A List of Problems in Real Analysis

W.Yessen & T.Ma

June 26, 2015

This document was first created by Will Yessen, who now resides at Rice University. Timmy Ma, who is still a student at UC Irvine, now maintains this document. Problems listed here have been collected from multiple sources. Many have appeared on qualifying exams from PhD granting universities. If you would like to add to this document, please contact Timmy.

Solutions are available upon request. Certainly, it is beneficial for the reader to attempt the problems first before seeking the solutions.

Should there be any problems, errors, questions, or comments please contact Timmy.

Contents

1 Measure Theory and Topology

1.1 Measurable Sets

Problem 1.1.1. Identify which statements are true, which are false. Give a counter-example for the false statements.
1) A finite union of open sets is an open set.
2) A countable union of open sets is an open set.
3) Any (even uncountable) union of open sets is an open set.
4) A finite union of compact sets is compact.
5) A countable union of compact sets is compact.
6) A countable union of measurable sets is a measurable set.
7) Any (even uncountable) union of measurable sets is a measurable set.
8) $\overline{\cup_1^n B_i} = \cup_1^n \overline{B_i}$
9) $\overline{\cup_1^\infty B_i} = \cup_1^\infty \overline{B_i}$
10) $\overline{\cup_\alpha B_\alpha} = \cup_\alpha \overline{B_\alpha}$

Problem 1.1.2. Let f_n be a sequence of continuous functions on a complete metric space X with $f_n(x) \geq 0$. Let $A = \{x \in X : \liminf f_n(x) = 0\}$. Prove that A is a countable intersection of open sets.

Problem 1.1.3. Let O_n be a countable collection of dense open subsets of a complete metric space. Prove that $\cap O_n$ is dense.

Problem 1.1.4. Does there exist a Lebesgue measurable subset A of $\mathbb{R}$ such that for every interval (a, b) we have $\mu_L(A \cap (a, b)) = \frac{b-a}{2}$? Either construct such a set or prove it does not exist.

Problem 1.1.5. Let $\mathcal{A}$ be a collection of pairwise disjoint subsets of a σ-finite measure space, each of positive measure. Show that $\mathcal{A}$ is at most countable.

Problem 1.1.6. Let $A \subset \mathbb{R}$ be a Lebesgue measurable set. Denote the length of an interval I by $|I|$.
(a) Suppose that $m(A \cap I) \leq \frac{2}{3}|I|$ for every interval I. Show that A has measure 0.
(b) Suppose that the measure of A is positive. Show that there is an interval I such that if $|d| < |I|/3$, then

$$((I \cap A) + d) \cap (I \cap A) \neq \emptyset$$

(c) Suppose that the measure of A is positive. Show that the set $A - A = \{x - y : x, y \in A\}$ contains an open interval about 0.

Problem 1.1.7. Suppose $A_j \subset [0, 1]$ is Lebesgue measurable with measure $\geq 1/2$ for each $j = 1, 2, \ldots$. Prove that there is a measurable set $S \subset [0, 1]$ with measure $\geq 1/2$ such that each $x \in S$ is in A_j for infinitely many j.

Problem 1.1.8. Are the irrationals an F_σ set? Justify.

Problem 1.1.9. Suppose $E \subset \mathbb{R}$ is such that $\mu_L(E) > 0$ (μ_L is the Lebesgue measure on $\mathbb{R}$). Prove that there exists an interval $I \subset \mathbb{R}$ satisfying

$$\mu_L(E \cap I) > \frac{3}{4}\mu_L(I)$$

Problem 1.1.10. Suppose f is a function of bounded variation on $[a, b]$. Define

$$G = \left\{(x, y) \in \mathbb{R}^2 : x \in [a, b], f(x) = y\right\}$$

Prove that G has Lebesgue measure 0.

