Boundary Behavior of Harmonic Functions in Metrics of Bergman Type on the Polydisc

Song-Ying Li* and Ezequias Simon

AMJ, 2002

1 Introduction and main result

Let $D$ be a bounded domain in $\mathbb{C}^n$. Let $K_D$ be the Bergman kernel function of $D$ with respect to the Lebesgue measure in $D$. Then the Bergman metric $g_{ij}dz_i\overline{dz}_j$ is defined as

$$(1.1) \quad g_{ij} = \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log K(z, z), \quad z \in D.$$ 

The Laplace-Beltrami operator with respect to the Bergman metric is defined as

$$(1.2) \quad \tilde{\Delta}_D = \frac{2}{g} \sum_{i,j=1}^{n} \left[ \frac{\partial}{\partial z_i} (gg^\overline{\gamma} \frac{\partial}{\partial \overline{z}_j}) + \frac{\partial}{\partial \overline{z}_j} (gg^\overline{\gamma} \frac{\partial}{\partial z_i}) \right] = 4 \sum_{i,j=1}^{n} g^{\overline{\gamma}} \frac{\partial^2}{\partial z_i \partial \overline{z}_j}.$$ 

Here $g = \det(g_{ij})$ and $(g^{\overline{\gamma}})$ is the inverse matrix of $(g_{ij})^t$. A function $u$ is said to be harmonic in the Bergman metric in $D$ if $\Delta_D u = 0$. The space of harmonic functions in the Bergman metric is invariant under composition by biholomorphic mappings, and it is closely related to, and includes the space of holomorphic functions on $D$. When $D$ is the unit ball $B_n$ in $\mathbb{C}^n$, we know that

$$K_{B_n}(z, z) = \frac{n!}{\pi^n (1 - |z|^2)^{n+1}}, \quad z \in B_n$$

*The author is supported in part by NSF Grant
and
\[ \tilde{\Delta}_{B_n} = \frac{4}{n+1} (1 - |z|^2) \sum_{i,j=1}^{n} (\delta_{ij} - z_i \overline{z}_j) \frac{\partial^2}{\partial z_i \partial \overline{z}_j}, \quad z \in B_n. \]

Thus \( u \) is harmonic in the Bergman metric in \( B_n \) (or invariant harmonic in \( B_n \)) if and only if \( \tilde{\Delta}_{B_n} u(z) = 0 \) in \( B_n \). It was proved by Hua [H] that the solution of the Dirichlet boundary value problem

\[ \tilde{\Delta}_{B_n} u = 0 \quad \text{in} \; B_n; \quad u = \phi \quad \text{on} \; \partial B_n \]

can be represented by integrating \( \phi \) against the Poisson-Szegő kernel on \( \partial B_n \)

\[ u(z) = P(\phi)(z) = \int_{\partial B_n} \frac{(1 - |z|^2)^n}{1 - \langle z, w \rangle |2n|} \phi(w) d\sigma(w). \tag{1.3} \]

When \( n = 1 \), it is well-known that \( P(\phi) \in C^{k-k\epsilon}(\overline{B_n}) \) if \( \phi \in C^k(\partial B_n) \) for any \( k \geq 0 \) and any small \( \epsilon > 0 \).

When \( n > 1 \), since the coefficients \( (\delta_{ij} - z_i \overline{z}_j) \) degenerate on \( \partial B_n \), it was proved by Graham in [G] that \( P(\phi) \in C^{k-k\epsilon}(\overline{B_n}) \) if \( \phi \in C^k(\partial B_n) \) if \( k \leq n \). However, \( \phi \in C^\infty(\partial B_n) \) cannot imply that \( u \in C^n(\overline{B_n}) \). An example was discovered by Garnett and Krantz in an unpublished note (see [K, pp. 153]).

In fact, a striking phenomenon was proved by Graham (Corollary 15 in [G]) that if \( u \in C^n(\overline{B_n}) \) is harmonic in the Bergman metric in \( B_n \) then \( u \) must be pluriharmonic (the real part of a holomorphic function). The statement and proof of this theorem can also be found in Theorem 6.8.12 in [K]. It was also proved by Graham and Lee [GL] that this phenomenon may not be true for many strictly pseudoconvex domains \( D \) which do not have enough symmetry.

