On the existence and regularity of Dirichlet problem for complex Monge-Ampère equations on weakly pseudoconvex domains

Song-Ying Li

Variation Calculus and PDEs, to appear

1 Introduction

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $C^1$ boundary and let $d = \partial + \overline{\partial}$ and $d^c = i(\overline{\partial} - \partial)$ in $\mathbb{C}^n$. For any plurisubharmonic function $u$ on $\Omega$, we let

$$(dd^c u)^n = dd^c u \wedge \cdots \wedge dd^c u, \quad dV = \left(\frac{i}{2}\right)^n \prod_{j=1}^{n} dz_j \wedge d\overline{z_j}.$$ 

When $u \in C(\Omega)$ and is plurisubharmonic $\Omega$, $dd^c u$ is a positive $(1,1)$ current and $(dd^c u)^n$ is well defined as a positive Borel measure on $\Omega$, see [2] for the detail. When $u \in C^2(D)$, we have $(dd^c u)^n = \det H(u) dV$ where $dV$ is the Lebesgue measure on $\mathbb{C}^n$ and

$$H(u)(z) = \left[\frac{\partial^2 u(z)}{\partial z_i \partial \overline{z_j}}\right]_{n \times n}$$

be the complex hessian matrix of $u$. We will consider the existence, uniqueness and regularity for the Dirichlet problem of the complex Monge-Ampère equations:

$$\text{(1.1)} \quad \det H(u)(z) = f(z) \geq 0, \quad z \in \Omega; \quad \text{and} \quad u = \phi(z), \quad z \in \partial \Omega.$$ 

Here $\det H(u) = f$ means that $(dd^c u)^n u = f dV$ when $u$ is not $C^2$. When $\Omega$ is a smooth, bounded strictly pseudoconvex domain in $\mathbb{C}^n$, the above problem
has been received considerable study. In a fundamental paper of Bedford and Taylor [2], they proved that if \( 0 < \alpha \leq 1 \), \( \phi \in C^{2\alpha}(\partial \Omega) \) and \( f^{1/n} \in C^\alpha(\Omega) \), then (1.1) has a unique plurisubharmonic solution \( u \in C^\alpha(\Omega) \). Their results are sharp and their method is very useful for Hölder regularity for Dirichlet problems for elliptic equations. When \( f \) in (1.1) is \( L^p \) function, there are several important results were obtained by S. Kołodziej in [22] and [23] and references therein. For global regularity, Caffarelli, Kohn, Nirenberg and Spruck [9] proved that if \( f(z) \in C^\infty(\Omega) \) is strictly positive and \( \phi \in C^\infty(\partial \Omega) \), then (1.1) has a unique plurisubharmonic solution \( u \in C^\infty(\Omega) \). When \( \Omega \) is weakly pseudoconvex, the problem becomes much more complicated. Bo Guan [16] generalized the theorem of Caffarelli-Kohn-Nirenberg-Spruck to weakly pseudoconvex domain under the assumption that the boundary data \( \phi \) has a plurisubharmonic extension \( u_0 \in C^\infty(\Omega) \) which is a subsolution of (1.1). In other words, \( u_0 \) satisfies

\[
(1.2) \quad \det H(u_0)(z) \geq f(z), \quad z \in \Omega; \quad \text{and} \quad u_0(z) = \phi(z), \quad z \in \partial \Omega.
\]

There are some known results for the existence and regularity for complex Monge-Ampère equation (1.1) on a more general domains than strictly pseudoconvex domains in \( \mathbb{C}^n \) without any restrictions on boundary data \( \phi \). In [7], Blocki give a characterization for existence of a continuous and plurisubharmonic solution on hyperconvex domains in \( \mathbb{C}^n \). However, there are no any Hölder continuous results on corresponing weakly pseudoconvex domains in \( \mathbb{C}^n \). It is obvious (1.1) has no \( C^{1,1}(\Omega) \) plurisubharmonic solution for \( f(z) = 1 \) on \( \Omega \) and \( \phi = 0 \) on \( \partial \Omega \) when \( \Omega \) is not strictly pseudoconvex. Main purpose of this article is to generalize Bedford and Taylor’s result from strictly pseudoconvex domains to weakly pseudoconvex domains of finite type.

In additional to the main purpose, we will provide a smoothly bounded pseudoconvex domain \( \Omega \) of infinite type in \( \mathbb{C}^2 \), which will be presented in Example 2 in Section 3. On which, for any positive integer \( \ell \), there is a plurisubharmonic function \( G \in C(\Omega) \) so that \( G|_{\partial\Omega} = \phi \in C^\infty \) and

\[
\sqrt{\det H(G)} = \sqrt{f(z)} \in C^\ell(\Omega), \quad \text{but} \quad G \notin C^\epsilon(\Omega) \quad \text{for any} \ \epsilon > 0.
\]

This implies that (1.1) does not have any Hölder continuous plurisubharmonic solution on \( \overline{\Omega} \) in general.

In order to prove the existence and regularity for the Dirichlet problem of complex Monge-Ampère equation (1.1) with Hölder continuous plurisubharmonic solution without any restriction on extension of the boundary data
\( \phi \), we may need to restrict our attention to certain pseudoconvex domains of ‘finite type’. Based on Catlin’s description (in discret sense) on a class of pseudoconvex domains of finite type in [10], and the main result of Fornaess and Sibony in [15] for D’Angelo finite type domain in \( \mathbb{C}^2 \) and convex finite type domain in \( \mathbb{C}^n \), one may introduce the following definition.

