore

\[ Y^\alpha(h_\alpha) + h^\alpha Y_\alpha g(z) - 2h^\alpha h_\alpha + \frac{\rho_{n+1}^{\eta+1}}{\rho_{n+1}^{\eta+1}} \frac{\rho_{n+1}^{\eta+1}}{\rho_{n+1}^{\eta+1}} \]

\[ = \frac{n-1}{|\partial \rho|^2} - \frac{1}{\rho_{n+1}} \partial_{n+1} \log J(\rho) \]

\[ + 2 \frac{\rho_{n+1}^{\eta+1}}{\rho_{n+1}^{\eta+1}} + \frac{2}{|\partial \rho|^2} - 2 \frac{\rho_{n+1}^{\eta+1}}{\rho_{n+1}^{\eta+1}} + E(\rho) \]

\[ = \frac{n+1}{|\partial \rho|^2} - \frac{1}{\rho_{n+1}} \partial_{n+1} \log J(\rho) + E(\rho) \]

Since \( R^\alpha_{\alpha} = \overline{R^\alpha_{\alpha}}, \) \( h_{\alpha \beta} = \overline{h_{\alpha \beta}} \) and \( E(\rho) = -E(\rho) \) by (2.37), we have

\[ (2.38) \quad R_{\alpha \gamma} = \frac{1}{2} (R_{\alpha \gamma} + \overline{R_{\alpha \gamma}}) \]
\[
Y_\gamma Y_\gamma = - \frac{1}{2} (Y_\gamma Y_\gamma + Y_\gamma Y_\gamma^\ast) \log J(\rho) \\
+ \left( \frac{n + 1}{|\partial \rho|^2} - \frac{1}{2} \left( \frac{1}{\rho_{n+1}} \partial_{n+1} + \frac{1}{\rho_{n+1}} \partial_{n+1}^\ast \right) \log J(\rho) \right) h_{\gamma \gamma}.
\]

Since
\[
Y_\gamma Y_\gamma = \frac{\partial^2}{\partial \bar{z}^\gamma \partial z^\gamma} - \frac{\partial}{\partial \bar{z}^\gamma} \left( \frac{\rho_\gamma}{\rho_{n+1}} \partial_{n+1} \right) - \frac{\rho_\gamma}{\rho_{n+1}} \frac{\partial^2}{\partial \bar{z}^\gamma} \partial_{n+1} + \frac{\rho_\gamma}{\rho_{n+1}} \partial_{n+1} \left( \frac{\rho_\gamma}{\rho_{n+1}} \partial_{n+1} \right)
\]
\[
- \frac{\partial^2}{\partial \bar{z}^\gamma} \partial_{n+1} \frac{\rho_{n+1}}{\rho_{n+1}} \partial_{n+1} - \frac{\rho_\gamma}{\rho_{n+1}} \frac{\rho_{n+1}}{\rho_{n+1}} \partial_{n+1} \partial_{n+1} + \frac{\rho_\gamma}{\rho_{n+1}} \partial_{n+1} \frac{\rho_{n+1}}{\rho_{n+1}} \partial_{n+1} \partial_{n+1} \partial_{n+1} \partial_{n+1} \partial_{n+1} \partial_{n+1}
\]
\[
\frac{\partial^2}{\partial \bar{z}^\gamma} \partial_{n+1} + \frac{\rho_\gamma}{\rho_{n+1}} \frac{\rho_{n+1}}{\rho_{n+1}} \partial_{n+1} \partial_{n+1} \partial_{n+1} \partial_{n+1} \partial_{n+1} \partial_{n+1}
\]
we have
\[
(2.39) \quad Y_\gamma Y_\gamma = - \frac{h_{\gamma \gamma}}{\rho_{n+1}} \partial_{n+1} + D_{\gamma \gamma}^p
\]
where
\[
(2.40) \quad D_{\gamma \gamma}^p = \frac{\partial^2}{\partial \bar{z}^\gamma \partial z^\gamma} - \frac{\rho_\gamma}{\rho_{n+1}} \frac{\partial^2}{\partial \bar{z}^\gamma \partial z^\gamma} - \frac{\rho_\gamma}{\rho_{n+1}} \frac{\partial^2}{\partial \bar{z}^\gamma \partial z^\gamma} \partial_{n+1} \partial_{n+1} \partial_{n+1} \partial_{n+1} \partial_{n+1} \partial_{n+1} \partial_{n+1} \partial_{n+1} \partial_{n+1} \partial_{n+1}
\]
Then
\[
(2.41) \quad h_{\alpha \beta} = D_{\alpha \beta}^p(\rho), \quad \overline{D_{\beta \alpha}^p} = D_{\alpha \beta}^p
\]
and
\[
(2.42) \quad Y_\beta Y_\beta + Y_\beta Y_\beta^\ast = - \frac{h_{\alpha \beta}}{\rho_{n+1}} \partial_{n+1} - \frac{h_{\alpha \beta}}{\rho_{n+1}} \partial_{n+1} + 2 D_{\alpha \beta}^p.
\]
By (2.38) and (2.42), we have
\[
(2.43) \quad R_{\alpha \beta} = - D_{\alpha \beta}^p \log J(\rho) + (n + 1) \frac{\det H(\rho)}{J(\rho)} D_{\alpha \beta}^p(\rho)
\]
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since $J(\rho) = |\partial \rho|^2 (\det H(\rho) - \rho)$.