The main purpose of this paper is to build up the above striking phenomenon for harmonic functions in the metrics of Bergman type, including the Bergman metric, in the unit polydisc \( U^n \) in \( \mathbf{C}^n \). It is well-known that the Bergman kernel for \( U^n \) is

\[ K_{U^n}(z, w) = \frac{1}{\pi^n} \prod_{j=1}^{n} (1 - z_j \overline{w}_j)^{-2}, \quad z, w \in U^n \tag{1.4} \]

and the Bergman metric on \( U^n \) is

\[ \sum_{i,j=1}^{n} \frac{2\delta_{ij}}{(1 - |z_j|^2)^2} dz_i d\overline{z}_j. \tag{1.5} \]
A complete metric $\sum_{i,j=1}^{n} r_{ij}dz_i d\overline{z}_j$ on $U^n$ is said to be a metric of the Bergman type if
\begin{equation}
\sum_{i,j=1}^{n} r_{ij}dz_i d\overline{z}_j = \sum_{i,j=1}^{n} \frac{\delta_{ij}}{r_j(z_j)^2} dz_i d\overline{z}_j.
\end{equation}

for $n$ positive defining functions $r_j(\lambda) (j = 1, \cdots, n)$ for the unit disk $U$. Then the Laplace-Beltrami operator with respect to this metric given by (1.6) is
\begin{equation}
\mathcal{L}_r = \sum_{j=1}^{n} r_j(z_j)^2 \frac{\partial^2}{\partial z_j \partial \overline{z}_j}, \quad r = (r_1, \cdots, r_n).
\end{equation}

We say that $u$ is harmonic in a metric of Bergman type or $\mathcal{L}_r$-harmonic in $U^n$ if $\mathcal{L}_r u = 0$ on $U^n$.

It is easy to see that if $r_1(\lambda) = r_2(\lambda) = \cdots = r_n(\lambda) = 1 - |\lambda|^2$ and $u$ is $\mathcal{L}_r$ harmonic in $U^n$ then $u \circ \phi$ is also $\mathcal{L}_r$ harmonic in $U^n$ for any biholomorphic self-map $\phi$ on $U^n$. However, for most $r$, a $\mathcal{L}_r$-harmonic function in $U^n$ is not biholomorphically invariant.

Since the coefficients of $\mathcal{L}_r$ degenerate on $\partial U^n$ and the variables are independent, one expects that harmonic functions in the metric of the Bergman type in $U^n$ are more rigid when $z$ approaches the boundary $\partial U^n$ for $n > 1$. To describe this phenomenon, we need another class of functions, the so-called, strongly harmonic (or $n$-harmonic) functions which play an important and useful role (see, for examples, [H], [R] and [S]) in the theory of harmonic analysis in the unit polydisc in $\mathbb{C}^n$. A function $u$ in $U^n$ is said to be strongly harmonic (or $n$-harmonic) if $u$ is harmonic in each variable, i.e.,
\begin{equation}
\frac{\partial^2 u}{\partial z_j \partial \overline{z}_j} = 0, \quad \text{for all } j = 1, 2, \cdots, n.
\end{equation}

The strongly harmonic function $u$ is uniquely determined by its value on the Silov boundary $\partial_0 U^n = (\partial U)^n \cong [0,2\pi)^n$ (the torus) through the integral formula:
\begin{equation}
u(z) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{j=1}^{n} \frac{1 - |z_j|^2}{1 - z_j e^{-i\theta_j}} u(e^{i\theta_1}, \cdots, e^{i\theta_n}) d\theta_1 \cdots d\theta_n.
\end{equation}

In particular, if $u|_{(\partial U)^n} \in C^k(\partial U)^n$ then $u \in C^{k-k\epsilon}(\overline{U^n})$ for any non-negative integer $k \geq 0$ and $\epsilon > 0$. 

3
It is obvious that any strongly harmonic function in $U^n$ is harmonic in any metric of Bergman type in $U^n$ and that the two coincide when $n = 1$. The converse is not true in general. When $n > 1$, the following example is constructed by M. Stoll in [S].

**EXAMPLE 1** Let $\alpha_i = \lambda_i + 1/2$ for $i = 1, 2$ and let

$$u(z_1, z_2) = P_1(z_1, e^{i\theta_1})^{\alpha_1} P_2(z_2, e^{i\theta_2})^{\alpha_2}$$

where $P_j$ is the Poisson kernel for the unit disc given by

$$P_1(\lambda, \eta) = P_2(\lambda, \eta) = (1 - |\lambda|^2)(1 - |\lambda\eta|^{-2}), \quad \lambda, \eta \in U.$$

Then

$$\tilde{\Delta}_j P_j(z_j, e^{i\theta_j})^{\alpha_j} = 2\alpha_j(\alpha_j - 1)P_j(z_j, e^{i\theta_j})^{\alpha_j},$$

where $\tilde{\Delta}_j = 2(1 - |z_j|^2)^2 \frac{\partial}{\partial z_j \partial \overline{z_j}}$; and

$$\tilde{\Delta}u(z_1, z_2) = 2(\alpha_1(\alpha_1 - 1) + \alpha_2(\alpha_2 - 1))u(z_1, z_2) = 2(\lambda_1^2 + \lambda_2^2 - \frac{1}{2})u(z_1, z_2),$$

where $\tilde{\Delta} = \sum_{j=1}^{n} \tilde{\Delta}_j$. Thus $u$ is harmonic in the Bergman metric if $\lambda_1^2 + \lambda_2^2 = 1/2$. Now if $\lambda_1$ and $\lambda_2$ are different from $1/2$ and $\lambda_1^2 + \lambda_2^2 = 1/2$, then $u$ is harmonic in the Bergman metric, but not strongly harmonic.