**Definition 1.1** Let \( \Omega \) be a bounded pseudoconvex domain in \( \mathbb{C}^n \) with \( C^1 \) boundary. Let \( m \geq 2 \) be a real number. We say that \( \Omega \) is of plurisubharmonic type \( m \) if there is a plurisubharmonic function \( \rho(z) \in C^2(\Omega) \cap C^{2/m}(\overline{\Omega}) \) such that \( z \in \Omega \) if and only if \( \rho(z) < 0, \partial \Omega = \{ z : \mathbb{C}^n : \rho(z) = 0 \} \) and \( H(\rho)(z) - I_n \) is positive semi-definite for all \( z \in \Omega \).

**Remark 1**
1) Every smoothly bounded strictly pseudoconvex domain in \( \mathbb{C}^n \) is of plurisubharmonic type 2.
2) It was proved by Fornaess and Sibony [15] that any smoothly bounded pseudoconvex domain of finite type \( m > 2 \) in \( \mathbb{C}^2 \) or convex domain of finite type \( m > 2 \) is a domain of plurisubharmonic type \( m \).
3) For any real number \( m \geq 1 \), if we let \( E_{2m} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1 \} \) be the complex ellipsoid then \( E_{2m} \) is a pseudoconvex domain of plurisubharmonic type \( 2m \) (here \( m \) may be non-integer).

In fact: Statement 1) follows from the fact that any smoothly bounded strictly pseudoconvex domains \( \Omega \) in \( \mathbb{C}^n \) admit strictly plurisubharmonic defining functions \( \rho \in C^\infty(\Omega) \).

Statement 2) is main result proved by Fornaess and Sibony in [15]. In fact, our statement here is a consequence of a combination of Theorem A and the proof of Proposition 5.1 in [15].

Statement 3) follows from considering \( \rho(z) = m(\rho(z) - (1 - |z_1|^2)^{1/m}) \) by checking that \( z \in E_{2m} \) if and only if \( \rho(z) < 0, \partial E_{2m} = \{ z \in \mathbb{C}^2 : \rho(z) = 0 \} \) and \( H(\rho) - I_2 \) is positive semi-definite on \( E_{2m} \).

We will prove the following theorem.

**Theorem 1.2** Let \( m \geq 2 \) be a real number, and let \( \Omega \) be a bounded pseudoconvex domain of plurisubharmonic type \( m \) in \( \mathbb{C}^n \) with \( C^2 \) boundary. For any \( 0 < \alpha \leq 2/m \), if \( \phi \in C^{\alpha}(\partial \Omega) \) and \( f \geq 0 \) on \( \Omega \) with \( f(z)^{1/n} \in C^\alpha(\overline{\Omega}) \), then the Dirichlet problem of complex Monge-Ampère equation (1.1) has a unique plurisubharmonic solution \( u \in C^\alpha(\overline{\Omega}) \) in weak sense.
In Section 3, we provide an example shows that results of Theorem 1.2 is sharp on $E_{2m}$, the complex ellipsoid in $\mathbb{C}^2$ for any $m \geq 1$.

Moreover, we will prove in Example 1 in Section 3 that if $m > 1$ then there is a plurisubharmonic function $u \in C^{1/m}(E_{2m}) \cap C^\infty(E_{2m})$ satisfies

(i) $u|_{\partial E_{2m}} = \phi(z) \in C^\infty(\partial E_{2m})$;
(ii) $\sqrt{\det H(u)(z)} = \sqrt{f(z)} \in C^\infty(E_{2m})$;
(iii) $u \notin C^\beta(E_{2m})$ for any $\beta > 1/m$.

This example provides an interesting gap phenomenon in regularity for the Dirichlet problem of complex Monge-Ampère on domains between strictly pseudoconvex and weakly pseudoconvex as follows:

If $m = 1$, then $E_{2m}$ is strictly pseudoconvex, the Dirichlet problem (1.1) on $E_{2m} = E_2$ is expected to have a unique plurisubharmonic solution $u \in C^{1,1}(E_2)$ when $\varphi(z) \in C^{3,1}(\partial E_2)$ and $g \geq 0$ with $f^{1/2} \in C^{1,1}(E_2)$ (this is a theorem for real Monge-Ampère equations with $f^{1/(n-1)} \in C^{1,1}(D)$ where $D$ is a smoothly bounded strictly convex domain in $\mathbb{R}^n$ proved by Guan, Trudinger and Wang in [21].)

If $m > 1$, then $E_{2m}$ is weakly (not strictly) pseudoconvex in $\mathbb{C}^2$ with smooth boundary when $m$ is integer. The above example shows that no matter how smooth is the boundary data $\varphi (\phi \in C^\infty(\partial E_{2m}))$ and no matter how smooth is $f(z)^{1/2}$ on $E_{2m}$ ($f^{1/2} \in C^\infty(E_{2m})$), the unique plurisubharmonic solution $u$ may not belong to $C^\beta(E_{2m})$ for any $\beta > 1/m$, where $1/m < 1$.

Finally, in Section 4, we provide a minor application of Theorem 2.5 to a proper holomorphic map $\phi$ between two smooth, bounded weakly pseudoconvex domains in $\mathbb{C}^n$.