Problem 1.1.11. Let $E_1, \ldots, E_n$ be measurable subsets of $[0, 1]$. Suppose almost every x in $[0, 1]$ belongs to at least k of these subsets. Prove that at least one of $E_1, \ldots, E_n$ has measure of at least k/n.

Problem 1.1.12. Suppose that A is a subset in $\mathbb{R}^2$. Define for each $x \in \mathbb{R}^2$ $\rho(x) = \inf\{|y - x| : y \in A\}$. Show that $B_r = \left\{x \in \mathbb{R}^2 : \rho(x) \leq r\right\}$ is a closed set for each non-negative r. Is the measure of B_0 equal to the outer measure of A? Why?

Problem 1.1.13. Let $\{E_n\}$ be a sequence of measurable sets in a measure space (X, Σ, μ). Suppose $\lim_{n\to\infty} E_n$ exists, i.e., $\liminf_{n\to\infty} E_n = \limsup_{n\to\infty} E_n$. Is it always true that $\mu(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$? Does it make a difference if the measure is finite? Prove or disprove by counterexample.

Problem 1.1.14. Given a measure space $(X, \mathfrak{A}, \mu)$, let $\{A_i\}$ be a sequence in $\mathfrak{A}$. Let $A = \cup_{i=1}^{\infty} A_i$ and suppose each $x \in A$ belongs to no more than k different A_i. Show that $\mu(A) \geq \frac{1}{k}\sum_{i=1}^{\infty} \mu(A_i)$
(a) for $k = 2$
(b) for $k = 1999$

Problem 1.1.15. Suppose μ_L (resp. μ_L^*) is the Lebesgue measure (resp. outer Lebesgue measure) on $\mathbb{R}$, and $E \subset \mathbb{R}$. Let $E^2 = \left\{e^2 : e \in E\right\}$.
(a) Show that, if $\mu_L^*(E) = 0$, then $\mu_L^*(E^2) = 0$.
(b) Suppose $\mu_L(E) < \infty$. Is it true that $\mu_L^*(E^2) < \infty$?

Problem 1.1.16. Suppose $(X, \mathcal{A})$ is a measurable space, and Y is the set of all signed measures ν on $\mathcal{A}$ for which $|\nu(A)| < \infty$ whenever $A \in \mathcal{A}$. For $\nu_1, \nu_2 \in Y$, define

$$d(\nu_1, \nu_2) = \sup_{A \in \mathcal{A}} |\nu_1(A) - \nu_2(A)|$$

Show that d is a metric on Y and that Y equipped with d is a complete measure space.

Problem 1.1.17. (a) For any $\epsilon > 0$, construct an open set $U_\epsilon \subset (0, 1)$ so that $U_\epsilon \supset \mathbb{Q} \cap (0, 1)$ and $m(U_\epsilon) < \epsilon$.
(b) Let $A = \cup_{n=1}^{\infty} U_{1/n}^c$. Find $m(A)$. Show that $A^c \cap (1/2, 5/16) \neq \emptyset$

Problem 1.1.18. Show that either a σ-algebra of subsets of a set X is finite, or else it has uncountably many elements.

Problem 1.1.19. Let $r_1, r_2, \ldots$ enumerate all positive rational numbers.
(a) Show that

$$\mathbb{R}^+ - \cup_n (r_n - \frac{1}{n^2}, r_n + \frac{1}{n^2}) \neq \emptyset$$

(b) Show with an example that the r_n may be enumerated so that

$$\mathbb{R}^+ - \cup_n (r_n - \frac{1}{n}, r_n + \frac{1}{n}) = \emptyset$$

Problem 1.1.20. Suppose that $A \subseteq [0,1]$ is a measurable set such that $m(I \cap A) \leq m(I)/2$, for all intervals $I \subseteq [0,1]$. Show that $m(A) = 0$.

Problem 1.1.21. Assume that E is a subset of $\mathbb{R}^2$ and the distance between any two points in E is a rational number. Show that E is a countable set. (Note that this is actually true in any dimension)

Problem 1.1.22. Does there exist a nowhere dense subset of $[0,1]^2 \subset \mathbb{R}^2$
(a) of Lebesgue measure greater than 9/10?
(b) of Lebesgue measure 1?