To state our theorem, we let $r(\lambda) = (r_1(\lambda), \cdots, r_n(\lambda))$ satisfy that

\[(1.10) \quad r_j(\lambda) > 0, \quad \lambda \in U; \quad r_j(\lambda) = 0 \quad \text{on} \quad \partial U.\]

The main result of this paper is the following theorem.

**THEOREM 1.1** Let $r(\lambda) = (r_1(\lambda), \cdots, r_n(\lambda)) \in C^2(\overline{U}, \mathbb{R}^n)$ satisfy (1.10). If $u \in C(\overline{U}^n)$ is $L_r$-harmonic in $U^n$ then $u$ is strongly harmonic in $U^n$.

In particular, if $r_j(\lambda) = 1 - |\lambda|^2$ ($j = 1, 2, \cdots, n$) then

**Corollary 1.2** Let $u \in C(\overline{U}^n)$ be harmonic in $U^n$ in the Bergman metric. Then $u$ is strongly harmonic.
2 Preliminary

For any \( r = (r_1, r_2, \ldots, r_n) \in C^2(\overline{U}, \mathbb{R}^n) \) satisfying (1.10), we let

\[
L_j = r_j(z_j) \frac{\partial^2}{\partial z_j \partial \overline{z}_j}, \quad j = 1, 2, \ldots, n.
\]

Then

\[
L_r = \sum_{j=1}^{n} L_j.
\]

Let \( W^{k,p}(D) \) be the usual Sobolev space over a bounded domain \( D \subset \mathbb{C}^n \).
Let \( \tilde{W}^{k,p}(U^n) \) be a weighted Sobolev space type with a weaker norm

\[
\|f\|_{\tilde{W}^{k,p}(U^n)} = \sum_{|\alpha| + |\beta| \leq k} \int_{U^n} \left| \frac{\partial^{\alpha+\beta} f(z)}{\partial z^\alpha \partial \overline{z}^\beta} \right| r(z) p^{p(\alpha + \beta)} \, dv(z).
\]

We will start with the following theorem.

**Theorem 2.1** Let \( r = (r_1, \ldots, r_n) \in C^2(\overline{U}, \mathbb{R}^n) \) satisfy (1.10). Let \( u \in C^2(U^n) \) be a function in \( U^n \) such that

\[
L_r u = f \quad \text{in} \quad U^n.
\]

Then for any non-negative integer \( k \), there is a constant \( C_k > 0 \) such that

\[
\sum_{|\alpha| + |\beta| = k} \left| \frac{\partial^k u(z)}{\partial z^\alpha \partial \overline{z}^\beta} \right| r(z) \prod_{j=1}^{n} r_j(z_j)^{\alpha_j + \beta_j} \leq C_k \left[ \|u\|_{L^\infty(U^n)} + C_\epsilon \sum_{|\alpha| + |\beta| = 0}^{k-2} \left| \frac{\partial^{\alpha+\beta} f(z)}{\partial z^\alpha \partial \overline{z}^\beta} \right| r(z) \prod_{j=1}^{n} r_j(z_j)^{\alpha_j + \beta_j} \right] \||C^\infty(U^n)||. \]

where \( C_\epsilon \) is a constant depending only on \( \epsilon \) for any \( 0 < \epsilon \leq 1 \). Moreover, for any \( 1 < p < \infty \), and \( 2 \leq k < \infty \), there is a constant \( C_{k,p} \) depending only on \( k \) and \( p \) so that

\[
\|u\|_{\tilde{W}^{k,p}} \leq C_{k,p} \left( \|u\|_{L^p} + \|f\|_{\tilde{W}^{k-2,p}} \right).
\]
Proof. Since \( r_j \in C^1(\mathcal{U}) \) satisfies (1.10), there is a constant \( C \geq 1 \) such that
\[
\frac{1}{C} \leq \frac{r_j(\lambda)}{1 - |\lambda|} \leq C, \quad \lambda \in \mathcal{U}, \quad j = 1, \cdots, n.
\]
For each \( z \in U^n \) and \( \lambda \in B_n \), we let
\[
\psi_z(\lambda) = \left( z_1 + \frac{\lambda_1 r_j(z_j)}{4C}, \cdots, z_n + \frac{\lambda_n r_n(z_n)}{4C} \right)
\]
and
\[
h(z, \lambda) = u \circ (\psi_z(\lambda))
\]
If we let \( w = \psi_z(\lambda) \) then
\[
r_j(w_j) = r_j(z_j + \frac{\lambda_j r_j(z_j)}{4C}) = r_j(z) \left[ \frac{r_j(z_j + \frac{\lambda_j r_j(z_j)}{4C})}{r_j(z_j)} \right] = r_j(z_j) a_j(z, \lambda)
\]
and \( a_j \in C^1(\overline{U}^n \times B_n) \) with the estimate
\[
\frac{1}{2C^2} \leq a_j(z, \lambda) \leq 2C^2
\]
uniformly for \( (z, \lambda) \in U^n \times B_n \). Let
\[
L(z, \lambda) = \sum_{j=1}^{n} a_j(z, \lambda)^2 \frac{\partial^2}{\partial \lambda_j \partial \lambda_j} u(z, \lambda).
\]
Then
\[
L(z, \lambda) h(z, \lambda) = \sum_{j=1}^{n} a_j(z, \lambda)^2 \frac{r_j(z_j)^2}{16C^2} \frac{\partial^2 u(\psi_z(\lambda))}{\partial w_j \partial \overline{w}_j}
\]
\[
= \frac{1}{16C^2} \sum_{j=1}^{n} r_j(w_j)^2 \frac{\partial^2 u(w)}{\partial w_j \partial \overline{w}_j}
\]
\[
= \frac{1}{16C^2} f(\psi_z(\lambda)).
\]