2 Preliminary results

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with $C^1$ boundary. Let $P(\Omega)$ denote the set of all bounded plurisubharmonic, continuous functions on $\Omega$. For $\phi \in C(\partial \Omega)$ and $f \in C(\Omega)$, we let

$$\mathcal{B}(\phi, f) = \{v \in P(\Omega) : \det H(v) \geq f \text{ and } \limsup_{z \to z_0} v(z) \leq \phi(z_0), z_0 \in \partial \Omega\}.$$  

Based on the arguments given in [2], we will prove the following proposition.
Proposition 2.1 Let $\Omega$ be a bounded pseudoconvex domain $\Phi^n$ with $C^1$ boundary. For any $0 < \alpha < 1$, we let $\phi \in C^\alpha(\partial \Omega)$ and $f \geq 0$ with $f^{1/n} \in C^\alpha(\Omega)$. If for each $\xi \in \partial \Omega$, there is $v_\xi \in C^\alpha(\Omega) \cap P(\Omega)$ such that the following four conditions hold.

(i) $v_\xi(z) \leq \phi(z), \ z \in \partial \Omega$;
(ii) $v_\xi(\xi) = \phi(\xi)$;
(iii) $\|v_\xi\|_{C^\alpha(\Omega)} \leq C_0$;
(iv) $\det(H(v_\xi))(z) \geq f(z)$.

Then there is a unique plurisubharmonic function $v \in C^\alpha(\overline{\Omega})$ satisfying (1.1) and there is a constant $C = C(\alpha, \|f^{1/n}\|_{C^\alpha}, C_0)$ depending only on $\alpha, \|f^{1/n}\|_{C^\alpha}$ and $C_0$ so that

\begin{equation}
\|v\|_{C^\alpha(\overline{\Omega})} \leq C(\alpha, \|f^{1/n}\|_{C^\alpha}, C_0).
\end{equation}

Proof. Let

\begin{equation}
v(z) = \sup \{v_\xi(z) : \xi \in \partial \Omega\}, \quad z \in \overline{\Omega}.
\end{equation}

Then $v(\xi) \geq v_\xi(\xi) = \phi(\xi)$ for all $\xi \in \partial \Omega$ and $v(z) \leq \phi(z)$ for all $z \in \partial \Omega$. Therefore, $v(z) = \phi(z)$ on $\partial \Omega$. Following the argument in [2], one has that $v$ is plurisubharmonic satisfying $\|v\|_{C^\alpha(\overline{\Omega})} \leq (C_0 + 1)$ and

$$\det(H(v)) \geq \inf \{\det(H(v_\xi))(z) : \xi \in \partial \Omega\} \geq f(z), \quad z \in \Omega.$$ 

Thus $v \in B(\phi, f) \cap C^\alpha(\overline{\Omega})$. Now we let

$$u(z) = \sup \left\{v(z) : v \in B(\phi, f) \cap C^\alpha(\overline{\Omega}), \|v\|_{C^\alpha(\overline{\Omega})} \leq (C_0 + 1)\right\}$$

and let

$$u^*(z) = \limsup_{h \to 0} u(z + h).$$

Then it was proved in [2] that $u^*$ is plurisubharmonic and $u^* \in B(\phi, f)$ and

$$\det H(u^*)(z) = f(z), \quad z \in \Omega, \quad \text{and} \quad u(z)^* = \phi(z) \text{ on } \partial \Omega.$$

Next we show that $u^* \in C^\alpha(\overline{\Omega})$. Without loss of generality, in the following argument, we may assume that $u(z) = u^*(z)$.

For any small vector $\tau \in \mathbb{C}^n$, we define

$$V(z, \tau) = \begin{cases} u(z) & \text{if } z + \tau \not\in \Omega, \ z \in \Omega, \\ \max \{u(z), u_\tau(z)\} & \text{if } z, \ z + \tau \in \Omega. \end{cases}$$
where
\[ u_\tau(z) = u(z + \tau) + K_1|\tau|^{\alpha}|z|^2 - K_2|\tau|^{\alpha} - K|\tau|^{\alpha}, \]

\[ K_1 = C K_0, \quad K_0 = \|f^{1/n}\|_{C^0(\overline{\Omega})}, \quad K_2 > K_1|z|^2 \quad \text{and} \quad K = C_0 \|\phi\|_{C^0(\partial \Omega)} + 1. \]

Then \( u_\tau(z) \) is plurisubharmonic, and so is \( V(z, \tau) \). Moreover, we have \( V(z, \tau) = u(z) \) on \( \partial \Omega \). In fact, for any \( z \in \partial \Omega \), we have either \( z + \tau \not\in \Omega \) or \( z + \tau \in \Omega \). If \( z + \tau \not\in \Omega \), then \( V(z, \tau) = u(z) \). Otherwise, \( z + \tau \in \Omega \). By the definition of \( u \), there is \( v \in C^\alpha(\overline{\Omega}) \cap P(\Omega) \) with \( \|v\|_{C^\alpha(\overline{\Omega})} \leq C_0 + 1 \) such that \( u(z + \tau) \leq |\tau| + v(z + \tau) \). Thus

\[
  u_\tau(z) - u(z) = u(z + \tau) - u(z) + K_1|\tau|^{\alpha}|z|^2 - K_2|\tau|^{\alpha} - K|\tau|^{\alpha} \\
  \leq v(z + \tau) + |\tau| - v(z) - K|\tau|^{\alpha} \leq 0
\]

This implies \( V(z, \tau) = \max\{u_\tau(z), u(z)\} = u(z) \). Thus \( V(z, \tau) = u(z) \) on \( \partial \Omega \).