Problem 1.1.23. Let $f_n : \mathbb{R} \to \mathbb{R}$ be Borel measurable for every $n \in \mathbb{N}$. Define E to be the set of points in $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x)$ exists and is finite. Show that E is a Borel measurable set.

Problem 1.1.24. Let A be a subset of $\mathbb{R}$ of positive Lebesgue measure. Prove that there exists $k, n \in \mathbb{N}$ and $x, y \in A$ with $|x - y| = \frac{k}{2^n}$

Problem 1.1.25. Is it possible to find uncountably many disjoint measurable subsets of $\mathbb{R}$ with strictly positive Lebesgue measure?

Problem 1.1.26. Let X be a non-empty complete metric space and let

$$\{f_n : X \to \mathbb{R}\}_{n=1}^{\infty}$$

be a sequence of continuous functions with the following property: for each $x \in X$, there exists an integer N_x so that $\{f_n(x)\}_{n>N_x}$ is either a monotone increasing or decreasing sequence. Prove that there is a non-empty open subset $U \subseteq X$ and an integer N so that the sequence $\{f_n(x)\}_{n>N}$ is monotone for all $x \in U$.

Problem 1.1.27. Give an example of a subset of $\mathbb{R}$ having uncountable many connected components. Can such a subset be open? Closed?

Problem 1.1.28. Let $\{f_n : [0,1] \to \mathbb{R}\}_{n=1}^{\infty}$ be a sequence of continuous functions and suppose that for all $x \in [0,1]$, $f_n(x)$ is eventually nonnegative. Show that there is an open interval $I \subseteq X$ such that for all n large enough, f_n is nonnegative everywhere on I.

Problem 1.1.29. Construct an open set $U \subset [0,1]$ such that U is dense in $[0,1]$, the Lebesgue measure $\mu(U) < 1$, and that $\mu(U \cap (a,b)) > 0$ for any interval $(a,b) \subset [0,1]$.

Problem 1.1.30. Is it possible for a continuous function $f : [0,1] \to \mathbb{R}$ to have
(a) infinitely many strict local minima?
(b) uncountably many strict local minima?
Prove your answers.

1.2 (Signed) Measure

Problem 1.2.1. Let μ be a measure and let $\lambda, \lambda_1, \lambda_2$ be signed measures on the measurable space $(X, \mathcal{A})$. Prove:
(a) if $\lambda \perp \mu$ and $\lambda \ll \mu$, then $\lambda = 0$.
(b) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then, if we set $\lambda = c_1\lambda_1 + c_2\lambda_2$ with c_1, c_2 real numbers such that λ is a signed measure, we have $\lambda \perp \mu$.
(c) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then, if we set $\lambda = c_1\lambda_1 + c_2\lambda_2$ with c_1, c_2 real numbers such that λ is a signed measure, we have $\lambda \ll \mu$.

Problem 1.2.2. Let μ be a positive Borel measure on $\mathbb{R}$ such that

$$\lim_{t\to 0} \chi_{E+t}(x) = \chi_E(x)$$

μ-a.e. for every borel set E. Show that $\mu = 0$.

Problem 1.2.3. Suppose a measure m is defined on a σ-algebra $\mathfrak{A}$ of subsets of X, and m^* is the corresponding outer measure. Suppose $A, B \subset X$. We say that $A \cong B$ if $m^*(A\Delta B) = 0$. Prove that $\cong$ is an equivalence relation.

Problem 1.2.4. Let μ be a positive measure and ν be a finite positive measure on a measurable space $(X, \mathfrak{A})$. Show that $\nu \ll \mu$ then for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $E \in \mathfrak{A}$ with $\mu(E) < \delta$ we have $\nu(E) < \epsilon$.