In order to prove (1.6), for any $z_0 \in M_1$, we let
\[ w = \sum_{j=1}^{n+1} w^j \frac{\partial}{\partial z^j}, \quad v = \sum_{j=1}^{n+1} v^j \frac{\partial}{\partial z^j} \in (T_{1,0})_{z_0}(M) \]
Then
\[ \sum_{j=1}^{n+1} w^j \rho_j(z_0) = \sum_{j=1}^{n+1} v^j \rho_j(z_0) = 0. \]
It is easy to see that
\[ w = \sum_{j=1}^n w^j Y_j, \quad v = \sum_{j=1}^n v^j Y_j. \]
Thus
\[ \text{Ric}(w, \nu)(z_0) \]
\[ = \sum_{\alpha, \beta = 1}^n R_{\alpha \beta} w^\alpha \nu^\beta \]
\[ = - \sum_{\alpha, \beta = 1}^n w^\alpha \nu^\beta \partial_{\alpha \beta} \log J(\rho) + (n + 1) \frac{\det H(\rho)}{J(\rho)} \sum_{\alpha, \beta = 1}^n h_{\alpha \beta} w^\alpha \nu^\beta \]
Applying (2.45), we have
\[ \sum_{\alpha, \beta = 1}^n w^\alpha \nu^\beta \partial_{\alpha \beta} \]
\[ = \sum_{\alpha, \beta = 1}^n w^\alpha \nu^\beta \frac{\partial^2}{\partial z^\alpha \partial z^\beta} - \sum_{\alpha, \beta = 1}^n \frac{\rho}{\rho_{n+1}} w^\alpha \nu^\beta \frac{\partial^2}{\partial z^{n+1} \partial z^{n+1}} - \sum_{\alpha, \beta = 1}^n \frac{\rho}{\rho_{n+1}} w^\alpha \nu^\beta \frac{\partial^2}{\partial z^{n+1} \partial z^{n+1}} \]
\[ = \sum_{\alpha, \beta = 1}^n w^\alpha \nu^\beta \frac{\partial^2}{\partial z^\alpha \partial z^\beta} + \sum_{\alpha = 1}^n w^{\alpha n+1} \frac{\partial^2}{\partial z^\alpha \partial z^{n+1}} + w^{n+1} \frac{\partial^2}{\partial z^{n+1} \partial z^{n+1}} \]
\[ = \sum_{k,j=1}^{n+1} w^k \nu^j \frac{\partial^2}{\partial z^k \partial z^j} \]
Similarly,

\[(2.48) \quad \sum_{\alpha, \beta = 1}^{n} h_{\alpha \beta} w^\alpha w^\beta = \sum_{k,j=1}^{n+1} \frac{\partial^2 \rho(z_0)}{\partial z^k \partial \overline{z}^j} w^k w^j.\]

Combining (2.46), (2.47) and (2.48), we have proved the formula (1.6).