Now we claim that \( \det H(V(z, \tau)) \geq \det H(u)(z)) \) on \( \Omega \). Since \( \det H(V(z, \tau)) \) (\( \det H(u)(z)) \) can be approximated by \( \det H(V_c(z, \tau)) \) (\( \det H(u_c)(z)) \) where \( V_c \) (\( u_c \)) are smooth plurisubharmonic function (see Page 25 in [2] for details). Without loss of generality we may assume that \( V \) and \( u \) are in \( C^2(\Omega) \) in following calculation. Thus for \( z, z + \tau \in \Omega \), by rotation, we may assume that

\[
  H(u)(z + \tau) = \text{diag}(\lambda_1(z + \tau), \ldots, \lambda_n(z + \tau)), \quad \lambda_1 \leq \cdots \leq \lambda_n
\]

Let \( g(z) = (f(z))^{1/n} \). Then \( g(z + \tau)^n = \prod_{j=1}^n \lambda_j(z + \tau) \). Since \( \lambda_1(z + \tau) \leq \lambda_2(z + \tau) \leq \cdots \leq \lambda_n(z + \tau) \) then \( \prod_{j=k}^n \lambda_j(z + \tau) \geq g(z + \tau)^{n+1-k} \) for all \( 1 \leq k \leq n \). Hence

\[
\det H(u_\tau)(z) \\
= \det(H(u)(z + \tau) + K_1|\tau|^{\alpha}I_n) \\
= \prod_{j=1}^n (\lambda_j(z + \tau) + K_1|\tau|^{\alpha}) \\
\geq \prod_{j=1}^n \lambda_j(z + \tau) + \sum_{k=1}^n (K_1|\tau|^{\alpha})^k \prod_{j=k+1}^n \lambda_j(z + \tau) \\
= g(z + \tau)^n + \sum_{k=1}^n (K_1|\tau|^{\alpha})^k g(z + \tau)^{n-k}
\]
\[
\begin{align*}
\geq g(z)^n - n! \sum_{k=1}^{n} (\|g\|_{C^\infty(\Omega)}|\tau|^\alpha)^k g(z + \tau)^{n-k} + \sum_{k=1}^{n} (K_1|\tau|^\alpha)^k g(z + \tau)^{n-k} \\
\geq g(z)^n \\
= f(z) \\
= \det(u)(z)
\end{align*}
\]

if we let \( K_1 \geq n!\|g\|_{C^\infty(\Omega)}. \)

Therefore, \( V(z, \tau) - u(z) \) attains its maximum on \( \partial\Omega \), and it equals zero on \( \partial\Omega \). Thus \( V(z, \tau) \leq u(z) \) on \( \Omega \). This implies that

\[
\begin{align*}
|u(z + \tau) - u(z)| &\leq V(z, \tau) - K_1|z|^2|\tau|^\alpha + (K_2 + K)|\tau|^\alpha - u(z) \\
&\leq -K_1|z|^2|\tau|^\alpha + (K_1 + K)|\tau|^\alpha \\
&\leq K_3|\tau|^\alpha
\end{align*}
\]

where \( K_3 = K_2 + K \). Reversing the order of \( z \) and \( z + \tau \), we have

\[
|u(z) - u(z + \tau)| \leq K_3|\tau|^\alpha, \quad z, \ z + \tau \in \Omega,
\]

This implies that \( u \in C^\alpha(\Omega) \), and the proof of the proposition is complete.

**Corollary 2.2** Let \( \Omega \) be a bounded pseudoconvex domain in \( \mathbb{C}^n \) with \( C^2 \) boundary. Assume that there is \( 0 < \alpha \leq 1 \) and a plurisubharmonic function \( \rho \in C^\alpha(\Omega) \) with \( \rho(z) = 0 \) on \( \partial\Omega \) so that \( H(\rho)(z) = cI_n \) is positive semi-definite for some \( c > 0 \). If \( f(z) \geq 0 \) on \( \Omega \) with \( f(z)^{1/n} \in C^\alpha(\Omega) \) and \( \phi \in C^{1,1}(\partial\Omega) \) then the Dirichlet boundary value problem (1.1) has a unique plurisubharmonic solution \( u \in C^\alpha(\Omega) \).

**Proof.** Since \( \phi \in C^{1,1}(\partial\Omega) \), we may extend \( \phi(z) \) to be a function on \( \overline{\Omega} \) so that \( \phi \in C^{1,1}(\Omega) \). For any \( \xi \in \partial\Omega \), we let

\[
v_\xi(z) = \phi(z) + K\rho(z).
\]

Then \( v_\xi(z) \leq \phi(z) \) on \( \partial\Omega \) and \( v_\xi(\xi) = \phi(\xi) \). Let \( K = \frac{1}{n}\|\phi(z)\|_{C^{1,1}(\Omega)} \). Then it is easy to show that \( v_\xi \) satisfies the all conditions in Proposition 2.1 (for this special case, \( v_\xi \) can be chosen independently on \( \xi \).) Therefore, the proof of the corollary is complete by Proposition 2.1.

The following theorem was first proved by Diederich and Fornaess [12] for \( \Omega \) having \( C^\infty \) boundary, and later by Range [30] for \( \Omega \) having \( C^3 \) boundary.
THEOREM 2.3 Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $C^3$ boundary. Then there are $\alpha = \alpha(\Omega) \in (0, 1)$ and a strictly plurisubharmonic defining function $\rho_0 \in C^\alpha(\overline{\Omega}) \cap C^3(\Omega)$ for $\Omega$ so that the followings hold:

(i) The smallest eigenvalue of $H(\rho_0)(z)$ is grater than $c|\rho_0(z)|$ where $c$ is constant depending only on $\Omega$ and $\alpha$;

(ii) $-\rho_0(z) \approx \delta_\Omega(z)^\alpha$ and $|\nabla \rho_0(z)| \approx \delta_\Omega(z)^{\alpha - 1}$, where $\delta_\Omega(z)$ is the distance function from $z$ to $\partial \Omega$.

We need the following proposition.