Problem 1.2.5. Suppose that μ and ν are two σ-finite measures on the Lebesgue σ-algebra of subsets of $\mathbb{R}^d$. Suppose that $\mu \ll \nu$ and $\nu \ll \mu$. Show that there is a measurable set A such that $\nu(A) = 0$ and Radon-Nikodym derivative $\frac{d\mu}{d\nu}(x) > 0$ for all $x \in \mathbb{R}^d \setminus A$.

Problem 1.2.6. Let (X, σ, μ) be a measure space with $\mu(X) < \infty$. Given sets $A_i \in \sigma$, $i \geq 1$, prove that

$$\mu(\cap_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(\cap_{i=1}^{n} A_i).$$

Give an example to show that this need not hold when $\mu(X) = \infty$.

2 Functions and Integration

2.1 Measurable Functions

Problem 2.1.1. Suppose $\{f_n\}$ is a sequence of measurable functions on $[0,1]$. For $x \in [0,1]$ define $h(x) = \#\{n : f_n(x) = 0\}$ (the number of indices n for which $f_n(x) = 0$). Assuming that $h < \infty$ everywhere, prove that the function h is measurable.

Problem 2.1.2. Let $x = 0.n_1 n_2 \dots$ be a decimal representation of $x \in [0,1]$. Let $f(x) = min\{n_i : i \in \mathbb{N}\}$. Prove that $f(x)$ is measurable and a.e constant.

Problem 2.1.3. Let f_n be a sequence of measurable functions on (X, μ) with $f_n \geq 0$ and $\int_X f_n d\mu = 1$ show that

$$\limsup_{n \to +\infty} f_n^{1/n} \leq 1 \text{ for } \mu \text{ a.e } x$$

Problem 2.1.4. Let $f : [0,1] \to \mathbb{R}$ be continuous function. Define the signed Borel measure μ on $[0,1]$ by $d\mu = f dm$ Assume

$$\int_{[0,1]} x^n d\mu = 0, \; n = 0, 1, 2, \dots$$

Prove that $\mu = 0$.

2.2 Integrable Functions

Problem 2.2.1. Let f be a nonnegative, Lebesgue measurable function on the real line, such that the function $g(x) = \sum_{n=1}^{\infty} f(x+n)$ is integrable on the real line. Show that $f = 0$ almost everywhere.

Problem 2.2.2. Suppose f is a bounded nonnegative function on (X, μ) with $\mu(X) = \infty$. Show that f is integrable if, and only if,

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \mu \left\{ x \in X : f(x) > \frac{1}{2^n} \right\} < \infty$$

Problem 2.2.3. Given a measure space $(X, \mathfrak{A}, \mu)$, let f be a nonnegative extended real-valued, $\mathfrak{A}$-measurable function on a set $D \in \mathfrak{A}$ with $\mu(D) < \infty$. Let $D_n = \{x \in D : f(x) \geq n\}$. Show that f is integrable if and only if $\sum_n \mu(D_n) < \infty$

Problem 2.2.4. Let f be a Lebesgue integrable function on the real line. Prove that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \sin(nx) dx = 0$$

Problem 2.2.5. Suppose $(X, \mathfrak{A}, \mu)$ is a measure space, and $\mu(X) < \infty$. Suppose furthermore, that f is a nonnegative measurable function on X. Prove that f is integrable if and only if the series

$$\sum_{n=0}^{\infty} 2^n \mu(\{x \in X : f(x) > 2^n\})$$

converges.

Problem 2.2.6. Let f be a real-valued integrable function on a measure space. Let $\{E_n\}$ be a sequence of measurable sets such that $\lim_{n\to\infty} \mu(E_n) = 0$. Show that $\lim_{n\to\infty} \int_{E_n} f d\mu = 0$.

Problem 2.2.7. Let f be a nonnegative Lebesgue measurable function on $[0,1]$. Suppose f is bounded above by 1 and $\int_{[0,1]} f dx = 1$. Show that $f = 1$ a.e. on $[0,1]$.