Let \( G^z(\lambda, \eta) \) be the Green’s function for \( L(z, \lambda) \) and \( P^z(\lambda, \eta) \) the Poisson kernel on \( B_n \times \partial B_n \) for \( L(z, \lambda) \). Then \( P^z(\lambda, \eta) \in C^2(B_n \times \partial B_n) \) and \( \int_{\partial B_n} P^z(\lambda, \eta) d\sigma(\eta) = 1 \) for all \( \lambda \in B_n \), and
\[
L(z, \lambda) P^z(\lambda, \eta) = 0, \quad (\lambda, \eta) \in B_n \times \partial B_n.
\]
For any fixed \( z \in U^n \), since

\[
L(z, \lambda)h(z, \lambda) = \frac{1}{16C^2} f \circ \psi_z(\lambda)
\]

we have

\[
h(z, \lambda) = \int_{\partial B_n} P^z(\lambda, \eta)h(z, \eta)d\sigma(\eta) + \int_{B_n} G^z(\lambda, \eta) \frac{1}{16C^2} f \circ \psi_z(\eta)d\mu(\eta)
\]

Therefore,

\[
\sum_{|\alpha|+|\beta|=k} \left| \frac{\partial^k h(z, 0)}{\partial \lambda^\alpha \partial \eta^\beta} \right| \leq C_k \left[ \|u\|_{L^\infty(U^n)} + C_\varepsilon \sum_{|\alpha|+|\beta|=0} \left\| \frac{\partial^{|\alpha|+|\beta|} f(\psi_z(\eta))}{\partial \eta^\alpha \partial \eta^\beta} \right\|_{C^0(\overline{U^n})} \right]
\]

for any \( 0 < \varepsilon \leq 1 \), where \( C_k \) is a numerical constant that depends only on \( k \), and \( C_\varepsilon \) is a constant that depends only on \( \varepsilon \). This implies that (2.5) holds for all \( z \in U^n \). Moreover,

\[
\|h(z, \cdot)\|_{W^{k, p}(B(0, 1/2))} \leq C_{k, p} \left( \|h(z, \cdot)\|_{L^p(\partial B(0, 1/2))} + \|f \circ \psi_z\|_{W^{k-2, p}(B(0, 1/2))} \right)
\]

where \( B(0, 1/2) \) is the ball in \( \mathbb{C}^n \) centered at 0 with radius 1/2. With the definition of \( h(z, \lambda) = u(\psi_z(\lambda)) \), one can easily see that (2.6) holds. Therefore, the proof of the theorem is complete. \( \square \)

3 Proof of Theorem 1.1.

In order to prove Theorem 1.1, we first prove some lemmas. For each \( 1 \leq j \leq n \), we shall use the following notation. Let

\[
z^{(j)} = (z_1, \cdots, \hat{z}_{j-1}, \hat{z}_{j+1}, \cdots, z_n) \in \mathbb{C}^{n-1},
\]

and

\[
z^{(j)}(\lambda) = (z_1, \cdots, \hat{z}_{j-1}, \lambda, \hat{z}_{j+1}, \cdots, z_n)
\]

for each \( \lambda \in \mathbb{C} \).
Lemma 3.1 Let \( r = (r_1, r_2, \ldots, r_n) \in C^2(\overline{U}, \mathbb{R}^n) \) satisfy (1.10). Let \( u \in C(\overline{U}^n) \) be \( L_r \)-harmonic in \( U^n \). Then

\[
(3.3) \quad \lim_{\lambda \to e^{i\theta}} \mathcal{L}_j u(z^{(j)}(\lambda))
\]

exists uniformly for all \( \theta \in [0, 2\pi) \), \( z^{(j)} \in U^{n-1} \) and \( 1 \leq j \leq n \).