**Proposition 2.4** Let $\Omega$ be bounded pseudoconvex domain in $\mathbb{C}^n$ with $C^3$ boundary. Then there is $\alpha = \alpha(\Omega) > 0$ such that for any positive constant $K > 0$, there is a plurisubharmonic defining function $\rho \in C^\alpha(\overline{\Omega}) \cap C^3(\Omega)$ for $\Omega$ such that

$$\det H(\rho) \geq K, \quad \text{on} \ \Omega.$$ 

**Proof.** Let $\rho_0(z)$ be given in the previous theorem. Let

$$\rho(z) = \rho_0(z) + C\rho_0(z)^2$$

Then

$$H(\rho) = (1 + C\rho_0)H(\rho_0) + 2C\partial \rho_0 \otimes \overline{\partial} \rho_0$$

and for all $z \in \Omega$ with $C\rho(z) > -1/2$ we have

$$\det H(\rho) \geq (1 + C\rho) \det(c(-\rho_0)I_n + 2C\partial \rho_0 \otimes \overline{\partial} \rho_0)$$

$$\geq \frac{1}{2}c^n(-\rho_0)^{n-2}2C|\partial \rho_0(z)|^2$$

$$\geq c^n(-\rho_0)^{n-2}C|\partial \rho_0(z)|^2$$

$$\geq c^nC\delta_\Omega(z)^{(n+1)\alpha} \delta_\Omega(z)^{-2}$$

$$= c^nC\delta_\Omega(z)^{(n+1)\alpha - 2}$$

Therefore, if we let $(n + 1)\alpha < 2$ and we have

$$\det H(\rho) \geq K$$

for all $z \in \Omega$ such that $C\rho_0(z) > -\frac{1}{2}$, and $c^nC\delta_\Omega(z)^{(n+1)\alpha - 2} > K$. Let $C = c^{-n}K$ then we have $\det H(\rho) \geq K$ on $\{z \in \Omega : \rho_0(z) \geq -\frac{c^n}{2K}\}$. Then we may modified the definition of $\rho$ on $\Omega \setminus \{z \in \Omega : \rho_0(z) \geq -\frac{c^n}{2K}\}$ so that $\det H(\rho) \geq K$ on $\Omega$. The proof of the proposition is complete. \[\square\]
THEOREM 2.5 Let Ω be a bounded pseudoconvex domain in \( \mathbb{C}^n \) with \( C^3 \) boundary. Let \( \alpha = \alpha(\Omega) \in (0, 1) \) be given by Theorem 2.3. For any \( 0 < \beta \leq \alpha \), if \( \phi \in C^\beta (\overline{\Omega}) \) there is a constant \( C > 0 \) such that \( H(\phi)(z) \geq -C\delta_\Omega(z)^\beta I_n \) in weak sense; and \( f \geq 0 \) with \( f^{1/m} \in C^\beta (\overline{\Omega}) \), then the Dirichlet boundary value problem for the Monge-Ampère equation

\[
(2.4) \quad \det(H(u))(z) = f(z), \quad \text{in } \Omega; \quad \text{and } u = \phi(z) \text{ on } \partial \Omega
\]

has a unique plurisubharmonic solution \( u \in C^\beta (\overline{\Omega}) \).

Proof. It suffices to prove the assumptions of Proposition 2.1 hold. Let

\[
v(z) = \phi(z) - K_0(-\rho(z))^{\beta/\alpha}
\]

Then \( v(z) = \phi(z) \) on \( \partial \Omega \), and \( \|v\|_{C^\alpha(\partial \Omega)} \leq C(1 + \|\phi\|_{C^\alpha(\partial \Omega)}) \). Moreover, since

\[
H(\rho) \geq c_0|\rho(z)|I_n, \quad z \in \Omega
\]

and

\[
H(-(-\rho)^{\beta/\alpha}) \geq c_0\frac{\alpha}{\beta}(-\rho(z))^{\beta/\alpha}I_n \geq c_0(\alpha/\beta)\delta_\Omega(z)^\beta I_n, \quad z \in \Omega.
\]

Let \( K = \|f\|_\infty \). Then, by Proposition 2.4, there is a defining function \( \rho \in C^3(\Omega) \cap C^\alpha(\Omega) \) such that \( \det H(\rho) \geq K \). If we choose \( K_0 \geq 1 + \frac{C\alpha}{c_0\beta} \) then \( v \) is plurisubharmonic in \( \Omega \). Moreover,

\[
det(H(v))(z) = \det(H(\phi) - K_0H(\rho)^{\beta/\alpha})(z)
\]

\[
\geq \det(-C\delta_\Omega(z)^\beta I_n + (K_0 - 1)c_0(\beta/\alpha)\delta_\Omega(z)^\beta I_n + H(\rho)(z))
\]

\[
\geq \det H(\rho)(z) \geq f(z).
\]

Therefore, the proof of Theorem 2.5 is complete. \( \Box \)

3 Dirichlet problem on domains of finite type

The main purpose of this section is to prove Theorem 1.2 and provide an example to show that Theorem 1.2 is sharp in two senses, which we will explain later. First let us prove Theorem 1.2.
Proof of Theorem 1.2.

Proof. When \( m = 2 \), the above theorem was proved by Bedford and Taylor in [2]. We shall prove the theorem when \( m > 2 \). Let \( \rho(z) \in C^{2/m}(\overline{\Omega}) \) satisfy Definition 1.1. Let

\begin{equation}
(3.1) \quad h(z) = -|z - \xi|^2 + m\rho(z).
\end{equation}

For any \( \xi \in \partial\Omega \), we define \( v_\xi \) in two cases in term of the value of \( \alpha \).