Problem 2.2.8. Let f be real-valued continuous function on $[0,\infty)$ such that the improper Riemann integral $\int_0^\infty f(x)dx$ converges. Is f Lebesgue integrable on $[0,\infty)$? Prove or disprove by counterexample.

Problem 2.2.9. (a) Show that an extended real valued integrable function is finite a.e.
(b) If f_n is a sequence of measurable functions such that

$$\sum_n \int |f_n| < \infty$$

show that $\sum_n f_n(x)$ converges a.e. to an integrable function f and

$$\int f = \sum_n \int f_n$$

Problem 2.2.10. Prove that the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

is well defined and continuous for $x > 0$.

Problem 2.2.11. Using the fact that Lebesgue measure m is translation invariant, that is, for every measurable set E and $x \in \mathbb{R}$, $m(E) = m(x+E)$, prove that for $f \in L^1(\mathbb{R})$, and for all $t \in \mathbb{R}$,

$$\int_{\mathbb{R}} f(x+t)dx = \int_{\mathbb{R}} f(x)dx$$

Problem 2.2.12. Let f be a real-valued uniformly continuous function on $[0,\infty)$. Show that if f is Lebesgue integrable on $[0,\infty)$, then $\lim_{x\to\infty} f(x) = 0$. Does this still hold if f is continuous, but not uniformly continuous? Prove or give a counterexample.

Problem 2.2.13. Consider the Lebesgue measure space $(\mathbb{R}, \mathcal{M}_L, \mu_L)$ on $\mathbb{R}$. Let f be a μ_L-integrable extended real-valued $\mathcal{M}_L$-measurable function on $\mathbb{R}$. Show that

$$\lim_{h\to 0} \int_{\mathbb{R}} |f(x+h) - f(x)| \mu_L(dx) = 0$$

Problem 2.2.14. Let $S \subset \mathbb{R}$ be closed and let $f \in L^1([0,1])$. Assume that for all measurable $E \subset [0,1]$ with $m(E) > 0$ we have $\frac{1}{m(E)} \int_E f \in S$. Prove that $f(x) \in S$ for a.e $x \in [0,1]$

Problem 2.2.15. Assume $f, g \in L^2(\mathbb{R})$. Define $A(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$. Show that $A(x) \in C(\mathbb{R})$ and that $\lim_{|x|\to+\infty} A(x) = 0$.

Problem 2.2.16. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f \in L^1$
(a) Show that the Lebesgue measure of $\{x : |f(x)| > t\}$ approaches 0 as $t \to \infty$.
(b) Show that the additional assumption that $f' \in L^1$ implies that $f(x) \to 0$ as $x \to \infty$.

Problem 2.2.17. Given f is integrable over $[0, 1]$. Show that there exists a decreasing sequence of positive numbers, a_n, converging to 0, such that $\lim_{n\to\infty} a_n|f(a_n)| = 0$.

Problem 2.2.18. Let f be nonnegative measurable function on $\mathbb{R}$. Suppose $\sum_{n=1}^{\infty} \int_{\mathbb{R}} f^n$ converges. Show that $f < 1$ a.e. and that $\dfrac{f}{1-f} \in L^1(\mathbb{R})$.

2.3 Convergence

Problem 2.3.1. Let $\{f_n\}$ be a sequence of continuous functions on $[0, 1]$ with $0 \le f_n(x) \le 1$ for all $x \in [0, 1]$ and all n. Prove that:
(a) If for each $x \in [0, 1]$ we have $\lim_{n\to\infty} f_n(x) = 0$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$$

(b) Construct a sequence of $\{f_n\}$ so that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$, but for each $x \in [0, 1]$ the sequence $\{f_n\}$ diverges.

