Complex Analysis Study Guide

1. Equivalences Of Holomorphicity For a domain $D \subset \mathbb{C}$, $f(z) = u + iv$ is holomorphic in D if and only if $\dfrac{\partial f}{\partial \bar{z}} = 0$:

(a) If and only if u and v satisfy the Cauchy-Riemann Equations

Proof:

$$0 = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(b) If and only if $f(z) \in C^1(D)$.

Proof:
First assume $f' \in C^1(D)$.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0, h \in \mathbb{R}} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h}$$

$$= \lim_{h \to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i\left(\lim_{h \to 0} \frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h}\right) = \left.\frac{\partial u}{\partial x}\right|_{(x_0, y_0)} + i\left.\frac{\partial v}{\partial x}\right|_{(x_0, y_0)}.$$

On the other hand

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0, h \in \mathbb{R}} \frac{f(z_0 + ih) - f(z_0)}{ih}$$

$$= \lim_{h \to 0} \frac{1}{i}\left[\frac{u(x_0, y_0 + h) - u(x_0, y_0)}{h} + i\left(\lim_{h \to 0} \frac{v(x_0, y_0 + h) - v(x_0, y_0)}{h}\right)\right] = (-i)\left.\frac{\partial u}{\partial y}\right|_{(x_0, y_0)} + \left.\frac{\partial v}{\partial y}\right|_{(x_0, y_0)}.$$

Equating the real and imaginary parts for $f'(z_0)$, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = i\frac{\partial v}{\partial x}$$

evaluated at (x_0, y_0). By part **(a)**, f is holomorphic. Now assume f is holomorphic on D. Fix $z_0 \in D$. For z near z_0 we may define $\gamma(t) = (1 - t)z_0 + tz$ and $\gamma : [0, 1] \to D$.

$$f(z) - f(z_0) = f(\gamma(1)) - f(\gamma(0)) = \oint_\gamma \frac{\partial f}{\partial z} dz = \int_0^1 \frac{\partial f}{\partial z}(\gamma(t))\frac{d\gamma}{dt} dt = \int_0^1 \frac{\partial f}{\partial z}(\gamma(t))(z - z_0) dt.$$

Dividing both sides by $z - z_0$ yields

$$\frac{f(z) - f(z_0)}{z - z_0} = \int_0^1 \frac{\partial f}{\partial z}(\gamma(t)) dt$$

$$= \int_0^1 \frac{\partial f}{\partial z}(z_0) dt + \int_0^1 \left[\frac{\partial f}{\partial z}(\gamma(t)) - \frac{\partial f}{\partial z}(z_0)\right] dt = \frac{\partial f}{\partial z}(z_0) + \int_0^1 \left[\frac{\partial f}{\partial z}(\gamma(t)) - \frac{\partial f}{\partial z}(z_0)\right] dt.$$

Let $\epsilon > 0$. As $\frac{\partial f}{\partial z}$ is continuous, there is a $\delta > 0$ such that

$$\left|\frac{\partial f}{\partial z}(w) - \frac{\partial f}{\partial z}(z_0)\right| < \epsilon$$

whenever $|w - z_0| < \delta$. Note that $|\gamma(t) - z_0| = t|z - z_0| \leq |z - z_0|$ for $t \in [0, 1]$, therefore

$$\left|\frac{\partial f}{\partial z}(\gamma(t)) - \frac{\partial f}{\partial z}(z_0)\right| < \epsilon$$

for $|z - z_0| < \delta$. Fixing $|z - z_0| < \delta$, we have

$$\left|\int_0^1 \left[\frac{\partial f}{\partial z}(\gamma(t)) - \frac{\partial f}{\partial z}(z_0)\right] dt\right| \leq \int_0^1 \left|\frac{\partial f}{\partial z}(\gamma(t)) - \frac{\partial f}{\partial z}(z_0)\right| dt \leq \int_0^1 \epsilon dt = \epsilon.$$

It follows that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{\partial f}{\partial z}(z_0).$$

2. Interaction of Holomorphic Functions

(a) $\mathcal{O}(D)$, the set of all holomorphic functions on D forms an algebra.

Proof:
Let $f, g, h \in \mathcal{O}(D)$ and $a, b \in \mathbb{C}$. To be an algebra, the following conditions must hold:

- $(f + g) \cdot h = f \cdot h + g \cdot h$.
- $f \cdot (g + h) = f \cdot g + f \cdot g$.
- $(af) \cdot (bg) = (ab)(f \cdot g)$.

As the algebra multiplication is standard commutative multiplication, it suffices to show that if f and g are any holomorphic function on D, then fg is as well.

$$\frac{\partial (fg)}{\partial \bar{z}} = \frac{\partial f}{\partial \bar{z}} g + f \frac{\partial g}{\partial \bar{z}} = 0g + f0 = 0.$$

So $\mathcal{O}(D)$ is in fact an algebra.

(b) Moreover, if $f : D \to E$ is holomorphic and $g : E \to C$ is holomorphic, then $g \circ f$ is holomorphic.

Proof:
As f is holomorphic, $f(x, y) = u_f(x, y) + v_f(x, y)$ and $g(x, y) = u_g(x, y) + v_g(x, y)$.

$$u(x, y) + iv(x, y) = g \circ f = g(u_f, v_f) = u_g(u_f, v_f) + iv_g(u_f, v_f).$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(u_g(u_f, v_f)\right) = \frac{\partial u_g}{\partial u_f}\frac{\partial u_f}{\partial x} + \frac{\partial u_g}{\partial v_f}\frac{\partial v_f}{\partial x}$$

and

$$\frac{\partial v}{\partial y} = \frac{\partial v_g}{\partial u_f}\frac{\partial u_f}{\partial y} + \frac{\partial v_g}{\partial v_f}\frac{\partial v_f}{\partial y}.$$

As g is holomorphic,

$$\frac{\partial u_g}{\partial u_f} = \frac{\partial v_g}{\partial v_f}, \frac{\partial u_g}{\partial v_f} = -\frac{\partial v_g}{\partial u_f}$$

and as f is holomorphic

$$\frac{\partial u_f}{\partial x} = \frac{\partial v_f}{\partial y}, \frac{\partial u_f}{\partial y} = -\frac{\partial v_f}{\partial x}$$

So

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

so $g \circ f$ is holomorphic.

(c) If u is harmonic on a domain E, $f : D \to E$ is holomorphic, then $u \circ f$ is harmonic in D.

Proof:
Let $z_0 \in E$. As E is a domain, there exists a $r > 0$ such that $D(z_0, r) \subset E$. As $D(z_0, r)$ is simply connected, on this set we can find some holomorphic g such that $\text{Re}(g) = u$. On $D(z_0, r)$, $g \circ f$ is holomorphic, so its real part is harmonic. Notice $\text{Re}(g \circ f) = u \circ f$.

3. Harmonic Conjugates and Antiderivatives

(a) If $f = u + iv$ is holomorphic, then u and v are harmonic. v is called the harmonic conjugate of u.

Proof:
By the Cauchy-Riemann Equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Thus

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right).$$

As $f \in C^\infty(D)$, order of differentiation can be changed, so

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial v}{\partial x \partial y} - \frac{\partial v}{\partial x \partial y} = 0.$$

The computation to show v is harmonic is identical.

Lemma 1. If f, g are C^1 functions on a simply connected set D and if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ on D, then there is a function $h \in C^2(D)$ such that

$$\frac{\partial h}{\partial x} = f, \frac{\partial h}{\partial y} = g$$

on D.

Proof:
For $(x, y) \in D$, set

$$h(x, y) = \int_a^x f(t, b)dt + \int_b^y g(x, s)ds,$$

where $(a, b) \in D$. As D is simply connected, these integrals exist. By the Fundamental Theorem of Calculus

$$\frac{\partial h}{\partial y}(x, y) = g(x, y).$$

Again by the Fundamental Theorem of Calculus and since $g \in C^1(D)$,

$$\frac{\partial h}{\partial x}(x, y) = f(x, b) + \frac{\partial}{\partial x}\int_b^y g(x, s)ds = f(x, b) + \int_b^y \frac{\partial}{\partial x} g(x, s)ds = f(x, b) + \int_b^y \frac{\partial}{\partial y} f(x, s)ds$$

$$= f(x, b) + f(x, y) - f(x, b) = f(x, y).$$

(b) Given a harmonic function u on a domain D, if D is simply connected u has a harmonic conjugate v such that $F = u + iv$ is holomorphic in D. Give a counterexample if D is not simply connected.

Proof:
Let $f = -\frac{\partial u}{\partial y}$, $g = \frac{\partial u}{\partial x}$. As u is harmonic, we have $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ on D. Since $f, g \in C^1(D)$, by Lemma 1, there exists a $v \in C^2(D)$ such that

$$\frac{\partial v}{\partial x} = f = -\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y} = g = \frac{\partial u}{\partial x}.$$

By the Cauchy-Riemann Equations, $F = u + iv$ is holomorphic.
Let D be the punctured unit disc. $u(x, y) = \ln(x^2 + y^2)$ is harmonic on D. Moreover, u along with the function $v(x, y) = 2\tan^{-1}(y/x)$ satisfies the Cauchy-Riemann equations. However, v is not defined everywhere on D, so $u + iv$ is not holomorphic on D.

(c) If f is holomorphic in a domain D, prove that if D is simply connected then there exists a holomorphic function F such that $F'(z) = f(z)$ on D. Give a counterexample if D is not simply connected.

Proof:

Let $f = u + iv$. Let $g = u$, $h = -v$, then by the Cauchy-Riemann equations we have $\dfrac{\partial g}{\partial y} = \dfrac{\partial h}{\partial x}$. By Lemma 1 there exists a real C^2 function f_1 such that

$$\frac{\partial f_1}{\partial x} = g = u, \frac{\partial f_1}{\partial y} = h = -v.$$

Now let $\hat{g} = v, \hat{h} = u$. Again by the Cauchy-Riemann equations, we get $\frac{\partial \hat{g}}{\partial y} = \frac{\partial \hat{h}}{\partial x}$ so by the Lemma there exists a function f_2 such that

$$\frac{\partial f_2}{\partial x} = \hat{g} = v, \frac{\partial f_2}{\partial y} = \hat{h} = u.$$

Let $F = f_1 + if_2$. F is C^2 and by the above equations F satisfies the Cauchy-Riemann equations, so F is holomorphic. Finally,

$$\frac{\partial}{\partial z}F = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_1 + f_2) = \frac{1}{2}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) = \frac{1}{2}(u+u) + \frac{i}{2}(v+v) = f.$$

Let D be the punctured unit disc. Let $f(z) = 1/z$. f is holomorphic on D. Assume there exists a holomorphic function F such that $F'(z) = f(z)$. Let $\gamma = \partial D(0,1)$. By calculation $\oint_\gamma f(z) = 2\pi i$. However, by the Fundamental Theorem of Calculus, $\oint_\gamma f(z) = F(\gamma(1)) - F(\gamma(0)) = 0$. So no F can exist.

4. Cauchy's Theorem, Morera's Theorem and Cauchy's Integral Formula

(a) Cauchy's Theorem

D is a bounded domain in $\mathbb{C}$ with piecewise C^1 boundary. If f is holomorphic in D and $f \in C(\overline{D})$, then

$$\oint_{\partial D} f(z)dz = 0.$$

Proof:

By Stokes' Theorem,

$$\oint_{\partial D} f(z)dz = \int_D d(f(z)dz) = \int_D \left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}\right) \wedge dz = \int_D \frac{\partial f}{\partial z}dz \wedge dz + 0d\overline{z} \wedge dz = 0.$$

(b) Morera's Theorem

Let D be connected. If $f \in C(D)$ and for any simply closed piecewise C^1 curve γ in D we have $\oint_\gamma f(z)dz = 0$, then f is holomorphic in D.

Proof:

Fix $z_0 \in D$. Define a function $F : D \to \mathbb{C}$ as follows. Given $z \in D$ choose a piecewise C^1 curve $\phi : [0,1] \to \mathbb{C}$ such that $\phi(0) = z_0$, $\phi(1) = z$. Set

$$F(z) = \int_\phi f(w)dw.$$

It is not yet clear that F is well-defined. Let τ be any piecewise C^1 curve such that $\tau(0) = z_0$, $\tau(1) = z$. Let μ be the closed curve $\phi \cup (-\tau)$. μ is piecewise C^1. By assumption we have

$$0 = \oint_\mu f(w)dw = \int_\phi f(w)dw - \int_\tau f(w)dw.$$

Let $F = U + iV$.

Fix $z = (x,y) \in D$ and ϕ going from z_0 to (x,y). Choose $h \in \mathbb{R}$ small enough so that $(x+t, y) \in D$ for $0 \le t \le h$. Let $l_h(t)$ be the line segment connecting (x,y) and $(x+h,y)$. Let $\phi_h = \phi \cup l_h$.

$$F(x+h,y) - F(x,y) = \int_{\phi_h} f(w)dw - \int_\phi f(w)dw = \int_{l_h} f(w)dw = \int_0^h f(z+s)ds.$$

Taking the real part of this equation yields

$$\frac{U(x+h,y)-U(x,y)}{h} = \frac{1}{h}\text{Re}\int_0^h f(z+s)ds = \frac{1}{h}\int_0^h \text{Re}f(z+s)ds.$$

Letting $h \to 0$ on both sides yields that

$$\frac{\partial U}{\partial x}(z) = \text{Re}f(z).$$

Similar calculations yield that

$$\frac{\partial U}{\partial y} = -\text{Im}f, \frac{\partial V}{\partial x} = \text{Im}f, \frac{\partial V}{\partial y} = \text{Re}f.$$

Thus F is holomorphic. Thus $F' = f$ is holomorphic.