In particular, if \(J(\rho) = 1\) on \(D\) then \(Y_\gamma^T Y_\gamma (\log J(\rho)) = 0\), and

\[(2.49) \quad R_{\gamma\tau}(z_0) = (n + 1) \det H(\rho) h_{\gamma\tau}(z_0).\]

This with (2.46) and (2.48) gives the formula (1.7). Therefore, Theorem 1.1 is proved when \(H(\rho)\) is positive definite at \(z_0 \in M_1\).

Next we consider the case when \(H(\rho)\) may not be positive definite. Instead of \(\rho\), we use \(r(z) = \rho + c \rho^2\) with \(c > 0\) being chosen so that \(H(r)\) is positive definite on \(M\). Moreover, the contact form \(\theta\) remains the same if we use \(r\) replacing \(\rho\). Then (1.6) holds by replacing \(\rho\) by \(r\). We will show the right side is the same for \(\rho\) and for \(r\). We know \(J(r) = J(\rho)\) on \(M\). However, \(J(r)\) will be different in \(D\). Notice that

\[J(r) = |r_{n+1}|^2 \det(h_{\alpha\beta}(r)) - r(z) \det(H(\rho))\]

and

\[h_{\alpha\beta}(r) = \frac{r_{\alpha\beta} - \frac{r_{\alpha n+1} r_{\beta n+1}}{r_{n+1}} r_{\alpha} r_{\beta} + r_{n+1} r_{\alpha} r_{\beta}}{|r_{n+1}|^2} \]

\[= \frac{(1 + 2c \rho) \rho_{\alpha\beta} + 2c \rho_{\alpha} \rho_{\beta} - (1 + 2c \rho) \rho_{\alpha n+1} \rho_{\beta n+1} + 2c \rho_{\alpha} \rho_{\beta n+1} (1 + 2c \rho) \rho_{\beta}}{(1 + 2c \rho) \rho_{n+1}}\]

\[- \frac{(1 + 2c \rho) \rho_{n+1 n+1} r_{\alpha} + 2c \rho_{n+1} \rho_{\alpha n+1} (1 + 2c \rho) \rho_{\beta}}{(1 + 2c \rho) \rho_{n+1}}\]

\[= \frac{(1 + 2c \rho) \rho_{\alpha n+1} + 2c \rho_{n+1} \rho_{\alpha}}{(1 + 2c \rho) \rho_{n+1}} - \frac{(1 + 2c \rho) \rho_{\alpha n+1} + 2c \rho_{n+1} \rho_{\alpha}}{(1 + 2c \rho) \rho_{n+1}} \rho_{\beta}^2 \]

\[= \frac{(1 + 2c \rho) \rho_{\alpha n+1} + 2c \rho_{n+1} \rho_{\alpha}}{(1 + 2c \rho) \rho_{n+1}} - \frac{(1 + 2c \rho) \rho_{\alpha n+1} + 2c \rho_{n+1} \rho_{\alpha}}{(1 + 2c \rho) \rho_{n+1}} \rho_{\beta}^2 \]

\[= (1 + 2c \rho) h_{\alpha\beta}(\rho).\]
Thus

\[ J(r) = (1 + 2c\rho)^{n+2} |\rho_{n+1}|^2 \det(h_{\alpha\beta}(\rho)) - \rho(1 + c\rho) \det H(\rho + c\rho^2) \]

\[ = (1 + 2c\rho)^{n+2} J(\rho) + (1 + 2c\rho)^{n+2} \rho(z) \det(H(\rho)) - \rho(1 + c\rho) \det H(\rho + c\rho^2) \]

\[ = (1 + 2c\rho)^{n+2} J(\rho) + \rho[\det H(\rho) - \det H(\rho + c\rho^2)] + O(\rho^2) \]

\[ = J(\rho) + \rho [2(n + 2)cJ(\rho) + \det H(\rho) - \det H(\rho + c\rho^2)] + O(\rho^2) \]