**Proof.** With the notation as in the proof of Theorem 2.1 and (2.10), one has

\[
L(z, \lambda)h(w, \lambda) = L(w, \lambda)h(w, \lambda) + (L(z, \lambda) - L(w, \lambda))h(w, \lambda)
= \sum_{j=1}^{n} (a_j(z, \lambda)^2 - a_j(w, \lambda)^2) \frac{\partial^2 h(w, \lambda)}{\partial \lambda_j \partial \lambda_j}.
\]

Thus

\[
\frac{\partial^2 h(w, 0)}{\partial \lambda_j \partial \lambda_j} - \frac{\partial^2 h(z, 0)}{\partial \lambda_j \partial \lambda_j}
= \int_{\partial B_n} \frac{\partial^2 P^z(z, \eta)}{\partial \lambda_j \partial \lambda_j} (h(w, \eta) - h(z, \eta))d\sigma(\eta)
\]

\[
+ \frac{\partial^2}{\partial \lambda_j \partial \lambda_j} \left[ \int_{B_n} G^*(\lambda, \eta) \sum_{k=1}^{n} (a_j(z, \eta)^2 - a_j(w, \eta)^2) \frac{\partial^2 h(w, \eta)}{\partial \eta_k \partial \eta_k} d\nu(\eta) \right]_{\lambda=0}
\]

Since

\[
a_j(z, \eta) = \frac{r_j(z + \frac{1}{\alpha} \eta r_j(z))}{r_j(z)} = 1 + \frac{r_j(z + \frac{1}{\alpha} \eta r_j(z)) - r_j(z)}{r_j(z)}, \quad z \in U^n \times B_n
\]

then

\[
|a_j(z, \eta) - a_j(w, \eta)| \leq C\|r_j\|_{C^2(\overline{\Omega})} |\eta| |z - w|.
\]

Therefore

\[
\left| \frac{\partial^2}{\partial \lambda_j \partial \lambda_k} G^*(\lambda, \eta) \right|_{\lambda=0} \leq \frac{C\|r_j\|_{C^2(\overline{\Omega})} |z - w|}{|\eta|^{2n-1}}
\]

for \( \eta \in B_n \setminus \{0\} \) (since \( B_n \) is the unit ball in \( \mathbb{C}^n = \mathbb{R}^{2n} \).) Thus

\[
\left| \int_{B_n} G^*(\lambda, \eta) \sum_{k=1}^{n} (a_k(z, \lambda)^2 - a_k(w, \lambda)^2) \frac{\partial^2 h(w, \eta)}{\partial \eta_k \partial \eta_k} d\nu(\eta) \right|
\]

8
\[
\begin{aligned}
&\leq C \sum_{k=1}^{n} \|r_k\|_{C^2(\partial^c \mathcal{M})} |w - z| \int_{B_n} |\eta|^{-2n+1} \left| \frac{\partial^2 h(w, \eta)}{\partial \eta_k \partial \eta^k} \right| \, d\nu(\eta) \\
&\leq C \sum_{k=1}^{n} \|r_k\|_{C^2(\partial^c \mathcal{M})} |z - w| \left\| \frac{\partial^2 h(w, \cdot)}{\partial \eta_k \partial \eta^k} \right\|_{L^\infty(U^n)} \\
&\leq C \sum_{k=1}^{n} \|r_k\|_{C^2(\partial^c \mathcal{M})} |z - w| \left\| \frac{\partial^2 u( \cdot )}{\partial z_k \partial \bar{z}_k} r_k(z_k)^2 \right\|_{L^\infty(U^n)} \\
&\leq C \sum_{k=1}^{n} \|r_k\|_{C^2(\partial^c \mathcal{M})} |z - w| \|u\|_{L^\infty(U^n)}
\end{aligned}
\]

by (2.5) in Theorem 2.1 with \( f \equiv 0 \).
Let \( s(\lambda, z) = \frac{1}{4C} (\lambda_1 r_1(z_1), \ldots, \lambda_n r_n(z_n)) \). Then
\[
u \left( z + \lambda_1 \frac{r_1(z_1)}{4C}, \ldots, z_n + \lambda_n \frac{r_n(z_n)}{4C} \right) = \nu (z + s(\lambda, z))
\]