(c) Cauchy's Integral Formula

Let D be a bounded open domain in $\mathbb{C}$ with piecewise C^1 boundary. Let $f(z)$ be holomorphic in D and $f \in C(\overline{D})$. Then

$$\frac{1}{2\pi i}\oint_{\partial D}\frac{f(w)}{w-z}dw = f(z), z \in D.$$

Proof:

$$\frac{1}{2\pi i}\oint_{\partial D}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\oint_{\partial D_\epsilon}\frac{f(w)}{w-z}dw + \frac{1}{2\pi i}\oint_{\partial D(z,\epsilon)}\frac{f(w)}{w-z}dw$$

where $D_\epsilon = D \backslash \overline{D(z,\epsilon)}$ for some $\epsilon > 0$ such that $D(z,\epsilon) \subset D$. By Cauchy's Integral Theorem we have

$$\frac{1}{2\pi i}\oint_{\partial D}\frac{f(w)}{w-z}dw = 0 + \frac{1}{2\pi i}\oint_{\partial D(z,\epsilon)}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z+\epsilon e^{i\theta})}{\epsilon e^{i\theta}}\epsilon i e^{i\theta}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z+\epsilon e^{i\theta})d\theta \overset{\epsilon\to 0}{\longrightarrow} \frac{1}{2\pi}\int_0^{2\pi} f(z)d\theta = f(z).$$

(c') Generalized Cauchy's Integral Formula

Let D be a bounded open domain in $\mathbb{C}$ with piecewise C^1 boundary. Let $f(z)$ be holomorphic in D and $f \in C(\overline{D})$. Then

$$\frac{k!}{2\pi i}\oint_{\partial D}\frac{f(w)}{(w-z)^{k+1}}dw = f^{(k)}(z), z \in D.$$

Proof:

For $z \in D$,

$$f^{(k)}(z) = \frac{\partial^k}{\partial z^k}\frac{1}{2\pi i}\oint_{\partial D}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\oint_{\partial D}\frac{\partial^k}{\partial z^k}\left(\frac{f(w)}{w-z}\right)dw = \frac{1}{2\pi i}\oint_{\partial D} f(w)\frac{\partial^k}{\partial z^k}\left(\frac{1}{w-z}\right)dw$$

$$= \frac{k!}{2\pi i}\oint_{\partial D}\frac{f(w)}{(w-z)^{k+1}}dw.$$

(d) Mean Value Property

If f is holomorphic in $D(z_0, R)$, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{z_0+ri\theta})d\theta$$

for $0 < r < R$.

Proof:
By Cauchy's Integral Formula

$$f(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{re^{i\theta}}rie^{i\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(re^{z_0+ri\theta})d\theta,$$

using the substitution $z = z_0 + re^{i\theta}$.

(d') If u is harmonic in $D(z_0, R)$, then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta.$$

Proof:
Since u is harmonic and $D(0, R)$ is simply connected, there exists a harmonic function v such that $f = u + iv$ is holomorphic. By the Mean Value Property,

$$u(z_0)+iv(z_0) = f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{z_0+ri\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(re^{z_0+ri\theta})d\theta + \frac{i}{2\pi}\int_0^{2\pi} v(re^{z_0+ri\theta})d\theta.$$

Equating the real parts of both sides yields the desired result as u and v are real-valued.

5. Applications of Cauchy's Integral Formula

(a) Liouville's Theorem
Any bounded entire function is constant.

Proof:
Let $z_0 \in \mathbb{C}$, $R > 0$. Let $|f| < M$.

$$f'(z_0) = \frac{1}{2\pi i}\oint_{|w-z_0|=R}\frac{f(w)}{(w-z_0)^2}dw.$$

$$|f'(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi}\frac{|f(z_0+Re^{i\theta})|}{|Re^{i\theta}|^2}|Rie^{i\theta}|d\theta \le \frac{M}{2\pi R}\int_0^{2\pi} d\theta = \frac{M}{R} \overset{R\to\infty}{\longrightarrow} 0.$$

So $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$. So f is constant.

(b) Linear and Sublinear Growth. Let $f(z)$ be entire.

i. If f is sub-linear growth (i.e. $\lim_{z\to\infty}\frac{f(z)}{|z|} = 0$), then f is a constant.

Proof:
By Cauchy's Integral Formula

$$f'(z) = \frac{1}{2\pi i}\oint_{|w|=R}\frac{f(w)}{(w-z)^2}dw$$

$$|f'(z)| \le \frac{1}{2\pi}\oint_{|w|=R}\frac{|f(w)|}{(R-|z|)^2}dw \le \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{|f(Re^{i\theta})|}{R}\right)\frac{R^2}{(R-|z|)^2}d\theta.$$

As $R \to \infty$, $\left(\frac{|f(Re^{i\theta})|}{R}\right) \to 0$ by assumption. So $f'(z) = 0$. So f is constant.

ii. If $f(z)$ is polynomial growth (i.e. there is some positive integer n such that $|f(z)| \le c_n(1 + |z|^n)$), then $f(z)$ is a polynomial.

Proof:
Let $z \in \mathbb{C}$ and $R >> 2|z| + 1$.

$$f^{(n+1)}(z) = \frac{(n+1)!}{2\pi i}\oint_{|w|=R}\frac{f(w)}{(w-z)^{n+2}}dw.$$

$$|f^{(n+1)}(z)| \leq \frac{(n+1)!}{2\pi}\int_0^{2\pi}\frac{|f(Re^{i\theta})|}{(R-|z|)^{n+1}}\frac{R}{R-|z|}d\theta \leq \frac{(n+1)!}{2\pi}\int_0^{2\pi}\frac{c_n(1+R^n)}{(R-|z|)^{n+1}}\frac{R}{R-|z|}d\theta \overset{R\to\infty}{\longrightarrow} 0.$$

So f is a polynomial of degree at most n.

(c) If f is holomorphic in a domain D, then f is analytic in D.

Proof:
Let $z_0 \in D$, $r_0 > 0$ such that $D(z_0, r_0) \subset D$. By Cauchy's Integral Formula, for $0 < R < r_0$, $z \in D(z_0, R)$,

$$f(z) = \frac{1}{2\pi i}\oint_{|w-z_0|=R}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\oint_{|w-z_0|=R}\frac{f(w)}{w-z_0-(z-z_0)}dw = \frac{1}{2\pi i}\oint_{|w-z_0|=R}\frac{f(w)}{(w-z_0)(1-\frac{z-z_0}{w-z_0})}dw.$$

Note that $\left|\frac{z-z_0}{w-z_0}\right| < 1$. So

$$f(z) = \frac{1}{2\pi i}\oint_{|w-z_0|=R}\frac{f(w)}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n dw = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_{|w-z_0|=R}\frac{f(w)}{w-z_0}\left(\frac{z-z_0}{w-z_0}\right)^n dw$$

since the sum converges uniformly on $\partial D(z_0, R)$. By Cauchy's Integral Formula

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_{|w-z_0|=R}\frac{f(w)}{(w-z_0)^{n+1}}dw(z-z_0)^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

So this sum converges uniformly on $D(z_0, R)$ and has radius of convergence at least r_0 and the radius of convergence is given by $\limsup_{n\to\infty}|\sqrt[n]{a_n}|$.

(d) Uniform Limits of Holomorphic Functions
Let $f_j : D \to \mathbb{C}$ for $j = 1, 2, \ldots$ be a sequence of holomorphic functions on an open set D in $\mathbb{C}$. Suppose that there is a function $f : D \to \mathbb{C}$ such that, for any compact subset K of D, the sequence $f_j \to f$ uniformly on K. Then f is holomorphic on D.

Proof:
Let $z_0 \in D$ be arbitrary. Choose $r > 0$ such that $\overline{D(z_0, r)} \subset D$. Since $\{f_j\}$ converges to f uniformly on $\overline{D(z_0, r)}$ and since each f_j is continuous, f is also continuous on $\overline{D(z_0, r)}$. For any $z \in D(z_0, r)$,

$$f(z) = \lim_{j\to\infty} f_j(z) = \lim_{j\to\infty}\frac{1}{2\pi i}\oint_{|w-z_0|=r}\frac{f_j(z)}{w-z}dw = \frac{1}{2\pi i}\oint_{|w-z_0|=r}\lim_{j\to\infty}\frac{f_j(z)}{w-z}dw = \frac{1}{2\pi i}\oint_{|w-z_0|=r}\frac{f(z)}{w-z}dw.$$

The interchange of integral and limit is justified by the fact that, for z fixed, $f_j(w)/(w-z)$ converges to $f(w)/(w-z)$ uniformly for w in the compact set $\{w : |w-z_0| = r\}$.

(d') If f_j, f, D are as in the Theorem above, then for any integer k,

$$\left(\frac{\partial}{\partial z}\right)^k f_j(z) \to \left(\frac{\partial}{\partial z}\right)^k f(z)$$

uniformly on compact sets.

Proof:
As we have shown that

$$f_j(z) = \frac{1}{2\pi i}\oint_{|w-z_0|=r}\frac{f_j(w)}{w-z}dw \to \frac{1}{2\pi i}\oint_{|w-z_0|=r}\frac{f(w)}{w-z}dw = f(z)$$

uniformly, we similarly have that

$$f_j^{(k)}(z) = \frac{k!}{2\pi i}\oint_{|w-z_0|=r}\frac{f_j(w)}{(w-z)^{k+1}}dw \to \frac{k!}{2\pi i}\oint_{|w-z_0|=r}\frac{f(w)}{(w-z)^{k+1}}dw = f^{(k)}(z)$$

as f is holomorphic.

(e) Fundamental Theorem of Algebra

Let $p_n(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$ be a polynomial of degree $n \geq 1$. Then $p_n(z)$ must have a zero in $\mathbb{C}$.

Proof:

Assume toward contradiction that $p_n(z) \neq 0$ on $\mathbb{C}$. Then $\dfrac{1}{p_n(z)}$ is entire. There exists $R \geq 1$ such that for $|z| \geq R$,

$$\frac{1}{2}|a_n||z|^n \leq |p_n(z)| \leq 2|a_n||z|^n.$$

So

$$\left|\frac{1}{p_n(z)}\right| \leq \frac{1}{\frac{1}{2}|a_n||z|^n} \leq \frac{2}{|a_n|}.$$

As $\dfrac{1}{p_m(z)}$ is continuous it has a maximum on the compact set $\overline{D(0,R)}$. Let M be this maximum. Therefore

$$\frac{1}{|p_n(z)|} \leq M + \frac{2}{|a_n|}$$

for $z \in \mathbb{C}$. Therefore $\dfrac{1}{p_n(z)}$ is constant, implying $p_n(z)$ is constant, contradicting that $n \geq 1$. Therefore $p_n(z)$ has a zero in $\mathbb{C}$.

(f) Uniqueness Theorem

If $f(z)$ is holomorphic in an open connected domain D and if $f(z) = 0$ on an open subset of D, then $f(z) \equiv 0$.

Proof:

Let z_0 be an accumulation point of $Z(f)$.

Claim: $\dfrac{\partial^n f}{\partial z^n}(z_0) = 0$ for all $n \in \mathbb{Z}^+$. Assume toward the contrary that this is not the case. Then there is some n_0 such that $\dfrac{\partial^{n_0} f}{\partial z^{n_0}}(z_0) \neq 0$. Then, on some disc $D(z_0, r) \subset D$, we have

$$f(z) = \sum_{j=n_0}^{\infty} \left(\frac{\partial^j}{\partial z^j} f(z_0)\right) \frac{(z-z_0)^j}{j!}.$$

Hence the function g defined by

$$g(z) = \sum_{j=n_0}^{\infty} \left(\frac{\partial}{\partial z}\right)^j f(z_0) \frac{(z-z_0)^{j-n_0}}{j!}$$

is holomorphic on $D(z_0, r)$. Notice that $g(z_0) \neq 0$ by our choice of n_0. As z_0 is an accumulation point, there exists a sequence $\{z_k\} \subset Z(f)$ such that $z_k \to z_0$. Note that $g(z_k) = 0$ for all z_k. By the continuity of g this implies $g(z_0) = 0$. So $f^{(n)}(z_0) = 0$ for all n.

As $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n (z-z_0)^n$ for $z \in D(z_0, r_0)$, where $r_0 = \text{dist}(z_0, \partial D)$. By definition of a_n, we have that $f(z) \equiv 0$ on $D(z_0, r_0)$. If $D(z_0, r_0) \neq D$, then there exists some $z_1 \in D(z_0, r_0)$, $z_1 \neq z_0$ such that there is a $r_1 > 0$ such that $D(z_1, r_1) \subset D$ and $D(z_1, r_1) \cap (D - D(z_0, r_0)) \neq \emptyset$ since D is open. So we may continue inductively as D is connected to find $\{z_i\}$ such that

$$\bigcup_{j=0}^{\infty} D(z_j, r_j) = D$$

and $f(z) \equiv 0$ on each $D(z_j, r_j)$. So $f(z) \equiv 0$ on D. Note that if $f(z) \not\equiv 0$, then $Z(f)$ must be discrete.

(f') Factorization of a Holomorphic Function.

Let $f(z)$ be holomorphic in D, $f \not\equiv 0$ on D. If $z_0 \in D$ such that $f(z_0) = 0$, then there exists some $k \in \mathbb{Z}^+$ and a holomorphic function g where $g(z_0) \neq 0$ such that $f(z) = (z - z_0)^k g(z)$.