It is easy to see from the definition that on \( M \) we have

\[ \mathcal{D}_r^{\alpha\beta} = \mathcal{D}_r^{\alpha\beta} \]

and

\[ \mathcal{D}_r^{\alpha\beta} \log J(r) = \mathcal{D}_r^{\alpha\beta} \log J(\rho) + (2(n + 2)c + \frac{\det H(\rho) - \det H(\rho + c\rho^2)}{J(\rho)}) \mathcal{D}_r^{\alpha\beta}(\rho) \]

\[ R_{\alpha\beta}(\tau)(\tau_0) = -\mathcal{D}_r^{\alpha\beta} \log J(r) + (n + 1) \frac{\det H(r)}{J(r)} h_{\alpha\beta} \]

\[ = -\mathcal{D}_r^{\alpha\beta} \log J(\rho) - 2(n + 2)c h_{\alpha\beta} + \frac{\det H(r)}{J(\rho)(\tau_0)} h_{\alpha\beta}(\tau) \]

\[ + (n + 1) \frac{\det H(r)}{J(\rho)} h_{\alpha\beta}(\tau) \]

\[ = -\mathcal{D}_r^{\alpha\beta} \log J(\rho) + (n + 1) \frac{\det H(r)}{J(\rho)} h_{\alpha\beta}(\tau) \]

\[ + \frac{(n + 2)}{J(\rho)} [\det H(\rho) - \det H(\rho + 2c \rho) - 2cJ(\rho)(\tau_0)] h_{\alpha\beta}(\tau_0) \]

\[ = -\mathcal{D}_r^{\alpha\beta} \log J(\rho) + (n + 1) \frac{\det H(\rho)}{J(\rho)} h_{\alpha\beta}(\tau_0) \]

provided

(2.50) \[ \phi(c) = \det H(\rho + c\rho^2) - \det H(\rho) - 2cJ(\rho) \equiv 0. \]

Let \( \tau_0 \in M_1 \). Then we will prove (2.50) holds in the following two cases:

a) If \( \det H(\rho)(\tau_0) \neq 0 \) then

\[ \det H(r)(\tau_0) = \det(H(\rho) + 2c\partial \rho \otimes \bar{\rho}) = \det H(\rho)(1 + 2c|\partial \rho|^2) \]
where $|d\rho|^2 = \rho^T \rho_i \rho_j(z_0)$ (it may not be positive). Thus

$$
\phi'(c) = \det H(\rho)(z_0)2|\partial \rho|^2 - 2J(\rho) = 2(J(\rho) - J(\rho)) = 0
$$

Thus

$$
\phi(c) = \phi(0) = 0
$$

b) If $\det H(\rho)(z_0) = 0$ then $\det H(\rho + \epsilon \rho^2)(z_0) \neq 0$ for any $\epsilon > 0$. Thus

$$
\phi(c) = \det H(\rho + \epsilon \rho^2 + (c - \epsilon)\rho^2) - 2cJ(\rho) - \det H(\rho)
$$

Thus

$$
\phi'(c) = 2J(\rho + \epsilon \rho^2) - 2J(\rho), \quad \text{for any small } \epsilon > 0.
$$

Let $\epsilon \to 0^+$, we have $\phi'(c) = 0$. Thus $\phi(c) = \phi(0) = 0$. The proof of the theorem is complete. \Box

Furthermore, we will show the pseudo scalar curvatures are the same for those $\rho$ with $J(\rho) = 1$ on $M$ and $\rho^0$ with $J(\rho^0) = 1$ in a neighborhood of $M$ we have the following stronger result.