Thus for \( z, w \in U^n \), since \( u \in C(\overline{U}^n) \), we have
\[
\left| \frac{\partial^2 h(w, 0)}{\partial \lambda_j \partial \lambda^j} - \frac{\partial^2 h(z, 0)}{\partial \lambda_j \partial \lambda^j} \right| \\
\leq C \|h(z, \cdot) - h(w, \cdot)\|_{L^\infty(B_n)} + C \sum_{k=1}^{n} \|r_k\|_{C^2(\partial^c \mathcal{M})} |z - w| \|u(\cdot)\|_{L^\infty(U^n)} \\
= C \sup \left\{ \left| u \left( z + s(\lambda, z) \right) - u \left( w + s(\lambda, w) \right) \right| : \lambda \in B_n \right\} \\
+C \sum_{k=1}^{n} \|r_k\|_{C^2(\partial^c \mathcal{M})} |z - w| \|u(\cdot)\|_{L^\infty(U^n)} \\
\to 0
\]
as \( z - w \to 0 \). Therefore
\[
\lim_{\lambda \to e^{i\theta}} \mathcal{L}_j u(z^{(j)}(\lambda))
\]
exists uniformly for \( \theta \in [0, 2\pi) \) and any \( z^{(j)} \in U^{n-1} \). \( \Box \)

**Lemma 3.2** Let \( u \in C(\overline{U}^n) \) be \( \mathcal{L}_r \)-harmonic in \( U^n \). Then

\[
\lim_{r \to 1^-} r_j(r)^2 \frac{\partial^2}{\partial z_j \partial \bar{z}_j} u(z_1, \ldots, z_{j-1}, r e^{i\theta}, z_{j+1}, \ldots, z_n) = 0.
\]

for any \( e^{i\theta} \in \partial U \) and \( z^{(j)} = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in U^{n-1} \).
Proof. By Lemma 3.1, if $u \in C(\overline{U}^n)$ is $\mathcal{L}_r$-harmonic in $U^n$ then for any $e^{i\theta} \in \partial U$

$\lim_{\lambda \to e^{i\theta}} r_j(\lambda)^2 \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j}(z^{(j)}(\lambda))$

exists for any $z^{(j)} \in U^{n-1}$, we denote the limit as $u(z^{(j)}(e^{i\theta}))$.

Therefore, to complete the proof of the lemma, it suffices to prove there is an $\epsilon > 0$ and a point $z^*_j$ with $|z_j - z^*_j| < \epsilon r_j(z_j)$) for any $z_j \in U$ such that

$\lim_{z_j \to e^{i\theta}} r_j(z^*_j)^2 \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j}(z^{(j)}(z^*_j)) = 0.$

To prove (3.6), we let $\phi \in C_{\partial_0}^\infty(U)$ be non-negative with $\phi(\lambda) = 1$ when $|\lambda| \leq 1/2$ and $\int_U \phi(\lambda)dA(\lambda) = 1$. Thus, for any fixed $0 < \epsilon < 1$, we let $\phi_\epsilon(\lambda) = \epsilon^{-2}\phi(\lambda/\epsilon)$. Then $\phi_\epsilon \in C^\infty(U(0, \epsilon))$, non-negative and $\int_U \phi_\epsilon(\lambda)dA(\lambda) = 1$. Moreover,

$\lim_{r \to 1^-} r_j(r)^2 \int_U \frac{\partial^2 u}{\partial w_j \partial \overline{w}_j}(z^{(j)}(w_j))\phi_{r_j(r)}(w_j - re^{i\theta})dA(w_j)$

$= \lim_{r \to 1^-} r_j(r)^2 \int_U u(z^{(j)}(w_j))\frac{\partial^2 \phi}{\partial w_j \partial \overline{w}_j}(w_j - re^{i\theta})dA(w_j)$

$= \lim_{r \to 1^-} \frac{1}{\epsilon^4 r_j(r)^2} \int_U u(z^{(j)}(w_j))\frac{\partial^2 \phi}{\partial w_j \partial \overline{w}_j}\left(\frac{w_j - re^{i\theta}}{\epsilon r_j(r)}\right)dA(w_j)$

$= \lim_{r \to 1^-} \frac{1}{\epsilon^2} \int_U u(z^{(j)}(re^{i\theta} + \epsilon r_j(r)w_j))\frac{\partial^2 \phi}{\partial w_j \partial \overline{w}_j}(w_j)dA(w_j)$

$= \lim_{r \to 1^-} \frac{1}{\epsilon^2} \int_U [u(z^{(j)}(re^{i\theta} + \epsilon r_j(r)w_j)) - u(z^{(j)}(e^{i\theta}))]\frac{\partial^2 \phi}{\partial w_j \partial \overline{w}_j}(w_j)dA(w_j)$

$+ \lim_{r \to 1^-} \frac{1}{\epsilon^2} \int_U u(z^{(j)}(e^{i\theta}))\frac{\partial^2 \phi}{\partial w_j \partial \overline{w}_j}(w_j)dA(w_j)$