Proof:

Since $f(z) \not\equiv 0$ on D, there is some k such that k is the first positive integer such that $f^{(k)}(z_0) \neq 0$. So

$$f(z) = \sum_{j=k}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j, \quad z \in D(z_0, r_0).$$

$$f(z) = (z - z_0)^k \sum_{j=0}^{\infty} \frac{f^{(j+k)}(z_0)}{(j+k)!}(z - z_0)^j$$

Let

$$g(z) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \frac{f^{(j+k)}(z_0)}{(j+k)!}(z - z_0)^j & \text{if} \quad z \in D(z_0, r) \\ \displaystyle\frac{f(z)}{(z - z_0)^k} & \text{if} \quad z \in D \backslash D(z_0, r) \end{cases}$$

$g(z)$ is holomorphic in $D(z_0, r)$ and, when $z \neq z_0$, $\dfrac{f(z)}{(z - z_0)^k}$ is holomorphic on $D \backslash \{z_0\}$. So $g \in C(D)$. So g is holomorphic in D and

$$g(z_0) = \frac{f^{(k)}(z_0)}{k!} \neq 0$$

and

$$f(z) = (z - z_0)^k g(z), z \in D.$$

6. Isolated Singularities and Laurent Series

(a) A Laurent Series for a meromorphic function $f(z)$ about an isolated singularity z_0 is given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$, where γ is a positively oriented simple closed curve enclosing z_0 in the annulus of convergence of $f(z)$. The annulus of convergence of a Laurent series is given by $A(z_0, r, R) = D(z_0, R) \backslash \overline{D(z_0, r)}$, where

$$r = \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|} \quad \text{and} \quad R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Proof:Let $r < s_1 < |z - z_0| < s_2 < r$.

$$f(z) = \frac{1}{2\pi i} \oint_{|w-z_0|=s_2} \frac{f(w)}{w - z} dw - \frac{1}{2\pi i} \oint_{|w-z_0|=s_1} \frac{f(w)}{w - z} dw.$$

Now

$$\oint_{|w-z_0|=s_2} \frac{f(w)}{w - z} dw = \oint_{|w-z_0|=s_2} \frac{f(w)}{1 - \frac{z-z_0}{w-z_0}} \cdot \frac{1}{w - z_0} dz =$$

$$\oint_{|w-z_0|=s_2} \frac{f(w)}{w - z_0} \sum_{j=0}^{\infty} \frac{(z - z_0)^j}{(w - z_0)^k} dw = \oint_{|w-z_0|=s_2} \frac{f(w)}{w - z_0} \sum_{j=0}^{\infty} \frac{(z - z_0)^j}{(w - z_0)^k} dw = \oint_{|w-z_0|=s_2} \sum_{j=0}^{\infty} \frac{f(w)(z - z_0)^j}{(w - z_0)^{j+1}} dw$$

where the geometric series converges since $|z - z_0|/s_2 < 1$. As this is independent of w, we may switch order of integration and summation to obtain

$$\oint_{|w-z_0|=s_2} \frac{f(w)}{w-z} dw = \sum_{j=0}^{\infty} \left(\oint_{|w-z_0|=s_2} \frac{f(w)}{(w-z_0)^{j+1}} dw \right) (z-z_0)^j.$$

For $s_1 < |z - z_0|$, a similar argument justifies that

$$\oint_{|w-z_0|=s_1} \frac{f(w)}{w-z} dw = -\sum_{j=-\infty}^{-1} \left(\oint_{|w-z_0|=s_1} \frac{f(w)}{(w-z_0)^{j+1}} dw \right) (z-z_0)^j.$$

Thus

$$f(z) = \sum_{j=-\infty}^{\infty} \left(\frac{1}{2\pi i} \oint_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{j+1}} dw \right) (z-z_0)^j.$$

To find the annulus of convergence,

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

converges on $D(z_0, R)$ where R is given by

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

$$\sum_{n=-\infty}^{-1} a_n (z-z_0)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}$$

converges when

$$\left| \frac{1}{z-z_0} \right| \leq R^- = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}} \iff |z-z_0| \geq \frac{1}{R^-} = r \iff r = \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}.$$

So our sum converges on $r \leq |z - z_0| \leq R = A(z_0, r, R)$.

(b) Isolated Singularities.

Let f be a meromorphic function in a domain D.

i. Removable: $z_0 \in D$ is a removable singularity of f if and only if $\lim_{z\to z_0} f(z) \in \mathbb{C}$ if and only if

(Riemann Lemma) $\lim_{z\to z_0} (z - z_0)f(z) = 0$ if and only if $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$.

Proof:
(Riemann Lemma for Removable Singularities)
Let

$$g(z) = \begin{cases} (z-z_0)^2 f(z) & \text{if} \quad z \neq z_0 \\ 0 & \text{if} \quad z = z_0 \end{cases}$$

As f is meromorphic, there exists some $\delta > 0$ such that g is holomorphic in $D(z_0, \delta)\backslash\{z_0\}$.

$$\lim_{z\to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z\to z_0} (z-z_0) f(z) = 0$$

by assumption. So $g'(z_0) = 0$. Therefore g is holomorphic in $D(z_0, \delta)$ and $g(z_0) = g'(z_0) = 0$. $g(z)$ is therefore analytic in $D(z_0, \delta)$ and for $z \in D(z_0, \delta)$,

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z-z_0)^n = \sum_{n=2}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z-z_0)^n = \sum_{n=2}^{\infty} a_n (z-z_0)^n = (z-z_0)^2 \sum_{n=0}^{\infty} a_{n+2} (z-z_0)^n.$$

Therefore, when $z \neq z_0$,

$$(z-z_0)^2 f(z) = (z-z_0)^2 \sum_{n=0}^{\infty} a_{n+2}(z-z_0)^n \implies f(z) = \sum_{n=0}^{\infty} a_{n+2}(z-z_0)^n.$$

Thus we have that $\lim_{z\to z_0} f(z) = a_2$. So z_0 is a removable singularity of f.
Assume z_0 is a removable singularity. Then $\lim_{z\to z_0} f(z) = c$ so $\lim_{z\to z_0}(z-z_0)f(z) = 0c = 0$.
If $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, then $\lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0}\sum_{n=0}^{\infty} a_n(z-z_0)^{n+1} = 0$. So z_0 is removable.

ii. Pole: $z_0 \in D$ is a pole of f if and only if $\lim_{z\to z_0} f(z) = \infty$ if and only if there is some k such that $\lim_{z\to z_0}(z-z_0)^{k+1} f(z) = 0$ if and only if $f(z) = \sum_{n=-k}^{\infty} a_n(z-z_0)^n$.

Proof:
Since $\lim_{z\to z_0} f(z) = \infty$, there exists some $\delta > 0$ such that $f(z)$ is holomorphic in $D(z_0,\delta)\backslash\{z_0\}$ and $|f(z)| \geq 1$ on $D(z_0,\delta)\backslash\{z_0\}$. Let $g(z) = \frac{1}{f(z)}$. Then $g(z)$ is holomorphic in $D(z_0,\delta)\backslash\{z_0\}$ and $|g(z)| \leq 1$ on $D(z_0,\delta)\backslash\{z_0\}$. By the Squeeze Theorem, $\lim_{z\to z_0}(z-z_0)g(z) = 0$. By Riemann's Lemma, z_0 is a removable singularity for g. So g is holomorphic in $D(z_0,\delta)$ and $\lim_{z\to z_0} g(z) = \lim_{z\to z_0}\frac{1}{f(z)} = 0$. $g(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ for $z \in D(z_0,\delta)$ and let k be the number such that a_k is the first coefficient not equal to 0. Note that $k \geq 1$. We have that

$$g(z) = (z-z_0)^k \sum_{n=0}^{\infty} a_{n+k}(z-z_0)^n = (z-z_0)^k h_1(z),$$

where h is holomorphic in $D(z_0,\delta)$ and $h(z_0) = a_k \neq 0$. Therefore

$$f(z) = \frac{1}{g(z)} = \frac{1}{(z-z_0)^k h_1(z)} = \frac{h(z)}{(z-z_0)^k}.$$

As $g(z) \neq 0$ on $z \in D(z_0,\delta)\backslash\{z_0\}$, $h_1(z) \neq 0$ in $D(z_0,\delta)$. So h is holomorphic and $h(z) \neq 0$. So for $z \in D(z_0,\delta)\backslash\{z_0\}$

$$f(z) = \frac{h(z)}{(z-z_0)^k},$$

so

$$\lim_{z\to z_0}(z-z_0)^{k+1} f(z) = \lim_{z\to z_0}(z-z_0)h(z) = 0.$$

iii. Essential Singularity: $z_0 \in D$ is an essential singularity of f if and only if $\lim_{z\to z_0} f(z)$ does not exist if and only if (Casorati-Weierstrass) $f(D(z_0,r)\backslash\{z_0\})$ is dense in $\mathbb{C}$ for any $r > 0$ if and only if $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ where there are infinitely many $n < 0$ such that $a_n \neq 0$.

Proof:
Suppose there is some $r > 0$ such that $f(D(z_0,r)\backslash\{z_0\})$ is not dense in $\mathbb{C}$. So there is some $w_0 \in \mathbb{C}$ and $\epsilon > 0$ such that $|f(z) - w_0| \geq \epsilon$ for all $z \in D(z_0,r)\backslash\{z_0\}$. Let $g(z) = \frac{1}{f(z)-w_0}$. g is holomorphic in $D(z_0,r)\backslash\{z_0\}$. So z_0 is an isolated singularity of g. Moreover,

$$|g(z)| = \frac{1}{|f(z)-w_0|} \leq \epsilon \implies \lim_{z\to z_0}(z-z_0)g(z) = 0$$

By Riemann's Lemma z_0 is a removable singularity of g. Therefore $\lim_{z\to z_0} g(z) = c$. If $c = 0$, then

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} w_0 + \frac{1}{g(z)} = \infty.$$

So z_0 is a pole of f.
If $c \neq 0$, then

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} w_0 + \frac{1}{g(z)} = w_0 + \frac{1}{c}.$$

So z_0 is a removable singularity of f. Both of these are a contradiction. Therefore w_0 must be a limit point of $f(D(z_0, r)\backslash\{z_0\})$.
If $f(D(z_0, r)\backslash\{z_0\})$ is dense in $\mathbb{C}$ for any $r > 0$, then there exists a sequence $\{z_j\} \subset D(z_0, r)\backslash\{z_0\}$ converging to z_0 such that $f(z_j) \to 0$ and a sequence $\{w_j\} \subset D(z_0, r)\backslash\{z_0\}$ converging to z_0 such that $f(w_j) \to 1$. So $\lim_{z\to z_0} f(z)$ does not exist.

(c) Residue Theorem

Suppose D is an open simply connected set in $\mathbb{C}$ and that $z_1, \ldots, z_n$ are distinct points of D. Suppose that $f : D\backslash\{z_1, \ldots, z_n\} \to \mathbb{C}$ is a holomorphic function and γ is a closed, piecewise C^1 curve in $D\backslash\{z_1, \ldots, z_n\}$. Then

$$\frac{1}{2\pi i}\oint_\gamma f(z)dz = \sum_{j=1}^{n} \mathrm{Res}(f(z); z = z_j)\left(\frac{1}{2\pi i}\oint_\gamma \frac{1}{z - z_j}dz\right).$$

Proof:

Let $S_j(z)$ be the singular part of f at $z = z_j$, i.e. if $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_j)^n$, then $S_j(z) = \sum_{n=-\infty}^{-1} a_n(z - z_j)^n$. Then $f(z) - \sum_{j=1}^{n} S_j(z)$ is holomorphic in D. By Cauchy's Theorem

$$\oint_\gamma f(z) - \sum_{j=1}^{n} S_j(z)dz = 0.$$

So

$$\frac{1}{2\pi i}\oint_\gamma f(z) = \sum_{j=1}^{n}\frac{1}{2\pi i}\oint_\gamma S_j(z)dz = \sum_{j=1}^{n}\frac{1}{2\pi i}\oint_\gamma \sum_{k=-\infty}^{-1} a_k^{(j)}(z - z_j)^k dz$$

$$= \sum_{j=1}^{n}\frac{1}{2\pi i}\left(\oint_\gamma \sum_{k=-\infty}^{-2} a_k^{(j)}(z - z_j)^k dz + a_{-1}^{(j)}\oint_\gamma \frac{1}{z - z_j}dz\right)$$

$$= \sum_{j=1}^{n}\sum_{k=-\infty}^{-2}\frac{a_k^{(j)}}{2\pi i}\oint_\gamma \frac{d(z - z_j)^{n+1}}{n+1} + a_{-1}^{(j)}\frac{1}{2\pi i}\oint_\gamma \frac{1}{z - z_j}dz = \sum_{j=1}^{n}\mathrm{Res}(f(z); z = z_j)\left(\frac{1}{2\pi i}\oint_\gamma \frac{1}{z - z_j}dz\right).$$

Note: $\frac{1}{2\pi i}\oint_\gamma \frac{1}{z - z_j}dz$ is an integer.

(c') Computing Residues

Let f be a function with a pole of order k at z_0. Then

$$\mathrm{Res}(f(z); z = z_0) = \frac{1}{(k-1)!}\left(\frac{\partial}{\partial z}\right)^{k-1}\left((z - z_0)^k f(z)\right)\Big|_{z=z_0}.$$

Proof:

By the Residue Theorem, for some $r > 0$,

$$\mathrm{Res}(f(z); z = z_0) = \frac{1}{2\pi i}\oint_{|z-z_0|=r} f(z)dz = \frac{1}{2\pi i}\oint_{|z-z_0|=r}\frac{f(z)(z - z_0)^k}{(z - z_0)^k}.$$

By the proof above, $f(z)(z-z_0)^k$ has a removable singularity at z_0, so it may be extended to a holomorphic function on $D(0,r)$. By Cauchy's Integral Formula

$$\frac{1}{2\pi i}\oint_{|z-z_0|=r}\frac{(f(z)(z-z_0)^k)}{(z-z_0)^k}dz = \frac{1}{(k-1)!}\left(\frac{\partial}{\partial z}\right)^{k-1}\left(f(z)(z-z_0)^k\right|_{z=z_0}.$$

7. Evaluating Improper Integrals

(a) $\displaystyle\int_0^{2\pi} R(\cos\theta,\sin\theta)d\theta$, R is a rational function.