(a) If \( m\alpha \leq 1 \) then we define

\begin{equation}
(3.2) \quad v_\xi(z) = \phi(\xi) - K\left[-h(z)\right]^{m\alpha/2}.
\end{equation}

(b) If \( 1 < m\alpha \leq 2 \) then we define

\begin{equation}
(3.3) \quad v_\xi(z) = \phi(\xi) - \sum_{j=1}^{n} 2\text{Re} \frac{\partial\phi(\xi)}{\partial z_j}(z_j - \xi_j) - K\left[-h(z)\right]^{m\alpha/2}.
\end{equation}

Here

\begin{equation}
(3.4) \quad K \geq \frac{2}{m\alpha}((\|h\|_\infty + 1)\|f\|_\infty^{1/n}).
\end{equation}

By the definition of \( \rho(z) \), it is easy to see that

\begin{equation}
(3.5) \quad (-h(z))^{m\alpha/2} = (|z - \xi|^2 - m\rho(z))^{m\alpha/2} \in C^\alpha(\overline{\Omega}).
\end{equation}

Let \( C_{m,\alpha} = \|\phi\|_{C^{m\alpha}(\partial\Omega)} \). Then

\begin{equation}
(3.6) \quad v_\xi(z) \leq \phi(z), \quad z \in \partial\Omega, \quad v_\xi(\xi) = \phi(\xi),
\end{equation}

and

\begin{equation}
(3.7) \quad \|v_\xi\|_{C^\alpha(\overline{\Omega})} \leq C_0\left(\|\phi\|_{C^{m\alpha}(\partial\Omega)} + 1\right)
\end{equation}

for some constant \( C_0 \) depending only on \( C_{m,\alpha} \) and \( K \). Moreover,

\begin{equation}
H(v_\xi)(z) = KH((-h(z))^{m\alpha/2}) = K\left[\frac{m\alpha}{2}(-h(z))^{m\alpha-1}h + \frac{m\alpha}{2}(1 - \frac{m\alpha}{2})h^2\right]
\end{equation}
Since $0 < \frac{ma}{2} \leq 1$ we have

\[
\det(H(v_{\xi}))(z) \geq K^n(\frac{ma}{2})^n(-h(z))^{n(\frac{ma}{2}-1)} \det \left[ h_{ij} \right](z) \\
\geq K^n(\frac{ma}{2})^n(-h(z))^{n(\frac{ma}{2}-1)} \det \left[ m \rho_{ij} - \delta_{ij} \right](z) \\
\geq K^n(\frac{ma}{2})^n(-h(z))^{n(\frac{ma}{2}-1)} \\
\geq f(z).
\]

By Proposition 2.1, we have that the Dirichlet problem (1.1) has a unique plurisubharmonic solution $u \in C^{\alpha}(\overline{\Omega})$. Therefore the proof of Theorem 1.2 is complete. $\Box$

Let $m > 1$, and let $E_{2m} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$ be the complex ellipsoid in $\mathbb{C}^2$. Next we will provide an example which shows that the regularity result of Theorem 1.2 is the best possible on $E_{2m}$.

**Example 1** Let $m > 1, 0 < \alpha \leq \frac{1}{m}$ and let $\gamma = 2k + 2 - \alpha$. Let

\[
u(z) = (1 - |z_1|^2)^\gamma|z_2|^2 - \alpha^{-2}\gamma(1 - |z_1|^2)^\alpha, \quad z \in E_{2m}.
\]

Then $u$ is plurisubharmonic in $E_{2m}$; $u \in (C^{\alpha}(E_{2m}) \setminus C^{\beta}(E_{2m}))$ for any $\beta > \alpha$ and

\[
\det H(u)(z) = \gamma(1 - |z_1|^2)^{2k}[-(1 - |z_1|^2)^2 - \frac{2}{\alpha} |z_2|^2 + \left(\frac{1}{\alpha} - |z_1|^2\right)] = f(z).
\]

Moreover, we have

(a) $f(z) \geq 0$ on $E_{2m}$, $\sqrt{f(z)} \in C^{k+1,1-\frac{1}{m}}$ and $f(z) \geq (\frac{1}{\alpha} - 1)\gamma$ if $k = 0$.

(b) If $\alpha = 1/m$ then on $\partial E_{2m}$,

\[
u(z) = (1 - |z_1|^2)^{2(k+1)} - m^2\gamma|z_2|^2 \in C^{\infty}(\partial E_{2m}),
\]

(c) If $0 < \alpha < 1/m$ then, on $\partial E_{2m}$,

\[
u(z) = (1 - |z_1|^2)^{2(k+1)} - \alpha^{-2}\gamma|z_2|^{2m} \in C^{2m\alpha}(\partial E_{2m}).
\]

**Proof.** Since

\[
\frac{\partial^2 u(z)}{\partial z_1 \partial z_1} = \gamma|z_2|^2(1 - |z_1|^2)^{\gamma-2}(\gamma|z_1|^2 - 1) + \alpha^{-1}\gamma(1 - |z_1|^2)^{\alpha-2}(1 - \alpha|z_1|^2)
\]

11
\[
\frac{\partial^2 u(z)}{\partial z_2 \partial \overline{z}_2} = (1 - |z_1|^2)\gamma, \quad \text{and} \quad \frac{\partial^2 u(z)}{\partial z_1 \partial \overline{z}_2} = -\gamma(1 - |z_1|^2)^{\gamma - 1}z_1z_2.
\]