**Corollary 2.2** Let $D$ be a strictly pseudoconvex domain in $\mathbb{C}^{n+1}$ with smooth boundary $M$. If $\rho^0$ is the defining function of $D$ with $J(\rho^0) \equiv 1$ in a neighborhood $U$ of $M$. Let $\rho \in C^0(\overline{D})$ is defining function for $D$ with $J(\rho) > 0$ on $\overline{D} \cap U$ and $\theta = (\partial \rho - \overline{\partial} \rho)/(2i)$. Then for $v, w \in T_{1,0}(M)$, we have two conclusions:

(a) If $J(\rho) = 1$ on $M$ or $J(\rho) = 1 + O(\rho^0)$ then

$$(2.51) \quad \text{Ric}(w, \overline{v}) = (n + 1) \det H(\rho^0)L_\theta(w, \overline{v})$$

(b) If $\log J(\rho)$ is pluriharmonic near $M$, then

$$(2.52) \quad \text{Ric}(w, \overline{v}) = (n + 1) \frac{\det H(\rho)}{J(\rho)}L_\theta(w, \overline{v}).$$

**Proof.** We can write $\rho = a(z)\rho^0$ with $a \neq 0$ on $M$. Since $1 = J(\rho) = J(a\rho^0) = a^{n+2} J(\rho^0)$ on $M$, we have $a \equiv 1$ on $M$. Thus we can write $a(z) = 1 + b\rho^0(z)$ and

$$(2.53) \quad \rho(z) = \rho^0(z) + b(z)\rho^0(z)^2$$
It is not difficult to see that the last argument of the proof of the last theorem remains true when $c$ is a function (or since $\theta(\rho) = \theta(\rho^0)$ on $M$, they must have the same Ricci curvature). Therefore,

$$R_{\alpha\beta} = (n + 1) \det H(\rho) h_{\alpha\beta}.$$ 

This with (2.46) and (2.48) give (2.51). Moreover, (2.52) follows directly from (1.6) which we have proved. Therefore, the proof of the corollary is complete. 

3 Proof of Theorem 1.2

In order to prove Theorem 1.2, we recall a theorem proved by Li (see Theorem 1.1 in [27]). Let

$$U(z) = -\log(-\rho(z)), \quad z \in D \subset \mathbb{C}^{n+1}$$

be strictly plurisubharmonic in $D$. Let

$$|\partial U|^2 = \sum_{i,j=1}^{n+1} U^\alpha U_i U_{\alpha j}, \quad H(U) = [U^\alpha]_{n \times n}, \quad [U^\alpha] = (H(U)^t)^{-1}$$

Then

$$\det H(\rho) = \det H(-e^{-U})$$

$$= e^{-(n+1)U} \det(H(U) - \partial U \otimes \bar{\partial} U)$$

$$= e^{-(n+1)U} \det(H(U))(1 - |\partial U|^2)$$

and

$$J(\rho) e^{(n+2)U} = -\det\left[ \begin{array}{cc} -1 & \bar{\partial} U \\ (\partial U)^* & H(U) - \partial U \otimes \bar{\partial} U \end{array} \right] = \det H(U)$$

The following theorem is proved in [27]:

THEOREM 3.1 Let $D$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Let $\rho \in C^3(D)$ be a defining function for $D$ with $J(\rho) >$
0 and \( \log J(\rho) \) being pluriharmonic in \( D \). Then the following conclusions holds:

(i) The function \( \det H(\rho)/J(\rho) \) attains its minimum over \( \partial D \) at some point in \( \partial D \).

(ii) If

\[
\det H(\rho)/J(\rho) \geq \det H(\rho)(z_0)/J(\rho)(z_0), \quad \text{on } \partial D,
\]

for some \( z_0 \in D \) satisfying \( \rho(z_0) = \min \{ \rho(z) : z \in \overline{D} \} \), then \( D \) is biholomorphically equivalent to the unit ball in \( \mathbb{C}^n \).

(iii) In addition to the condition (3.5), if we assume that \( J(\rho) \equiv 1 \) on \( D \) then \( D \) is biholomorphically equivalent to the unit ball in \( \mathbb{C}^n \) with a constant Jacobian biholomorphic map.