We use the parameterization, $e^{i\theta}=z$. As $|z|=1$, $\overline{z}=\frac{1}{z}$, so we have

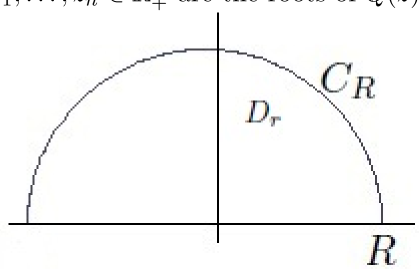
$$\cos z = \frac{1}{2}\left(z+\frac{1}{z}\right), \qquad \sin z = \frac{1}{2i}\left(z-\frac{1}{z}\right).$$

As $z=e^{i\theta}, dz = ie^{i\theta}d\theta$, so $\frac{1}{iz}dz = d\theta$. Our integral can then be written

$$\int_{|z|=1} R\left(\frac{1}{2}\left(z+\frac{1}{z}\right),\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{1}{iz}dz.$$

(b) $\displaystyle\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}dx$, $Q(x)\neq 0$ on $\mathbb{R}$, $\deg(Q)\geq\deg(P)+2$.

If $z_1,\ldots,z_n\in\mathbb{R}^2_+$ are the roots of $Q(z)$, then consider the curve

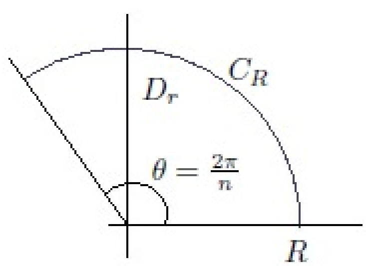


where R is large enough so that $\{z_j\}\subset D_R$.

$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}dx = \oint_{\partial D_R}\frac{P(z)}{Q(z)}dz = 2\pi i\sum_{k=1}^{n}\text{Res}\left(\frac{P(z)}{Q(z)};z=z_k\right).$$

(c) $\displaystyle\int_0^{\infty}\frac{P(x)}{Q(x)}dx$, $Q(x)\neq 0$ on $(0,\infty)$, $\deg(Q)\geq\deg(P)+2$.

Let $Q(x)=1+a_1x^{n_1}+\cdots+a_kx^{n_k}$, $a_j\neq 0$. Let $n=\gcd(n_1,\ldots,n_k)$. Consider the curve



where R is large enough so that all roots of $Q(x)$ with argument between 0 and $2\pi/n$ are in D_R.

$$2\pi i\sum_{k=1}^{n}\text{Res}\left(\frac{P(z)}{Q(z)};z=z_k\right)=\oint_{\partial D_R} = \int_0^R\frac{P(x)}{Q(x)}dx+\int_{C_R}\frac{P(z)}{Q(z)}dz+\int\frac{P(re^{i2\pi/n})}{Q(re^{i2\pi/n})}e^{i2\pi/n}dr$$

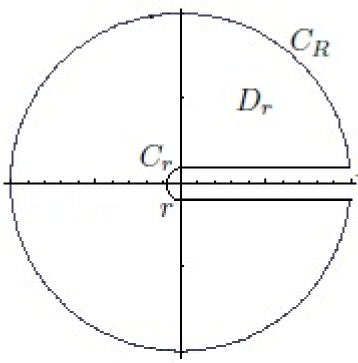
$$\overset{R\to\infty}{\longrightarrow} \left(1-e^{2\pi/n}\right)\int_0^\infty \frac{P(x)}{Q(x)}dx + 0$$

where $z_1, \ldots, z_k$ are the zeros of $Q(z)$ in D_R for R large enough.

(d) $\displaystyle\int_{-\infty}^{\infty} R(x)\cos(x)dx = \text{Re}\left(\int_{-\infty}^{\infty} R(x)e^{ix}dx\right)$, where $R(x)$ is a rational function.

Fact: $\displaystyle\int_{-\infty}^{\infty} f(x)e^{ix}dx = 2\pi i\sum_{k=1}^{n}\text{Res}\left(f(z)e^{iz}; z=z_k\right)$, for $z_k \in \mathbb{R}^2_+$ so long as $f(re^{i\theta}) \to 0$ as $r\to\infty$ uniformly for $\theta\in(0,\pi)$.

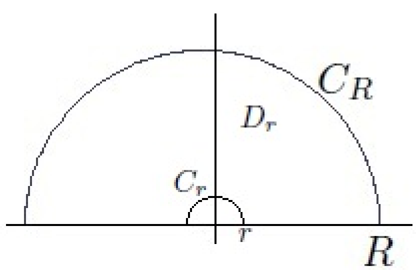
(e) $\int_0^\infty R(x)x^\alpha dx$, $\alpha \in ((-1,0)\cup(0,1))\cap\mathbb{Q}$, R a rational function.
For $\deg R \le -2$, consider the curve



$$2\pi i\sum_{k=1}^{n}\text{Res}\left(\frac{P(z)}{Q(z)}; z=z_k\right) = \int_r^R R(x)x^\alpha dx + \int_{C_R} R(z)z^\alpha dz + \int_R^r R(xe^{2\pi i})x^\alpha e^{i\alpha 2\pi}dx - \int_{C_r} R(x)x^\alpha dx.$$

$$\overset{r\to 0^+, R\to\infty}{0} + \int_0^\infty R(x)x^\alpha dx - e^{i2\pi i\alpha}\int_0^\infty R(x)x^\alpha dx - 0 = \left(1-e^{i2\pi\alpha}\right)\int_0^\infty R(x)x^\alpha dx.$$

(f) $\displaystyle\int_0^\infty R(x)\ln x dx$. Consider the curve



$$\int_{\partial D_{r,R}} R(z)\ln z dz = \int_r^R R(x)\ln x dx + \int_{-R}^{-r} R(x)(\ln|x| + i\pi)dx - \int_{C_r} R(z)\ln z dz + \int_{C_R} R(z)\ln z dz.$$

8. Argument Principle

(a) Argument Principle

Let f be a meromorphic function on a domain D, f continuous on $\overline{D}$, ∂D has piecewise C^1 boundary. If F has neither poles nor zeros on ∂D, then

$$\frac{1}{2\pi i}\oint_{\partial D}\frac{f'(z)}{f(z)}dz$$

(b) Rouché's Theorem

Let D be a bounded domain with piecewise C^1 boundary. Let f, g be holomorphic in D and continuous on ∂D. If

$$|f(z)+g(z)| \leq |f(z)|+|g(z)|, z \in \partial D,$$

then $\#Z_D(f) = \#Z_D(g)$.

(c) Hurwitz's Theorem

Let $\{f_n\}, f$ be holomorphic functions in a domain $D \subset \mathbb{C}$. $f_n(z)$ is nonzero for $z \in D$. If $f_n \to f$ uniformly on any compact subset of D, then either

$$f(z) \equiv 0 \qquad \text{or} \qquad f(z) \neq 0 \text{ for } z \in D.$$

Corollary If $f_n \to f$ uniformly on $\overline{D}$ and $f(z) \neq 0$ on $\overline{D}$, then there exists some N such that $f_n(z) \neq 0$ for $n \geq N, z \in D$.

Proof:

Let $z_0 \in D$. As D is open, there exists $r > 0$ such that $\overline{D(z_0, r)} \subset D$. Let $\delta = \min\{|f(z)| : z \in \partial D(z_0, r)\} > 0$. By uniform continuity, for $z \in \partial \overline{D(z_0, r)}$, there exists some N such that for any $n \geq N$,

$$|f(z) - f_n(z)| < \frac{\delta}{2}.$$

We also have

$$\frac{\delta}{2} < |f(z)| \leq |f(z)| + |f_n(z)|.$$

Therefore

$$|f(z) + (-f_n(z))| < |f(z)| + |-f_n(z)|.$$

So $\#Z_{D(z_0,r)}(f) = \#Z_{D(z_0,r)}(f_n)$. So, in particular, as $f(z_0) \neq 0, f_n(z_0) \neq 0$. As z_0 was arbitrary, $f_n(z) \neq 0$ for $z \in D$.

(d) Gauss-Lucas Theorem

Let $p(z)$ be a polynomial. Then all the zeros of $p'(z)$ lie in the convex hull of the zero set of $p(z)$.

Proof:

Let $p(z) = a \prod_{j=1}^{n} (z - a_j)$.

$$\frac{p'(z)}{p(z)} = \sum_{j=1}^{n} \frac{1}{z - a_j}.$$

If z is a zero of p' and $p(z) \neq 0$, then

$$\sum_{j=1}^{n} \frac{1}{z - a_j} = 0.$$

Multiplying top and bottom of each term by $\overline{z - a_j}$ respectively, we get

$$\sum_{j=1}^{n} \frac{\overline{z} - \overline{a_j}}{|z - a_j|^2} = 0.$$

Rewriting this, we have

$$\left(\sum_{j=1}^{n} \frac{1}{|z - a_j|^2}\right) \overline{z} = \sum_{j=1}^{n} \frac{1}{|z - a_j|^2} \overline{a_j}.$$

Taking the conjugate of both sides, we have z is a weighted sum with positive coefficients that sum to one. So z is in the convex hull of the roots of p.

Note that if $p(z) = 0$ also, then as z is a root of p it is in the convex hull of the zero set of $p(z)$ already.

(e) Open Mapping Theorem

If f is holomorphic and nonconstant in a domain D, then $f : D \to \mathbb{C}$ is an open mapping.

Proof:

Let O be open in D. Let $w_0 \in f(O)$. Then there exists a $z_0 \in O$ such that $f(z_0) = w_0$, i.e. $f(z_0) - w_0 = 0$. By the Uniqueness Theorem, there exists a $\delta > 0$ such that $\overline{D(z_0, \delta)} \subset O$ and $f(z) - w_0 \neq 0$ on $\overline{D(z_0, \delta)} \setminus \{z_0\}$. Let $\epsilon = \min\{|f(z) - w_0| : |z - z_0| = \delta\} > 0$, which exists by compactness and choice of δ. It suffices to show $D(w_0, \epsilon) \subset f(O)$, that is, for any $w \in D(w_0, \epsilon)$, $f(z) - w = 0$ has a solution in $D(z_0, \delta)$, then $D(w_0, \epsilon) \subset f(D(z_0, \delta)) \subset f(O)$. It suffices to show, for $w \in D(w_0, \epsilon)$,

$$\#Z_{D(z_0,\delta)}(f(z) - w) = \frac{1}{2\pi i} \oint_{\partial D(z_0,\delta)} \frac{(f(z) - w)'}{f(z) - w} dz = \frac{1}{2\pi i} \oint_{\partial D(z_0,\delta)} \frac{f'(z)}{f(z) - w} dz > 0.$$

Set

$$g(w) = \frac{1}{2\pi i} \oint_{\partial D(z_0,\delta)} \frac{f'(z)}{f(z) - w} dz.$$

By the Argument Principle, $g(w)$ is integer-valued on $D(w_0, \epsilon)$. From the definition, $g(w)$ is continuous on $D(w_0, \epsilon)$ as $\epsilon = \min\{|f(z) - w_0| : z \in \partial D(z_0, \delta)\}$, so for z such that $|z - z_0| = \delta$, $|f(z) - w| > 0$. Moreover, $f(z_0) = w_0$. So $g(w_0) \geq 1$. By continuity, $g(w) \equiv g(w_0) \geq 1$. So $D(w_0, \epsilon) \subset f(O)$. So f is an open mapping.

(f) Maximum Modulus Theorem

Let f be holomorphic in a domain D. If there is some $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$, then f is constant.

Proof:

Let z_0 be as above and assume toward contradiction f be nonconstant . Then f is an open mapping, so there exists some $\delta > 0$ such that $\overline{D(f(z_0), \delta)} \subset f(D)$. However, there exists a point on $\partial D(f(z_0), \delta)$, for instance, with modulus greater than $|f(z_0)|$. This contradicts our choice of z_0. So f must be constant.

Corollary If f is holomorphic in a bounded domain D and f is continuous on $\overline{D}$, then the maximum of f occurs on ∂D.

Minimum Modulus Theorem If f is holomorphic in a domain and $f \neq 0$ on D, then if there is some $z_0 \in D$ such that $|f(z_0)| \leq |f(z)|$ for $z \in D$, then $f(z)$ is constant.

Proof:Define $g(z) = \dfrac{1}{z}$ and apply the Maximum Modulus Theorem to $g(z)$.

9. Conformal Maps

(a) Schwarz Lemma

Let $f : D(0, 1) \to D(0, 1)$ be holomorphic and $f(0) = 0$. Then

(i) $|f(z)| \leq z$ for $z \in D(0, 1)$

(ii) If $f'(0) = 1$ or $|f(z_0)| = |z_0|$ for $z_0 \in D(0, 1) \setminus \{0\}$, then $f(z) = e^{i\theta} z, \theta \in [0, 2\pi)$.

Proof:

(i) Let $g(z) = \dfrac{f(z)}{z}$. By Riemann's Lemma, g is holomorphic in $D(0, 1)$, $|g(z)| = \dfrac{1}{|z|}|f(z)| \leq \dfrac{1}{|z|}$ for $z \in D(0, 1)$.