$= 0$

since $\int_U u(z^{(j)}(e^{i\theta}))\frac{\partial^2 \phi}{\partial w_j \partial \overline{w}_j}(w_j)dA(w_j) = 0$ and $[u(z^{(j)}(re^{i\theta} + \epsilon r_j(r)w_j)) - u(z^{(j)}(e^{i\theta}))] \to 0$ as $r \to 1^-$ (here we used $u \in C(\overline{U}^n)$). Therefore, there is $z^*_j \in U(z_j, \epsilon r_j(r))$ such that (3.6) holds. Therefore, the proof of the lemma is complete.
Lemma 3.3 Let $r = (r_1, \cdots, r_n) \in C^2(\overline{U}, \mathbb{R}^n)$ satisfy (1.10). Let $u(z) \in C(\overline{U})$ be $\mathcal{L}_r$-harmonic in $U^n$. Then for each $\theta \in [0, 2\pi)$ and $1 \leq j \leq n$, $u(z^{(j)}(e^{i\theta}))$ is $\mathcal{L}_{r(j)}$-harmonic for $z^{(j)} \in U^{n-1}$ where

$$r^{(j)} = (r_1, \cdots, r_{j-1}, r_{j+1}, \cdots, r_n).$$

Proof. For convenience, we prove the case $j = 1$, the other cases being handled similarly. Let us write $z^{(1)}(e^{i\theta}) = (e^{i\theta}, z')$. If $u(e^{i\theta}, z') \in C^2(U^{n-1})$ and $\mathcal{L}_j = r_j(z_j)^2 \frac{\partial^2}{\partial z_j \partial \overline{z}_j}$ then

$$\mathcal{L}_{r(1)} u(e^{i\theta}, z^{(1)}) = \sum_{j=2}^n \mathcal{L}_j u(e^{i\theta}, z')$$

$$= \lim_{r \to 1^-} \sum_{j=2}^n \mathcal{L}_j u(re^{i\theta}, z')$$

$$= \lim_{r \to 1^-} [\mathcal{L}_r u(re^{i\theta}, z') - \mathcal{L}_1 u(re^{i\theta}, z')]$$

$$= - \lim_{r \to 1^-} \mathcal{L}_1 u(re^{i\theta}, z')$$

$$= 0$$

for all $z' \in U^{n-1}$ by the previous lemma. Otherwise, we may use the distribution derivatives. Namely, for any $f \in C_0^\infty(U^{n-1})$, we have

$$\int_{U^{n-1}} u(e^{i\theta}, z') \sum_{j=2}^n \frac{\partial^2}{\partial z_j \partial \overline{z}_j} \left( r_j(z_j)^2 f(z') \right) dv(z')$$

$$= \lim_{r \to 1^-} \int_{U^{n-1}} u(re^{i\theta}, z') \sum_{j=2}^n \frac{\partial^2}{\partial z_j \partial \overline{z}_j} \left( r_j(z_j)^2 f(z') \right) dv(z')$$

$$= \lim_{r \to 1^-} \sum_{j=2}^n \mathcal{L}_j u(re^{i\theta}, z') f(z') dv(z')$$

$$= \lim_{r \to 1^-} \int_{U^{n-1}} - \mathcal{L}_1 u(re^{i\theta}, z') f(z') dv(z')$$

$$= 0$$

by Lemma 3.2. Therefore, the proof of the lemma is complete. \( \Box \)

Now we are ready to prove Theorem 1.1.

Proof. Let $u \in C(\overline{U}^n)$. Then we shall prove Theorem 1.1 by mathematical induction.
1) It is clear that Theorem 1.1 holds for \( n = 1 \);
2) We assume that Theorem 1.1 holds for \( n - 1 \), and we shall prove it is true for \( n > 1 \). Let
\[
v(z) = P_1(u(\cdot, z'))(z_1), \quad z' = (z_2, \ldots, z_n) \in U^{n-1},
\]
where
\[
P_1(u(\cdot, z'))(z_1) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z_1|^2}{|1 - z_1 e^{-i\theta}|^2} u(e^{i\theta}, z') \, d\theta
\]
By Lemma 3.3, we know that \( u(e^{i\theta}, z') \) is \( \mathcal{L}_{r(1)} \)-harmonic for \( z' \in U^{n-1} \), and \( u(e^{i\theta}, z^{(1)}) \in C([0, 2\pi] \times \mathbb{R}^{n-1}) \). By the induction assumption, we have that \( u(e^{i\theta}, z') \) is strongly harmonic in \( z' \in U^{n-1} \) for all \( \theta \in [0, 2\pi) \). Thus \( v(z) \) is strongly harmonic in \( U^n \). Moreover,
\[
\lim_{r \to 1^-} v(re^{i\theta}, z') = u(e^{i\theta}, z')
\]
and for any \( 2 \leq j \leq n \), for simplicity, we let \( j = n \). If we let \( z'' = (z_2, \ldots, z_{n-1}) \) then for any \( z^{(n)} = (z_1, z'') \in U^{n-1} \)
\[
\lim_{r \to 1^-} v(z^{(n)}(re^{i\theta})) = P_1(u(\cdot, z'', e^{i\theta})(z_1)) = v(z^{(n)}, e^{i\theta})
\]
since \( u(z^{(n)}(e^{i\theta})) = u(z^{(n)}, e^{i\theta}) \) is strongly harmonic in \( z^{(n)} \in U^{n-1} \) from the induction assumption. Therefore \( u(z^{(j)}(e^{i\theta})) = v(z^{(j)}(e^{i\theta})) \) for all \( \theta \in [0, 2\pi) \) and \( 1 \leq j \leq n \). Therefore, \( u = v \) on \( \partial U^n \). By the Maximum principle, we have \( u(z) \equiv v(z) \) on \( U^n \). Therefore, \( u \) is strongly harmonic in \( U^n \), and the proof of Theorem 1.1 is complete.