Since \( \gamma = 2k + 2 - \alpha \) then \( 2\gamma - 2 = 2k + \gamma - \alpha \) and \( \gamma + \alpha - 2 = 2k \). Thus

\[
\det \left[ \frac{\partial^2 u(z)}{\partial z_i \partial \overline{z}_j} \right] = \begin{pmatrix}
(1 - |z_1|^2)^{\gamma - 2}(\gamma|z_1|^2 - 1) + \alpha^{-1}\gamma(1 - |z_1|^2)^{\alpha - 2}(1 - \alpha|z_1|^2) \\
-\gamma^2(1 - |z_1|^2)^{2\gamma - 2}|z_2|^2|z_2|^2
\end{pmatrix} 
= \gamma(1 - |z_1|^2)^{2\gamma - 2}|z_2|^2(\gamma|z_1|^2 - 1 - \gamma|z_1|^2) + \gamma(1 - |z_1|^2)^{\alpha - 2+\gamma}(\frac{1}{\alpha} - |z_1|^2) 
= \gamma(1 - |z_1|^2)^{2k}\left[ - (1 - |z_1|^2)^{\gamma - \alpha}|z_2|^2 + (\frac{1}{\alpha} - |z_1|^2) \right] 
\]

Since \( \gamma \geq 1 \) and \( |z_1|^2 \in [0, 1] \), then

\[
-(1 - |z_1|^2)^{\gamma - \alpha}|z_2|^2 + (\frac{1}{\alpha} - |z_1|^2) 
\geq (\frac{1}{\alpha} - |z_1|^2) - (1 - |z_1|^2)^{\gamma} 
= (\frac{1}{\alpha} - |z_1|^2) - (1 - |z_1|^2) 
= (\frac{1}{\alpha} - 1) > 0.
\]

Therefore, it is easy to see that if \( k = 0 \) then \( f(z) \in C^{1,1-\alpha}(\overline{E}_{2m}) \) and \( f(z) > 0 \) on \( \overline{E}_{2m} \); if \( k > 0 \) then \( f(z) \geq 0 \) and \( f(z)^{1/2} \in C^{k+1,1-\alpha}(\overline{E}_{2m}) \).

Hence, Part (a) is proved. Parts (b) and (c) follow immediately from the fact that \( |z_2|^2m = (1 - |z_1|^2) \) on \( \partial E_{2m} \). Therefore, the proof of the all conclusions of the example is complete. \[ \square \]

**REMARK 2** (i) By parts (a) and (c) of the previous example, one can see that Theorem 1.2 is sharp for \( 0 < \alpha \leq 2/m \).
(ii) By parts (a) and (b) in the previous example, the Dirichlet problem (1.1) cannot have better than global $C^{2/m}$ plurisubharmonic solution when type of the domain is bigger than 2 no matter how smooth of $\phi$ and $f(z)^{1/n}$ are.

**EXAMPLE 2** Let

$$D = \{(z_1, z_2) \in \mathbb{C}^2 : r(z) = |z_1|^2 + \exp(2 - |z_2|^2) < 1\}$$

Then $D$ is a bounded pseudoconvex domain in $\mathbb{C}^2$ with $C^\infty$ boundary. Let

$$g(t) = (1 - t)^k, \quad h(t) = -\frac{10k}{h_0(t)}, \quad \text{and} \quad h_0(t) = 2 - \log(1 - t),$$

and let

$$G(z) = |z_2|^2g(|z_1|^2) + h(|z_1|^2).$$

Then $G \in C^{(D)}$ is plurisubharmonic in $D$ and $G \not\in C^\infty(D)$ for any $\epsilon > 0$. Moreover,

$$G(z) = |z_2|^2(1 - |z_1|^2)^k - 10k|z_2|^2, \quad z = (z_1, z_2) \in \partial D,$$

and

$$\sqrt{\det H(G)} \in C^{\frac{k}{2} - 1 - \epsilon}(D), \quad \text{for any} \quad k \geq 4, \quad \epsilon > 0.$$

**Proof.** Since

$$G_{11} = |z_2|^2[g'(|z_1|^2) + g''(|z_1|^2)|z_1|^2] + h'(|z_1|^2) + |z_1|^2h''(|z_1|^2)$$

and

$$G_{12} = \bar{z}_2z_1 g'(|z_1|^2), \quad G_{\bar{z}z} = g(|z_1|^2).$$

Since

$$g'(t) = -k(1 - t)^{k-1}, \quad g''(t) = k(k - 1)(1 - t)^{k-2},$$

then

$$g(t)g'(t) + t[g(t)g''(t) - g'(t)^2] = -k(1 - t)^{2k-2}[1 - (k-1)t + kt] = -k(1 - t)^{2k-2}.$$ 

Since

$$h'(t) = \frac{10k}{h_0(t)^2} \frac{1}{1 - t}, \quad h''(t) = \left[\frac{10k}{h_0(t)^2} - \frac{20k}{h_0(t)^3}\right] \frac{1}{(1 - t)^2}.$$
and
\[
\nu'(t) + t\nu''(t) = \frac{10k}{h_0(t)(1-t)} \left[ \frac{t}{h_0(t)} + \frac{t}{h_0(t)(1-t)} - \frac{2t}{h_0(t)^2(1-t)} \right] = \frac{10k}{h_0(t)^2(1-t)^2} \left[ 1 - \frac{2t}{h_0(t)} \right].
\]