For $r < 1$, $|g(z)| \leq \dfrac{1}{r}$ for $z \in D(0, r)$ by the Maximum Modulus Principle. Letting $r \to 1^-$, we have $|g(z)| \leq 1$, i.e. $|f(z)| \leq |z|$. Note also that this implies that $g : D(0, 1) \to \overline{D(0, 1)}$.

(ii) If $z_0 \neq 0$ and $|f(z)| = |z_0|$, then $|g(z_0)| = 1$, which by the Maximum Modulus Theorem implies $|g(z)| = 1$, i.e. $g(z) = e^{i\theta}, \theta \in [0, 2\pi)$. So $f(z) = e^{i\theta} z$.

If $f'(0) = 1$, then

$$f'(0) = \lim_{z \to 0} \left| \frac{f(z) - f(0)}{z - 0} \right| = \lim_{z \to 0} \left| \frac{f(z)}{z} \right| = 1,$$

which implies $|g(0)| = 1$. Again by the Maximum Modulus Theorem, $g(z) = e^{i\theta}$, so $f(z) = e^{i\theta} z$.

(b) Schwarz-Pick Lemma

Let $f : D(0,1) \to D(0,1)$

(i) $\left|\dfrac{f(z)-f(0)}{1-\overline{f(0)}f(z)}\right| \le |z|$

(ii) Let $a \in D(0,1)$. Then $\left|\dfrac{f(z)-f(a)}{1-\overline{f(a)}f(z)}\right| \le \left|\dfrac{z-a}{1-\overline{a}z}\right|$ for $z \in D$.

Proof:

(i) Let $g(z) = \dfrac{f(0)-f(z)}{1-\overline{f(0)}f(z)}$. Then $g : D(0,1) \to D(0,1)$. Note $g(0) = 0$, so by Schwarz Lemma, $|g(z)| \le |z|$, i.e.

$$\left|\frac{f(z)-f(0)}{1-\overline{f(0)}f(z)}\right| \le |z| \text{ for } z \in D(0,1).$$

(ii) Let $a \in D(0,1)$ and define $\phi_a(z) = \dfrac{a-z}{1-\overline{a}z}$. Then $\phi_a(0) = a$. Note that $f \circ \phi_a : D(0,1) \to D(0,1)$. By the above argument,

$$\left|\frac{f\circ\phi_a(z) - f\circ\phi_a(0)}{1-\overline{f\circ\phi_a(0)}f\circ\phi_a(z)}\right| \le |z| \text{ for } z \in D,$$

or after simplification

$$\left|\frac{f\circ\phi_a(z) - f(a)}{1-\overline{f(a)}f\circ\phi_a(z)}\right| \le |z| \text{ for } z \in D.$$

As this holds for all $z \in D(0,1)$, we may plug in $\phi_a^{-1}(z) = \phi_a(z)$ into this function. Therefore

$$\left|\frac{f\circ\phi_a\circ\phi_a(z) - f(a)}{1-\overline{f(a)}f\circ\phi_a\circ\phi_a(z)}\right| = \left|\frac{f(z)-f(a)}{1-\overline{f(a)}f(z)}\right| \le |\phi_a(z)| = \left|\frac{a-z}{1-\overline{a}z}\right| \text{ for } z \in D.$$

Corollary If $f : D(0,1) \to D(0,1)$, then $|f'(z)| \le \dfrac{1-|f(z)|^2}{1-|z|^2}$ for $z \in D(0,1)$

Proof:

By Schwarz-Pick, after rearrangement

$$\left|\frac{f(z)-f(a)}{z-a}\right| \le \left|\frac{1-\overline{f(a)}f(z)}{1-\overline{a}z}\right|.$$

Letting $a \to z$, we have

$$|f'(z)| \le \left|\frac{1-|f(z)|^2}{1-|z|^2}\right| = \frac{1-|f(z)|^2}{1-|z|^2}$$

as $|f(z)|, |z| < 1$.

Corollary If there is a $z_0 \in D(0,1)$ such that $|f'(z_0)| = \dfrac{1-|f(z_0)^2}{1-|z|^2}$, then

$$\left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right| = \left|\frac{z-z_0}{1-\overline{z_0}z}\right|.$$

Proof:

Proof follows from the Maximum Modulus Theorem.

(c) Image of Boundary under Biholomorphic Maps

Let D_1, D_2 be domains in $\mathbb{C}$. Let $f : D_1 \to D_2$ be biholomorphic. Then $f(\partial D_1) \subset \partial D_2$.

(d) Characterization of $\text{Aut}(D(0,1))$

Let $\phi \in \text{Aut}(D(0,1))$, $a \in D(0,1)$ such that $\phi(a) = 0$.

Let $g(z) = \phi(z)\dfrac{1-\overline{a}z}{a-z}$. Then $g : D(0,1) \to D(0,1)$ is holomorphic, $g(z) \neq 0$ on D and $|g(z)| \to 1$ as $|z| \to 1$. By the Maximum and Minimum Modulus Theorem, $\min\{|g(z)| : |z| = r\} \leq |g(z)| \leq \max\{|g(z)|L|z| = r\}$ for $z \in D(0,r)$. Letting $r \to 1^-$, we have $1 \leq |g(z)| \leq 1$ for $z \in D(0,1)$. So $g(z) = e^{i\theta}$ for $\theta \in [0, 2\pi)$. So $\phi(z) = e^{i\theta}\phi_a(z) = e^{i\theta}\dfrac{a-z}{1-\overline{a}z}$.

(e) Möbius Transfoms

A Möbius Transfom is a rational function of the form $f(z) = \dfrac{az+b}{cz+d}$, where $ad - bc \neq 0$. For any two circles $\Gamma_1, \Gamma_2 \subset \overline{\mathbb{C}}$, there exists a Möbius Transfom T that maps $\Gamma_1 \to \Gamma_2$ where $z_1, z_2, z_3 \in \Gamma_1, z_i \neq z_j$, $w_1, w_2, w_3 \in \Gamma_2$, $T(z_j) = w_j$.

Proof:

We use the real line as an intermediate step. Consider first

$$S(z) = \frac{z - z_1}{z - z_3} / \left(\frac{z_2 - z_1}{z_2 - z_3}\right)$$

Then

$$S(z_1) = 0$$
$$S(z_2) = 1$$
$$S(z_3) = \infty$$

So $S : \Gamma_1 \to \mathbb{R}$.

Consider now

$$\tilde{S}(w) = \frac{w - w_1}{w - w_3} / \left(\frac{w_2 - w_1}{w_2 - w_3}\right)$$

Then

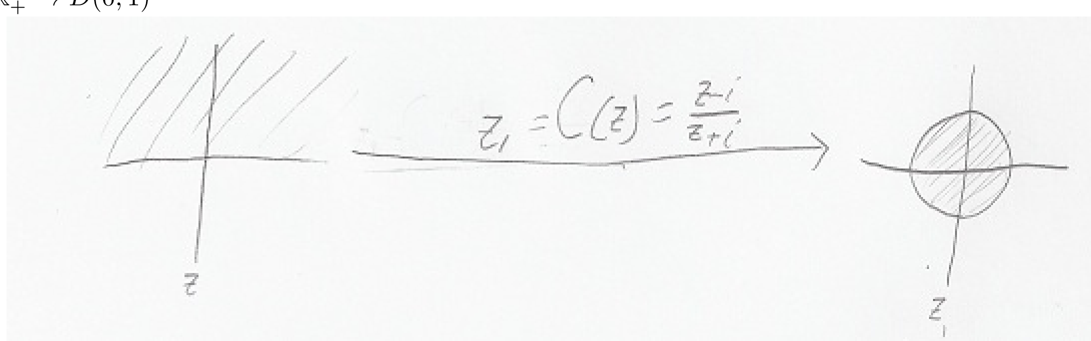
$$\tilde{S}(w_1) = 0$$
$$\tilde{S}(w_2) = 1$$
$$\tilde{S}(z_3) = \infty$$

Define $T(z) = \tilde{S}^{-1} \circ S$. Then

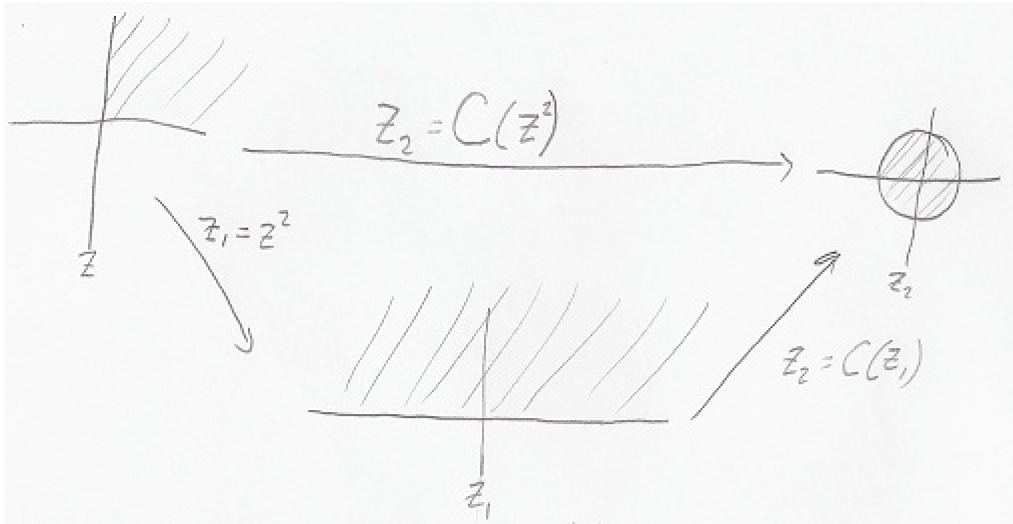
$$T(z_1) = \tilde{S}^{-1} \circ S(z_1) = \tilde{S}^{-1}(0) = w_1$$
$$T(z_2) = \tilde{S}^{-1} \circ S(z_2) = \tilde{S}^{-1}(1) = w_2$$
$$T(z_3) = \tilde{S}^{-1} \circ S(z_3) = \tilde{S}^{-1}(\infty) = w_3$$

(f) Conformal Maps

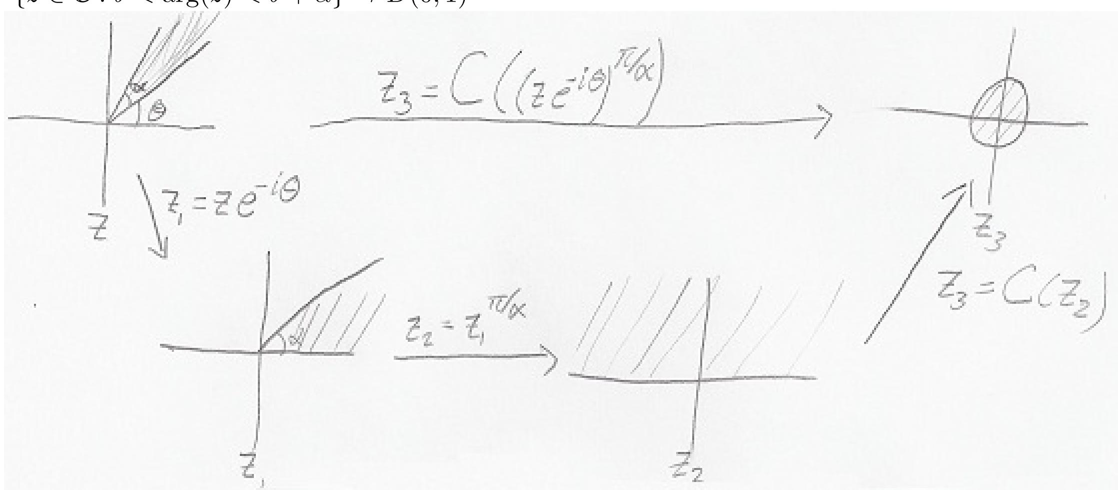
I. $\mathbb{R}^2_+ \to D(0,1)$



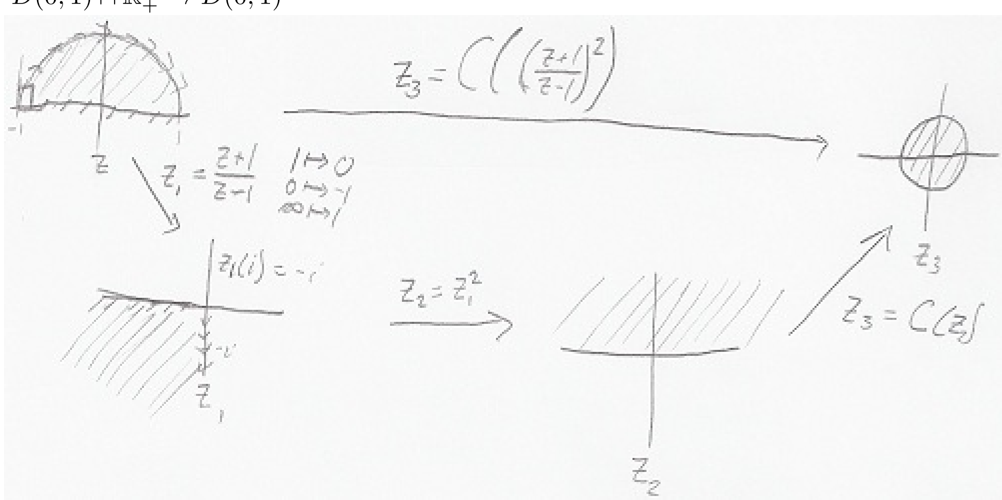
II. $\{z \in \mathbb{C} : \mathrm{Re}(z), \mathrm{Im}(z) > 0\} \to D(0,1)$