As a corollary of Theorem 1.1, we have

**Corollary 3.4** Let \( r = (r_1, r_2, \ldots, r_n) \in C^\infty(\overline{U}, \mathbb{R}^n) \) satisfy (1.10). If \( \phi \in C^\infty(\partial U^n) \) and \( u \in C(\overline{U}^n) \) is \( \mathcal{L}_r \)-harmonic in \( U^n \) such that \( u = \phi \) on \( \partial U^n \) then \( u \in C^\infty(\overline{U}^n) \).

As a corollary of the proof of Theorem 1.1 and Lemmas 3.1–3.3, we have

**Corollary 3.5** If \( r = (r_1, \ldots, r_n) \in C^2(\overline{U}, \mathbb{R}^n) \) satisfy (1.10). Let \( u \) be a bounded \( \mathcal{L}_r \)-harmonic function in \( U^2 \). Let
\[
(3.8) \quad u_{h,1}(\lambda) = \int_U u(\lambda, \eta) h(\eta) dA(\eta) \quad \text{and} \quad u_{h,2}(\lambda) = \int_U u(\eta, \lambda) h(\eta) dA(\eta).
\]
If for any \( h \in C^\infty_c(U) \), \( u_{h,1}(\lambda) \) and \( u_{h,2}(\lambda) \) have non-tangential limits for almost all \( e^{i\theta} \in \partial U \), then \( u \) is strongly harmonic.
Proof. For any $h \in C_c^\infty(U)$, let $e^{i\theta} \in \partial U$ be such that the non-tangential limit of $u_{h,1}(\lambda)$ and $u_{h,2}(\lambda)$ exist, say, $u_{h,1}(e^{i\theta})$ and $u_{h,2}(e^{i\theta})$. Then the arguments of the proofs of Lemmas 3.1 and 3.2 show that

$$\lim_{r \to 1^{-}} L_r u_{h,1}(re^{i\theta}) = \lim_{r \to 1^{-}} L_r u_{h,2}(re^{i\theta}) = 0$$

This implies $u(e^{i\theta}, \lambda)$ and $u(\lambda, e^{i\theta})$ are harmonic for $\lambda \in U$ for almost all $\theta \in [0, 2\pi)$. The argument of the proof of Lemma 3.3 implies $u$ is strongly harmonic in $U^2$. \[\]

Finally, we provide the following example.

EXAMPLE 2 Let $\phi(z_1, z_2) = (2 - |z_1|^2 - |z_2|^2)$ on $\partial U^2$. There is no function $u \in C(\overline{U}^n)$ harmonic in the Bergman metric in $U^2$ such that $u = \phi$ on $\partial U^2$.

Proof. If $u \in C(\overline{U}^2)$ is harmonic in the Bergman metric and $u = \phi$ on $\partial U^2$. Then $u$ is strongly harmonic by Theorem 1.1, and

$$u(z) = P_1 P_2(f)(z)$$

$$= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{(1 - |z_1|^2)}{|1 - z_1 e^{-i\theta_1}|^2} \frac{(1 - |z_2|^2)}{|1 - z_2 e^{-i\theta_2}|^2} \phi(e^{i\theta_1}, e^{i\theta_2}) d\theta_1 d\theta_2$$

$$= 0$$

This contradicts with $u(1, 2) = f(1, 2) = 1 - |z_2|^2 \neq 0$ if $|z_2| < 1$. \[\]

The example shows that we cannot expect continuous up to the boundary solutions of the Dirichlet problem

$$\Delta u = 0, \text{ in } U^n, \text{ and } u = \phi \text{ on } \partial U^n.$$  

In fact, from Corollary 3.5, we cannot expect a harmonic solution $u$ having non-tangential limit equaling $\phi$ in each variable in the sense that

$$u_{h,j}(e^{i\theta}) = \phi_{h,j}(e^{i\theta}), \quad \theta \in [0, 2\pi) \text{ and } j = 1, 2$$

for any $h \in C_c^\infty(U)$. Therefore, the question about in what sense, one can have existence of the solution of the Dirichlet problem (3.8) in $U^n (n > 1)$ remains open.
References


Mailing address:
Department of Mathematics, University of California, Irvine, CA 92697-3875

E-mail address
sli@math.uci.edu (for Song-Ying Li),
esimon@math.uci.edu (for Ezequias Simon)