We are ready to prove Theorem 1.2.

**Proof.** Let \( \rho_0(z) \) be the plurishubharmonic solution of the Monge-Ampère equation:

\[
\det H(\rho_0) = J(\rho) \quad \text{in } D, \quad \rho_0 = |z|^2 \quad \text{on } \partial D.
\]

In particular, when \( J(\rho) = 1 \), we have \( \rho_0(z) = |z|^2 \). Let \( \theta_0(z) = (1/2i)(\partial \rho_0 - \overline{\partial} \rho_0) \). Let \( \theta = (\partial \rho - \overline{\partial} \rho)/(2i) \). Since \( \log J(\rho) \) is pluriharmonic, by the formula (2.43), we have

\[
R = n(n + 1)\det H(\rho)/J(\rho), \quad \text{on } \partial D.
\]

Combining this with the assumption (1.9), we have

\[
\frac{\det H(\rho)}{J(\rho)} \geq \frac{\int_{\partial D} \theta \wedge (d\theta)^n}{\int_{\partial D} \theta_0 \wedge (d\theta_0)^n}, \quad \text{on } \partial D.
\]

We claim that

\[
\frac{\det H(\rho)}{J(\rho)} = \text{constant on } \partial D.
\]

In fact, since \( \log J(\rho) \) is pluriharmonic, by main theorem in [27] stated in Theorem 3.1, we have that \( \frac{\det H(\rho)}{J(\rho)} \) attains its minimum over \( \partial D \) at some point
in \( \partial D \). Thus, (3.8) holds for all \( z \in \overline{D} \). Suppose that (3.9) fails, then there exist \( z_0 \in D \) and \( \delta > 0 \) such that

\[
(3.10) \quad \det H(\rho) > \frac{\int_{\partial D} \theta \wedge (d\theta)^n}{\int_{\partial D} \theta_0 \wedge (d\theta_0)^n} \quad \text{in } B(z_0, \delta).
\]

Let

\[
(3.11) \quad \theta_s = \frac{1}{2i} \sum_{j=1}^{n+1} (\overline{\omega}^j dz^j - z^j d\overline{\omega}^j),
\]

Then by Stoke’s Theorem

\[
(3.12) \quad \int_{\partial D} \theta_0 \wedge (d\theta_0)^n = \int_D (d\theta_0)^{n+1} = \int_D \det H(\rho_0)(d\theta_s)^{n+1} = \int_D J(\rho)(d\theta_s)^{n+1}.
\]

By the fact that (3.8) holds for all \( z \in D \) and Stokes Theorem

\[
C = \int_{\partial D} \theta \wedge (d\theta)^n \\
= \int_D (d\theta)^{n+1} \\
= \int_D \det H(\rho)(d\theta_s)^{n+1} \\
> \frac{\det H(\rho)}{J(\rho)} \int_D J(\rho)(d\theta_s)^{n+1} \\
= \frac{C}{J(\rho)} \int_D \theta_0 \wedge (d\theta_0)^n \\
= \int_{\partial D} \theta_0 \wedge (d\theta_0)^n \\
= C.
\]

This is a contradiction. Therefore

\[
(3.13) \quad \det H(\rho) = \text{constant} = \frac{\int_{\partial D} \theta \wedge (d\theta)^n}{\int_{\partial D} \theta_0 \wedge (d\theta_0)^n}.
\]

By Theorem 3.1 and (3.13), there is a biholomorphic map \( \phi \) from \( D \) onto \( B_{n+1} \), the unit ball in \( \Phi^{n+1} \).

In addition that \( J(\rho) \equiv 1 \), then \( \rho_0(z) = |z|^2 \), and Theorem 3.1 and (3.13) implies that there is a biholomorphic map \( \phi : D \to B_{n+1} \) so that

\[
\det \phi'(z) \equiv c, \text{ a constant. This completes the proof of Theorem 1.2.} \]

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