III. $\{z \in \mathbb{C} : \theta < \arg(z) < \theta + \alpha\} \to D(0,1)$

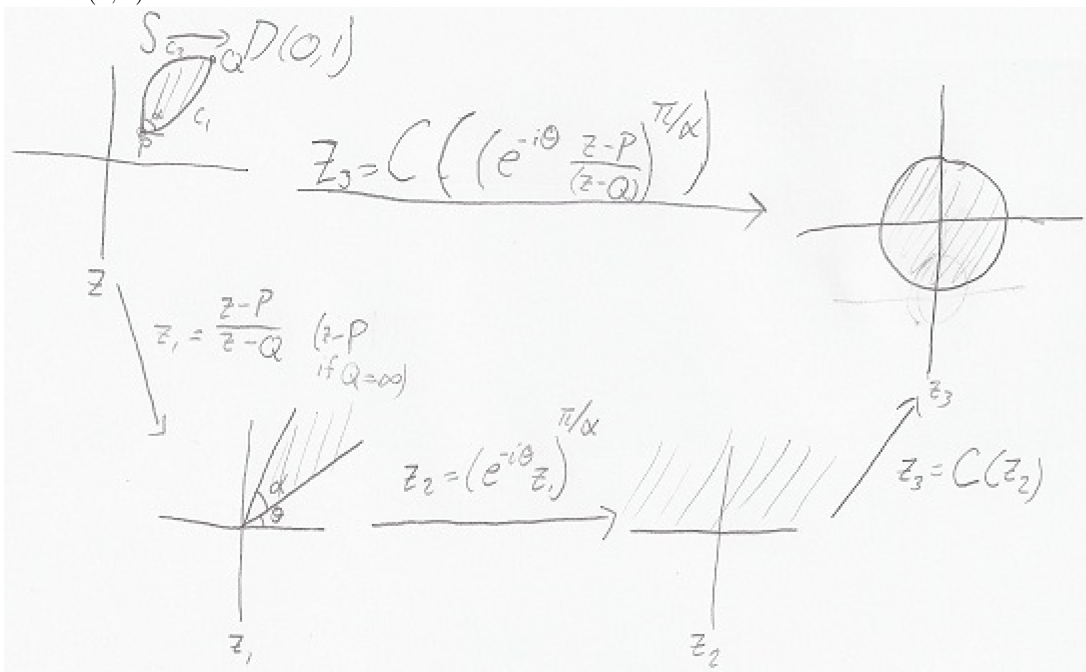


IV. $D(0,1) \cap \mathbb{R}^2_+ \to D(0,1)$

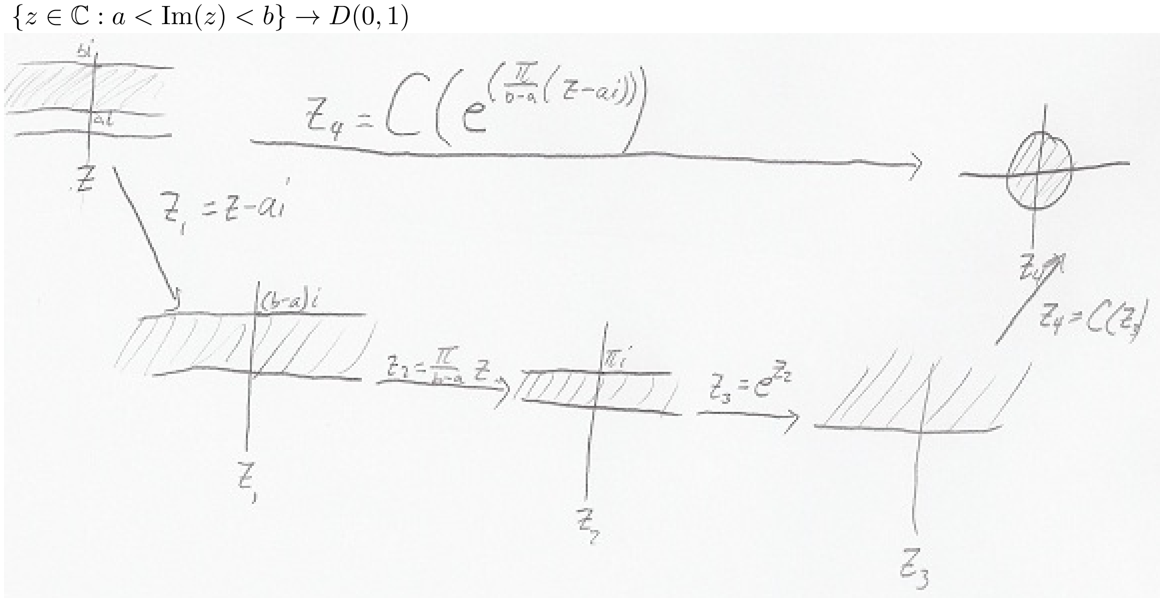


V. Given two circles C_1, C_2 in $\overline{\mathbb{C}}$ that intersect at P, Q, where $P \in \mathbb{C}$. Let α denote the interior angle between C_1 and C_2 at P. Let S be the area bounded by C_1 and C_2.

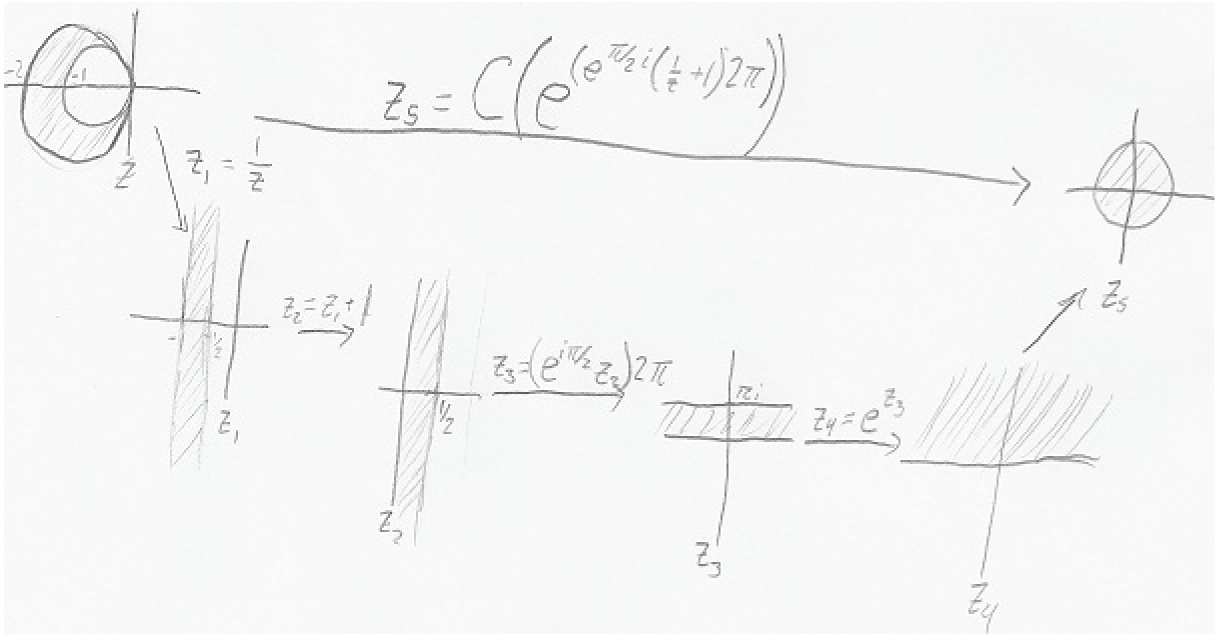
$S \to D(0,1)$



VI. $\{z \in \mathbb{C} : a < \mathrm{Im}(z) < b\} \to D(0,1)$



VII. $D(-1,1)\cap\overline{D(-1/2,1/2)}\to D(0,1)$



(g) Normal Families

(i) Let D be a domain in $\mathbb{C}$ and let $\{f_n\}$ be a sequence of continuous functions on D and f is continuous on D. f_n is said to converge to f normally in D, if, for any compact $K\subset D$, $f_n\to f$ uniformly on K.

(ii) Let D be a domain and $\mathcal{F}$ be a family of continuous functions on D. $\mathcal{F}(D)$ is a normal family if, for any sequence $\{f_n\}\subset\mathcal{F}$, there is a subsequence $\{f_{n_k}\}$ and $f\in C(D)$ such that $f_{n_k}\to f$ normally on D.

(iii) Montel's Theorem
Let D be a domain in $\mathbb{C}$, $\mathcal{F}(D)$ is a family of holomorphic functions on D. $\mathcal{F}$ is a normal family if and only if $\mathcal{F}$ is uniformly bounded on compact subsets of D.

(h) Riemann Mapping Theorem
Let D be a simply connected domain in $\mathbb{C}$, $D\neq\mathbb{C}$. Then, for any given point $p_0\in D$, there is a unique biholomorphic map ϕ on D such that $\phi: D\to D(0,1)$, $\phi(p_0)=0, \phi'(p_0)>0$.

Uniqueness of Riemann Mapping Theorem Let $\phi_1,\phi_2: D\to D(0,1)$, $\phi_j(p_0)=0, \phi_j'(p_0)>0$ for $j=1,2$. It suffices to show $f=\phi_2\circ\phi_1^{-1}: D(0,1)\to D(0,1)$ is the identity. Note first that $f(0)=\phi_2\circ\phi_1^{-1}(0)=\phi_2(p_0)=0$. Second note that

$$f'(0)=\phi_2'(\phi^{-1}(0))\left(\phi_1^{-1}\right)'(0)=\phi_2'(p_0)\frac{1}{\phi_1'(\phi_1^{-1}(0))}=\phi_2'(p_0)\frac{1}{\phi_1'(p_0)}>0.$$

So $f(z)=e^{i\theta}z$ by Schwarz's Lemma. As $f'(0)>0$, $\theta=0$, so f is the identity, as desired.

Lemma Let f be a one-to-one holomorphic function $f: D\to D(0,1)$, $f(p_0)=0, f'(p_0)>0$, f not onto, then there exists a holomorphic $g: D\to D(0,1)$, g one-to-one, $g(p_0)=0$, $g'(p_0)>f'(p_0)$.

Proof:
As f is not onto, there is a $w\notin f(D)$. The function $\dfrac{f(z)-w}{1-\overline{w}f(z)}$ is nonzero on D. As D is simply connected, we can define $h(z)^2=\dfrac{f(z)-w}{1-\overline{w}f(z)}$, where $h(z)$ is a holomorphic function. Let

$$g(z)=\frac{|h'(z_0)|}{h'(z_0)}\frac{h(z)-h(z_0)}{1-\overline{h(z_0)}h(z)}.$$

Notice that $g(z_0) = 0$, $g : D \to D(0, 1)$ and g is one-to-one. Also,

$$g'(z_0) = \frac{|h'(z_0)|}{1 - |h(z_0)|^2} = \frac{1 + |w_0|}{2\sqrt{|w_0|}} f'(p_0) > f'(p_0).$$

10. Reflection Principle (a) Let $f(z)$ be holomorphic on D, $D \subset \mathbb{R}^2_+$, $(a, b) \subset \overline{D} \cap \mathbb{R}$. If $f(z)$ is continuous on $D \cup (a, b)$ and $f(z)$ is real-valued for $z \in (a, b)$, then $f(z)$ can be extended to a holomorphic function on $D_e = D \cup (a, b) \cup D^*$ where $D^* = \{z : \overline{z} \in D\}$.

Corollary Let D be a domain contained in $D(z_0, r)$ so that $\Gamma = \{z_0 + re^{i\theta} : \theta_1 < \theta < \theta_2\}$, $\Gamma \subset \overline{D} \cap \overline{D(z_0, r)}$. Assume f is holomorphic on D, continuous on $D \cup \Gamma$ and f is real valued on Γ. Then f extends to a holomorphic function on $D \cup \Gamma \cup D^*$ where $D^* = \{z_0 + \frac{r^2}{\overline{z - z_0}} : z \in D\}$.

11. Infinite Products (a) $\prod_{j=1}^{\infty}(1 + z_j)$ converges to $z \neq 0$ if and only if $\sum_{j=1}^{\infty} \ln(1 + z_j)$ converges in $\mathbb{C}$.

(b) $\prod_{j=1}^{\infty}(1 + z_j)$ <u>converges absolutely</u> if $\sum_{j=1}^{\infty} |\ln(1 + z_j)|$ converges. $\prod_{j=1}^{\infty}(1 + z_j)$ converges absolutely if and only if $\sum_{j=1}^{\infty} |z_n|$ converges, for $|z_n| \leq 1$.

(c) Weierstrass Factorization Theorem
Let D be a simply connected domain in $\mathbb{C}$, let $f(z) \neq 0$ be meromorphic on D. Then

$$f(z) = e^{g(z)} \frac{h_1(z)}{h_2(z)}$$

where g, h_1, h_2 are holomorphic on D, where $Z_D(h_1) = Z_D(f)$ counting multiplicity and $Z_D(h_2) = P(f)$ counting order.

Example Prove that if D is a simply connected domain and $f(z)$ is holomorphic and nonzero on D, then $f(z) = e^{g(z)}$, where $g(z)$ is holomorphic on D.

Solution:
As $f(z)$ is nonzero on D, the function $\frac{f'(z)}{f(z)}$ is holomorphic on D. Let $F'(z) = \frac{f'(z)}{f(z)}$.
Consider the function $e^{-F(z)}f(z)$.
We claim this is a constant function. As it is holomorphic, it suffices to check its derivative is zero.

$$\left(e^{-F(z)}f(z)\right)' = e^{-F(z)} - F'(z)f(z) + e^{-F(z)}f'(z) = e^{-F(z)}\left(-F'(z)f(z) + f'(z)\right)$$

$$= e^{-F(z)}\left(-\frac{f'(z)}{f(z)}f(z) + f'(z)\right) = e^{-F(z)}(0) = 0.$$

Therefore $e^{-F(z)}f(z) = e^{i\alpha}$, or after rearrangement

$$f(z) = e^{F(z)+i\alpha}.$$

As $F(z)$ is holomorphic on D, setting $g(z) = F(z) + i\alpha$ yields the desired result.