Thus
\[
\det[G_{ij}](z) = g(|z_1|^2) \left[ |z_2|^2 g'(|z_1|^2) + |z_2|^2 |z_1|^2 g''(|z_1|^2) + h'(|z_1|^2) + |z_1|^2 h''(|z_1|^2) \right]
\]
\[
- |z_1|^2 |z_2|^2 g'(|z_1|^2)^2
\]
\[
= |z_2|^2 \left[ g(|z_1|^2) g'(|z_1|^2) + |z_1|^2 g(|z_1|^2) g''(|z_1|^2) - g'(|z_1|^2)^2 \right]
\]
\[
+ g(|z_1|^2) (h'(|z_1|^2) + |z_1|^2 h''(|z_1|^2))
\]
\[
= -k |z_2|^2 (1 - |z_1|^2)^{2k-2} + 10k (1 - |z_1|^2)^{k-2} \left[ \frac{1}{h_0(|z_1|^2)^2} - \frac{2|z_1|^2}{h_0(|z_1|^2)^3} \right]
\]
\[
= (1 - |z_1|^2)^{k-2} \left[ -k |z_2|^2 (1 - |z_1|^2)^k + \frac{10k}{h_0(|z_1|^2)^2} - \frac{20k|z_1|^2}{h_0(|z_1|^2)^3} \right]
\]
\[
= f(|z_1|^2, |z_2|^2) \geq 0.
\]

It is easy to see that \( \sqrt{f(z)} \in C^{\frac{3}{2} - 1 - \epsilon} \) for any \( \epsilon > 0 \) and any integer \( k \geq 4 \). Therefore, the proof is complete. \( \square \)

4 Applications

THEOREM 4.1 Let \( \Omega_1 \) and \( \Omega \) be two bounded pseudoconvex domains in \( \mathbb{C}^n \) with \( C^3 \) boundaries. Let \( \phi : \Omega_1 \rightarrow \Omega \) be a proper holomorphic map. Then there are \( \alpha = \alpha(\Omega_1) \in (0, 1] \) and a constant \( C > 1 \) such that
\[
\delta_{\Omega}(\phi(z)) \leq C \delta_{\Omega_1}(z)^{\alpha}, \quad z \in \Omega_1.
\]

Proof. We choose \( z_0 \in \Omega \) and we choose a non-negative function \( f(z) \in C^\infty(\mathbb{C}^n) \) so that \( f(z) = 1 \) on \( B(z_0, \delta_{\Omega}(z_0)/2) \) and \( f = 0 \) on \( \Omega \setminus B(z_0, 3\delta_{\Omega}(z_0)/4) \) such that \( f^{1/n} \in C^1(\overline{\Omega}) \). For any \( U \in C^\alpha(\overline{\Omega}) \) be subharmonic function in a domain \( \Omega \) so that \( U(z) = 0 \) on \( \partial\Omega \) and \( \Delta U(z) \geq f(z) \geq 0 \) on \( \Omega \). We have
\[
\int_\Omega -G(z, w)f(w)dv(w) \leq -U(z) \leq C \delta_{\Omega}(z)^{\alpha}
\]

14
where $G(z, w)$ is Green’s function for $\Delta$ on $\Omega \subset \mathbb{R}^{2n}$.

In fact, $U = 0$ on $\partial\Omega$. Thus

$$U(z) = \int_{\Omega} G(z, w) \Delta U(w) dv(w)$$

in sense of distribution. Since $\Delta U(w) \geq f(w) \geq 0$ and $-G(z, w) \geq 0$, we have

$$-U(z) = \int_{\Omega} -G(z, w) \Delta U(w) dv(w) \geq \int_{\Omega} -G(z, w)f(w) dv(w)$$

and it was know that

$$-G(z, w) \geq \frac{1}{C} \delta_{\Omega}(z) \delta_{\Omega}(w), \quad z, w \in \Omega$$

Therefore,

$$-U(z) \geq \frac{\delta_{\Omega}(z)}{C} \int_{\Omega} \delta_{\Omega}(w)f(w) dv(w) = \frac{\delta_{\Omega}(z)}{C_1}. \quad \text{Thus}$$

$$\frac{\delta_{\Omega}(z)}{C_1} \leq -U(z) \leq C \delta_{\Omega}(z)\alpha.$$

Now let $U$ be the solution of the complex Monge-Ampère equation

$$\det(U_{ij}) = f \quad \text{in } \Omega; \quad U = 0 \quad \text{on } \partial\Omega.$$

Then, by Theorem 2.5, $U \in C^\alpha(\overline{\Omega})$ for some $\alpha = \alpha(\Omega) \in (0, 1)$, and $\Delta U(z) \geq f(z)^{1/n}$. Thus

$$\frac{\delta_{\Omega}(z)}{C_1} \leq -U(z) \leq C \delta_{\Omega}(z)\alpha.$$

Since $\phi : \Omega_1 \to \Omega$ is a proper holomorphic map. We have that $V(z) = U(\phi)$ is the unique plurisubharmonic solution of the Monge-Ampère equation

$$\det(V_{ij}) = f(\phi(z))|\det \phi'(z)|^2 \quad \text{in } \Omega_1; \quad V = 0 \quad \text{on } \partial\Omega_1.$$

Let $g(z) = f(\phi(z))|\det \phi'(z)|^2 \geq 0$ and $g^{1/n} \in C^1(\overline{\Omega}_1)$. By Theorem 2.5, we have $V(z) \in C^\alpha(\Omega_1)$ for some $\alpha = \alpha(\Omega_1) \in (0, 1]$. Thus

$$-V(z) \leq C \delta_{\Omega_1}(z)\alpha.$$

Therefore

$$\delta_{\Omega}(\phi(z)) \leq -C_1 U(\phi(z)) = -C_1 V(z) \leq C_1 C \delta_{\Omega_1}(z)\alpha$$

and the proof is complete.  \[\blacksquare\]
References