(d) Mittag-Leffler Theorem
Let D be a domain in $\mathbb{C}$, $\{z_n\}$ a sequence in D without a limit point in D. Let

$$S_n = \sum_{k=-N_n}^{-1} a_{k_n}(z - z_n)^k \text{ for some } N_n \geq 1.$$

Then there is a meromorphic function f on D such that $f - S_n$ is holomorphic at a neighborhood of z_n.

12. Harmonic Functions

(a) Harmonic Function

$u : D \to \mathbb{R}$ is harmonic if $\Delta u = 0$ in D.

(b) Dirichlet Boundary Value Problem

For a continuous ϕ on $\partial D(0,1)$, consider the conditions

$$\begin{cases} \Delta u(z) = 0 & \text{if} \quad z \in D(0,1) \\ u = \phi & \text{on} \quad \partial D(0,1) \end{cases}$$

This has the unique solution

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} \phi(e^{i\theta}) d\theta.$$

Proof:
We first show that $u(z)$ is harmonic. We begin by writing

$$\frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} = \frac{e^{i\theta}}{e^{i\theta} - z} + \frac{e^{-i\theta}}{e^{-i\theta} - \overline{z}} - 1.$$

Therefore, for $z \in D(0,1)$,

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\theta}) \frac{e^{i\theta}}{e^{i\theta} - z} d\theta + \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\theta}) \frac{e^{-i\theta}}{e^{-i\theta} - \overline{z}} d\theta - \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\theta}) d\theta.$$

The first integral is a holomorphic function (so it is harmonic) on $D(0,1)$ since $e^{i\theta}/(e^{i\theta} - z)$ is holomorphic in z on $D(0,1)$. The second integral is harmonic, since its derivative with respect to z is 0. The final integral is a constant, so it is also harmonic on $D(0,1)$. So u is harmonic.
It remains to show that u is continuous at the boundary, i.e.

$$\lim_{z \to z_0} u(z) = \phi(z_0)$$

for $z_0 \in \partial D(0,1)$.
Notice first that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} d\theta = 1,$$

therefore

$$\phi(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} \phi(z_0) d\theta$$

where $z_0 \in \partial D(0,1)$. Consider now the function

$$E(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} \left(\phi(e^{i\theta}) - \phi(z_0)\right) d\theta.$$

As ϕ is continuous, for any $\epsilon > 0$, there exists some $\delta > 0$ such that for $|\theta - \theta_0| < \delta$ we have $\phi(e^{i\theta}) - \phi(e^{i\theta_0})| < \epsilon$. Therefore,

$$|E(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} \left|\phi(e^{i\theta}) - \phi(z_0)\right| d\theta$$

$$\leq \frac{1}{2\pi} \int_{|\theta - \theta_0| \geq \delta} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} 2M d\theta + \int_{|\theta - \theta_0| < \delta} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} \epsilon d\theta \leq \frac{M}{\pi} \int_{|\theta - \theta_0| \geq \delta} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} d\theta + \epsilon,$$

where $M = \max_{z \in \partial D(0,1)} \{\phi(z)\}$.
Notice $|1 - ze^{-i\theta}| \geq \delta.33 - |e^{i\theta_0} - |z|e^{i\theta_0}|$. So

$$|E(z)| \leq \frac{1}{2\pi} \frac{1}{2\pi} \int_0^{2\pi} \frac{2(\delta/6)^2 \epsilon}{\delta/6} + \epsilon \to 0,$$

as desired.

(c) Harnack's Inequality

Let u be a nonnegative harmonic function on $D(0,R)$. Then for $z \in D(0,R)$,

$$\frac{R-|z|}{R+|z|}u(0) \leq u(z) \leq \frac{R+|z|}{R-|z|}u(0).$$

Proof:
Without loss of generality, we may assume u is continuous on the boundary. Recall

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}d\theta.$$

Now

$$\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2} \leq \frac{R^2-|z|^2}{(R-|z|)^2} = \frac{R+|z|}{R-|z|}.$$

From these two above equations, we have

$$u(z) \leq \frac{R+|z|}{R-|z|}\frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})d\theta = \frac{R+|z|}{R-|z|}u(0).$$

Similarly, note that

$$\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2} \geq \frac{R^2-|z|^2}{(R+|z|)^2} = \frac{R-|z|}{R+|z|},$$

which yields the other inequality.

(d) Corollary

Let u be a positive harmonic function in a domain D. For any compact $K \subset D$ there exists a constant C_K dependent only on K and D such that

$$\max\{u(z) : z \in K\} \leq C_K \min\{u(z) : z \in K\}.$$

Proof:
Let $1 > \delta = \operatorname{dist}(\partial D, K)$. As K is compact, there exist finitely many points $z_1, \ldots, z_n \in K$ such that $\cup D(z_j, \delta)$ covers K. For any z_j and $z \in D(z_j,\delta) \cap K$, we have

$$u(z) \leq \frac{1+|z}{1-|z|}u(z_j) \leq \frac{1+\delta}{1-\delta}u(z_j),$$

$$u(z) \geq \frac{1-|z|}{1+|z|}u(z_j) \geq \frac{1-\delta}{1+\delta}u(z_j).$$

Therefore

$$u(z) \leq \frac{1+\delta}{1-\delta}u(z_j) = \left(\frac{1+\delta}{1-\delta}\right)^2 \frac{1-\delta}{1+\delta}u(z_j) \leq \left(\frac{1+\delta}{1-\delta}\right)^2 u(z).$$

Let $C_K = \left(\frac{1+\delta}{1-\delta}\right)^2$. Then, we have

$$\max\{u(z) : z \in K\} \leq C_K \min\{u(z) : z \in K\}.$$

(e) Harnack's Principle

Let $\{u_j\}$ be a sequence of harmonic functions on D such that for $z \in D$, $u_j(z) \leq u_{j+1}(z)$. Then either $u_j \to \infty$ uniformly on compact sets, or there is a harmonic function u on U such that $u_j \to u$ uniformly on compact sets.

Proof:
Without loss of generality $u_n(z) \geq 0$, as otherwise we can replace this sequence with $v_n(z) = u_n(z) - u_1(z)$.

Let $z_0 \in D$. We have two cases.
Case 1: $\lim_{n\to\infty} u_n(z_0) = \infty$.
Then, for any $K \subset D$, by Harnack's Inequality, there is some $C_K > 0$ such that

$$u_n(z) \geq \frac{1}{C_K} u_n(z_0), z \in K,$$

So $\lim_{n\to\infty} u_n(z) = \infty$ uniformly on K.
Case 2: $dis \lim_{n\to\infty} u_n(z_0) = c_0 < \infty$.
In this case, for any $K \subset D$ there exists some C'_K such that

$$u_n(z) \leq C'_K u_n(z_0) \leq C'_K c_0 < \infty.$$

Then $\lim_{n\to\infty} u_n(z) =: u(z)$ is finite. Let $K \to D$. Then there exists a function u on D such that $\lim_{n\to\infty} u_n(z) = u(z)$, for $z \in D$.
Notice if $m > n$, $u_m(z) - u_n(z) \geq 0$, so

$$0 \leq u_m(z) - u_n(z) \leq C_K \left(u_m(z) - u_n(z)\right) \to 0$$

as $m, n \to \infty, m > n$. By Cauchy's Test we have uniform convergence.
It remains to show u is harmonic in D.
Let $z_0 \in D$, $r < \text{dist}(z_0, \partial D)$. Consider $D(z_0, r)$.

$$u_n(z_0 + rz) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} u_n(z_0 + re^{i\theta}) d\theta, z \in D(0,1)$$

. Letting $n \to \infty$, we have by uniform convergence,

$$u(z_0 + rz) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} u(z_0 + rei\theta) d\theta$$

is harmonic at z_0. Therefore u is harmonic on D.

(f) Example

Show that if $f : D(0,1) \to D(0,1)\backslash\{0\}$ is a holomorphic function, then $\max\{|f(z)|^2 : |z| \leq 1/5\} \leq \min\{|f(z)| : |z| \leq 1/7\}$ and furthermore find a $f(z)$ to show that this inequality is sharp.

Solution:
As f is holomorphic and nonzero, $\ln|f(z)|$ is harmonic. Consider the harmonic function $u(z) = -ln|f(z)| \geq 0$ is harmonic on $D(0,1)$. We apply Harnack's Inequality to u to obtain

$$\frac{1 - |z|}{1 + |z|}(-\ln|f(z)|) \leq -\ln|f(z)| \leq \frac{1 + |z|}{1 - |z|}(-\ln|f(0)|).$$

Exponentiating this yields

$$e^{\frac{1-|z|}{1+|z|}} \frac{1}{|f(0)|} \leq \frac{1}{|f(z)|} \leq e^{\frac{1+|z|}{1-|z|}} \frac{1}{|f(0)|}.$$

Notice then we have

$$|f(z)| \geq e^{-\frac{1+|z|}{1-|z|}} |f(0)|,$$

so for $|z| \leq 1/7$, as a minimal value

$$\min\{|f(z)|\} \geq e^{-\frac{1+1/7}{1-1/7}} |f(0)| = e^{-4/3} |f(0)|.$$

Similarly, we have

$$\frac{1 - |z|}{1 + |z|}(-2\ln|f(z)|) \leq -\ln|f(z)^2| \leq \frac{1 + |z|}{1 - |z|}(-2\ln|f(0)|),$$

or

$$|f(z)|^2 \le e^{-2\frac{1-|z|}{1+|z|}}|f(0)|,$$

which on $|z| \le 1/5$, reduces to

$$\max\{|f(z)|^2\} \le e^{-2\frac{1-1/5}{1+1/5}}|f(0)| = e^{-4/3}|f(0)|,$$

which gives us the desired result.
To show that the bound is sharp, consider the function

$$g(z) = e^{-\frac{1-z}{1+z}}.$$

Then $\min\{g(z) : |z| \le 1/7\} = e^{-\frac{1+1/7}{1-1/7}} = e^{-4/3}$. Similarly, $\max\{g(z)^2 : |z| \le 1/5\} = e^{-2\frac{1-1/5}{1+1/5}} = e^{-4/3}$.

(g) Maximum Principle for Harmonic Functions

If u is harmonic in D and there is some $z_0 \in D$ such that $u(z_0) \ge u(z)$ for $z \in D$ then $u(z) \cong u(z_0)$.

Proof:
By the Mean-Value Property

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta, 0 \le r < \operatorname{dist}(\partial D, z_0),$$

so

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \left(u(z_0 + re^{i\theta}) - u(z_0)\right)d\theta.$$

As $u(z_0) \ge u(z_0 + re^{i\theta})$ for all r, θ, by continuity, we have

$$u(z_0) = u(z_0 + re^{i\theta})$$

for all r, θ. Therefore, for any $z \in D$, as D is open and connected, there exists a path $\gamma_z : [0, 1] \to D$ such that $\gamma_z(0) = z_0$, $\gamma_z(1) = z$ and overlapping discs on points $z_1, \ldots, z_k$ of γ_z with radii r_i respectively. As we get equality in each disc, by transitivity, we get $u(z_0) = u(z)$.

(h) Automorphisms of Annuli

Define $A(0; 1, R) = \{z \in \mathbb{C} : 1 < |z| < R\}$. $A(0; 1, R_1)$ and $A(0; 1, R_2)$ are biholomorphically equivalent if and only if $R_1 = R_2$, where $R_1, R_2 > 1$.

Proof:
If we assume $R_1 = R_2$ the result holds as the identity map suffices.
Conversely, assume there exists a biholomorphic map $\phi : A(0; 1, R_1) \to A(0; 1, R_2)$. ϕ^{-1} is continuous, so ϕ is proper, therefore $\phi : \partial A(0; 1, R_1) \to \partial A(0; 1, R_2)$. We claim $\phi(\partial D(0, 1)) = \partial D(0, 1)$ and $\phi(\partial D(0, R_1)) = \partial D(0, R_2)$ or $\phi(\partial D(0, 1)) = \partial D(0, R_2)$ and $\phi(\partial D(0, R_1)) = \partial D(0, 1)$. This holds as the continuous image of a connected set is connected since if otherwise, by connectedness, without loss of generality (the other case is identical) $\phi(\partial D(0, 1)) \subset \partial D(0, 1)$. Then if $\partial D(0, 1) \cap \phi(\partial D(0, R_1)) \ne \emptyset$, we have $\phi(\partial D(0, R_1))$ is not connected as $\partial D(0, R_2) \subset \phi(\partial D(0, R_1))$.
Assume first the case where $|\phi(z)| = 1$ for $|z| = 1$ and $|\phi(z)| = R_2$ for $|z| = R_1$. Consider

$$u(z) = \log|\phi(z)| - \frac{\log R_2}{\log R_1}\log|z|.$$

Then $u(z) = 0$ when $|z| = 1$ or $|z| = R_1$. By the Maximum and Minimum Principles, $u(z) \equiv 0$ on $A(0; 1, R_1)$. Therefore

$$\log\left|\frac{\phi(z)}{z^{\frac{\log R_2}{\log R_1}}}\right| = 0 \implies \left|\frac{\phi(z)}{z^{\frac{\log R_2}{\log R_1}}}\right| = 1,$$

or after simplification, $\phi(z) = e^{i\theta}z^{\frac{\log R_2}{\log R_1}}$. As ϕ is holomorphic, $\log R_2/logR_1$ is an integer. As ϕ is injective, $\log R_2/\log R_1 = 1$, i.e. $R_1 = R_2$ as desired. So biholomorphic maps in this case are of the form $\phi(z) = e^{i\theta}z$.
Assume now that $|\phi(z)| = R_2$ for $|z| = 1$ and $|\phi(z)| = 1$ for $|z| = R_1$. Consider $\psi(z) = \phi(R_1/z)$. Then $|\psi(z)| = 1$ for $|z| = 1$ and $|\psi(z)| = R_2$ for $|z| = R_1$. As we have seen above, $\psi(z) = e^{i\theta}z$ and $R_1 = R_2$. Therefore $\phi(z) = e^{i\theta}R_1/z$.

13. Sub-harmonic Functions (a) Definition 1

$u \in C^2(D)$ is sub-harmonic if $\Delta u \geq 0$.

(b) Definition 2

u is sub-harmonic if

i. u is Upper Semi-Continuous,

ii. $u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta$ for appropriate r.

(c) Sub-harmonic Functions

For $u \in C^2(D)$, these two definitions are equivalent.

Proof:
Assume first that $\Delta u \geq 0$. By Green's Theorem, for n normal to ∂D at z,

$$\int_{\partial D} \frac{\partial u}{\partial n} v d\sigma - \int_{\partial D} \frac{\partial v}{\partial n} u d\sigma = \int_D \Delta uv - \Delta vudA.$$

So

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial n}\log\frac{|z-z_0|}{r}\frac{1}{r}d\theta - \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial n}u(z+re^{i\theta})\log\frac{|z-z_0|}{r}\frac{1}{r}d\theta$$
$$= \frac{1}{r}\frac{1}{2\pi}\int_{D(z_0,r)} \Delta\log\frac{|z-z_0|}{r}u(z) - \Delta u\log\frac{|z-z_0|}{r}dA.$$

We have $\frac{1}{2\pi}\log\frac{|z-z_0|}{r} = \delta_{z_0}$, so

$$\frac{1}{r}\frac{1}{2\pi}\int_{D(z_0,r)} \Delta\log\frac{|z-z_0|}{r}u(z) - \Delta u\log\frac{|z-z_0|}{r}dA = u(z_0) + \frac{1}{2\pi}\int_{D(z_0,r)}\Delta u\log\frac{|z-z_0|}{r}DA \geq u(z_0).$$

Assume now that $u(z_0) \leq \frac{1}{2\pi}\int_0^2 \pi u(z_0+re^{i\theta})d\theta$ for all $r < \text{dist}(z_0,\partial D)$. By the identity established in what we have shown above,

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})d\theta = u(z_0) + \frac{1}{2\pi}\int_{D(z_0,r)}\Delta u\log\frac{r}{|z-z_0|}dA.$$

Therefore we know that

$$0 \leq \frac{1}{2\pi}\int_{D(z_0,r)}\Delta u\log\frac{r}{|z-z_0|}dA.$$

We claim $\Delta u(z_0) \geq 0$ for all $z_0 \in D$. If not, then fix z_0 such that $\Delta u(z_0) < 0$. Then there exists an $r_0 < r$ such that $\Delta u(z) < -\epsilon_0$ for $z \in D(z_0, r_0)$. Then we have

$$0 \leq \int_{D(z_0,r_0)}\Delta u\log\frac{r_0}{|z-z_0|}dA \leq -\epsilon_0\int_{D(z_0,r_0)}\log\frac{r_0}{|z-z_0|}dA < 0,$$

a contradiction. This proves our claim.

(d) Jensen's Inequality

If u is subharmonic and ϕ is convex on the range of u then

$$\phi\left(\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta\right) \leq \frac{1}{2\pi}\int_0^{2\pi}\phi\circ u(z+re^{i\theta})d\theta.$$

Proof:
We have

$$\phi\left(\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta\right) = \phi\left(\sum_{k=1}^n u(z+re^{i\theta_k}\Delta\theta_k)\right) \leq \sum_{k=1}^\infty \phi\left(\sum_{k=1}^n u(z+re^{i\theta_k})\right)\Delta\theta_k = \frac{1}{2\pi}\int_0^{2\pi}\phi\circ u(z+re^{i\theta})d\theta.$$

(e) Examples of Subharmonic functions

$\log|f|$ is subharmonic

(f) Hadamard Three-Line Theorem

Let f be holomorphic on $S(a,b) = \{x+iy : a < x < b\}$ and define $M(x,f) = \max\{|f(z)| : \mathrm{Re}(z) = x\}$. Assume $M(a,f), M(b,f) < \infty$. Then if $|f(z)| \leq Be^{A|z|}$ on $S(a,b)$, for real numbers A, B, then $M(x,f)$ is finite and $\log(M(x,f))$ is convex.

Proof:
The proof proceeds by cases.
Case 1: Assume $f(z)$ is bounded on $S(a,b)$. Then there is some $M < \infty$ such that $|f(z)| < M$. Therefore $M(x,f) < M$ for $x \in [a,b]$. It remains to show that $\log M(x,f)$ is convex. $\log M(x,f)$ is convex if, since we may write $x = \dfrac{b-x}{b-a}a + \dfrac{x-a}{b-a}b$,

$$\log M(x,f) \leq \frac{b-x}{b-a}\log M(a,f) + \frac{x-a}{b-a}\log M(b,f).$$

Notice that this is equivalent to

$$|f(x+iy)| \leq M(x,f) \leq M(a,f)^{\frac{b-x}{b-a}} M(b,f)^{\frac{x-a}{b-a}},$$

for $y \in \mathbb{R}, x \in [a,b]$. This, in turn, is equivalent to

$$\left| f(z)M(a,f)^{-\frac{b-z}{b-a}} M(b,f)^{\frac{z-a}{b-a}} \right| \leq 1$$

since $|a^z| = a^x$.
Define $g(z) = f(z)M(a,f)^{\frac{b-z}{b-a}} M(b,f)^{\frac{z-a}{b-a}}$. Then

$$|g(a+iy)| = |f(a+iy)|M(a,f)^{-1} \leq 1,$$

$$|g(a+iy)| = |f(b+iy)|M(b,f)^{-1} \leq 1.$$

Define now $g_\epsilon(z) = g(z)\dfrac{1}{1+\epsilon z}$ for some $\epsilon > 0$. Then $|g_\epsilon(a+iy)| \leq 1$, $|g_\epsilon(b+iy)| \leq 1$. For $x \in [a,b]$, $|y| >> 1$. Then

$$g_\epsilon(x+iy)| \leq MM(a,f)^{-\frac{b-x}{b-a}} M(b,f)^{-\frac{a-x}{b-a}} \frac{1}{\epsilon|y|} \leq M\min\{M(a,f), M(b,f)\}^{-1}\frac{1}{\epsilon|y|},$$

for $y \geq \frac{1}{\epsilon}M\min\{M(a,f), M(b,f)\}^{-1}$. Then, by the Maximum Modulus Principle, $|g_\epsilon(z)| \leq 1$ on $S(a,b)$. Letting $\epsilon \to 0^+$, we have $|g(z)| \leq 1$, which proves our claim.
Case 2: $|f(z)| \leq Be^{A|z|}$.
Consider the function

$$g_\epsilon(z) = f(z)M(a,f)^{-\frac{b-z}{b-a}} M(b,f)^{-\frac{a-z}{b-a}} e^{\epsilon z^2}.$$

Then we have

$$|g_\epsilon(a+iy)| \leq 1|e^{\epsilon(a+iy)^2} = e^{\epsilon(a^2-y^2)} \leq e^{\epsilon a^2},$$

and similarly

$$|g_\epsilon(b+iy)| \leq e^{\epsilon b^2}.$$

For $x \in [a,b]$, $|y| >> 1$ we have

$$|g_\epsilon(x+iy)| = |f(z)|M(a,f)^{-\frac{b-x}{b-a}} M(b,f)^{-\frac{x-a}{b-a}} e^{\epsilon(x^2-y^2)} \leq Be^{A|z|}\min\{M(a,f), M(b,f)\}^{-1}e^{\epsilon b^2 - \epsilon y^2}$$

$$= Be^{A(b+|y|)+\epsilon b^2 - \epsilon y^2}\min\{M(a,f), M(b,f)\}^{-1} \leq 1$$

for y appropriately large. Therefore, $|g_\epsilon(z)| \leq e^{\epsilon b^2}$ for $z \in S(a,b)$. Letting $\epsilon \to 0^+$ yields

$$|f(z)|M(a,f)^{-\frac{b-z}{b-a}} M(b,f)^{-\frac{a-z}{b-a}} \leq 1$$

for $z \in S(a,b)$ as desired.

(g) Hadamard Three-Circle Theorem

Let $f(z)$ be holomorphic in $A(0;r,R)$ and define $M(r,f) = \max\{|f(re^{i\theta})| : \theta \in [0,2\pi)\},$. If $M(r,f), M(R,f) < \infty$, then $\log M(e^t, f)$ is convex on $(\log r, \log R)$.

Proof:
Let $S = \{x+iy : y \in \mathbb{R}, \log r < x < \log R\}$ and define F to be

$$F(z) = f(e^z),$$

so F is holomorphic on S and $|F(z)| = |f(e^z)|$. Then we have $|F(\log r + iy)| = |f(re^{iy})| \le M(r,f)$ and similarly $|F(\log R + iy)| \le M(R,f)$. Then, by the previous theorem, we have

$$|f(r_0 e^{it})| = |F(\log r_0 + it)| \le M(a,f)^{\frac{\log R - \log r_0}{\log R - \log r}} M(b,f)^{\frac{\log r_0 - \log r}{\log R - \log r}}$$

14. Jensen's Formula (a) Jensen's Formula

Let f be holomorphic in $\overline{D(0,R)}$, $f(z) \neq 0$ for $z \in \partial D(0,R)$ and $f(0) \neq 0$. Then, if $a_1, \ldots, a_n$ are the zeros of f in $D(0,R)$ counting multiplicity, then

$$\log|f(0)| + \sum_{k=1}^{n} \log \frac{R}{|a_k|} = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| d\theta.$$

Proof:
Let

$$g(z) = \frac{f(z)}{\prod_{j=1}^{n} \left(\frac{R(z-a_j)}{R^2 - \overline{a_j} z} \right)}.$$

Then $g(z) \neq 0$ for $z \in \overline{D(0,R)}$ and

$$|g(Re^{i\theta})| = \frac{|f(Re^{i\theta})|}{\prod_{j=1}^{n} \left| \frac{R(z-a_j)}{R^2 - \overline{a_j} z} \right|} = |f(Re^{i\theta}).$$

By the Mean-Value Property,

$$\log|g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log|g(Re^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})| d\theta.$$

Further

$$\log|g(0)| = \log \left| \frac{f(0)}{\prod_{j=1}^{n} \frac{|a_j|}{R}} \right| = \log|f(0)| + \sum_{j=1}^{n} \log \frac{R}{|a_j|},$$

completing the proof.

(b) Application of Jensen's Formula

Let $f : D(0,1) \to D(0,1)$ be holomorphic and $f(0) = 2^{-10}$. What is the best upper bound for the number of zeros of f in $\overline{D(0,1/2)}$?

Solution:
Let $1/2 < r < 1$ and $a_1, \ldots, a_n$ be all the zeros of f in $D(0,r)$ such that $f(re^{i\theta}) \neq 0$. By Jensen's Formula

$$\log|f(0)| + \sum_{j=1}^{n} \log \frac{r}{|a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| d\theta < 0,$$

so

$$\sum_{j=1}^{n} \log \frac{r}{|a_j|} < -\log|f(0)| = 10 \log 2.$$

Without loss of generality, the zeros of f in $\overline{D(0,1/2)}$ are $a_1,\ldots,a_k$, $k\le n$. Then

$$\sum_{j=1}^{k}\frac{r}{|a_j|}\le\sum_{j=1}^{n}\log\frac{r}{|a_j|}\le 10\log 2.$$

For $a_j\in\overline{D(0,1/2)}$, $\log\frac{r}{|a_j|}\ge\log\frac{r}{1/2}$, so

$$10\log 2\ge\sum_{j=1}^{k}\log\frac{r}{|a_j|}\ge k\log\frac{r}{1/2},$$

or after simplification, we have

$$k\le 10\frac{log2}{\log 2r}, \text{for all } 1/2<r<1.$$

Letting $r\to 1^-$, we have $k\le 10$.

15. Special Functions

(a) The Gamma Function

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt,$$

which is holomorphic for $\mathrm{Re}(z)>0$. Note by integrating by parts, we get

$$\Gamma(z)=\frac{1}{z}\Gamma(z+1),$$

and continuing in this method, we get

$$\Gamma(z)\frac{1}{z(z+1)\cdots(z+n)}\Gamma(z+n+1),$$

which defines a meromorphic function on $\mathrm{Re}(z)>-(n+1)$ with simple poles at $z=0,-1,-2,\ldots,-n$ for any positive integer n.
Note also, that $\Gamma(n)=(n-1)!$ for positive integer values n, so

$$\mathrm{Res}(\Gamma(z);z=-k)=\lim_{z\to -k}\frac{z-k}{z\cdots(z+k)}\Gamma(z+k+1)=\frac{1}{(-k)(-k+1)\cdots(-k+k-1)}\Gamma(1)=\frac{(-1)^k}{k!}.$$

(b) The Riemann Zeta Function

$$\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z},$$

for $\mathrm{Re}(z)>1$.
$\zeta(z)$ can be extended to a meromorphic function with a simple pole only at $z=1$ and with residue 1 there.
$\zeta(z)$ also satisfies the following functional equation:

$$\zeta(z)=2^z\pi^{z-1}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta(1-z).$$