Problem 2.3.2. Let $\{f_n\}$ be a sequence of real-valued functions such that $\int_{-\infty}^{\infty} |f_n(x)|^{1.5}dx \le 1$. Show that there is a subsequence f_{n_k} such that $\int_1^{\infty} x^{-0.5} f_{n_k}(x)dx$ converge when $k \to \infty$.

Problem 2.3.3. Let $(X, \mathcal{U}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of nonnegative extended real-valued $\mathcal{U}$-measurable functions on X. Suppose $\lim_{n\to\infty} f_n = f$ exists μ-a.e. on X and moreover $f_n \le f$ μ-a.e. on X. Show that

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu$$

Problem 2.3.4. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $\{f_n\}$ and f be real-valued, $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$. Suppose that $f_n \to f$ in measure μ on D. Let F be a real-valued, continuous function on $\mathbb{R}$. Show that $F \circ f_n \to F \circ f$ in measure μ.

Problem 2.3.5. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\{f_n\}$ and f be real-valued, $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$. Suppose that $f_n \to f$ in measure μ on D. Let F be a real-valued, uniformly continuous function on $\mathbb{R}$. Show that $F \circ f_n \to F \circ f$ in measure μ.

Problem 2.3.6. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\{f_n\}$ and f be real-valued, $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$. Suppose that $f_n \to f$ in measure μ on D. Let F be a real-valued, continuous function on $\mathbb{R}$. And that for all n, f_n, f are uniformly bounded on D. Show that $F \circ f_n \to F \circ f$ in measure μ.

Problem 2.3.7. Let f be an element and $\{f_n\}$ be a sequence in $L^p(X, \mathfrak{A}, \mu)$, where $p \in [1, \infty)$, such that $\lim_{n\to\infty} \|f_n - f\|_p = 0$. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ we have

$$\int_E |f_n|^p d\mu < \epsilon \text{ for every } E \in \mathfrak{A} \text{ such that } \mu(E) < \delta$$

Problem 2.3.8. Let f be an integrable function on a measure space $(X, \mathcal{B}, \mu)$. Show that:
(a) The set $\{f \neq 0\}$ is of σ-finite measure.
(b) If $f \geq 0$, then $f = \lim_n \phi_n$ pointwise for some increasing sequence of simple functions ϕ_n each of which vanishes outside a set of finite measure.
(c) For every $\epsilon > 0$ there is a simple function ϕ such that

$$\int_X |f - \phi| d\mu < \epsilon$$

Problem 2.3.9. Let $\{f_n\}$ and $\{g_n\}$ be sequences of measurable functions on the measurable set E, and let f and g be measurable functions on set E. Suppose $f_n \to f$ and $g_n \to g$ in measure. Is it true that $f_n^3 + g_n \to f^3 + g$ in measure if
(a) $m(E) < \infty$
(b) $m(E) = \infty$

Problem 2.3.10. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\{f_n\}$ and f be extended real-valued $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$ such that $\lim_{n\to\infty} f_n = f$ on D. Then for every $\alpha \in \mathbb{R}$ we have
(1) $\mu\{D : f > \alpha\} \leq \liminf \mu\{D : f_n \geq \alpha\}$
(2) $\mu\{D : f < \alpha\} \leq \liminf \mu\{D : f_n \leq \alpha\}$

Problem 2.3.11. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\{f_n\}$ and f be a sequence of extended real-valued $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$ with $\mu(D) < \infty$. Show that f_n converges to 0 in measure on D if and only if

$$\lim_{n\to\infty} \int_D \frac{|f_n|}{1 + |f_n|} d\mu = 0$$

Problem 2.3.12. Given a measure space $(X, \mathfrak{A}, \mu)$, let $\{f_n\}$ and f be extended real-valued $\mathfrak{A}$-measurable functions on a set $D \in \mathfrak{A}$ and assume that f is real-valued a.e. on D. Suppose there exists a sequence of positive numbers ϵ_n such that
1. $\sum_{n\in\mathbb{N}} \epsilon_n < \infty$
2. $\int_D |f_n - f|^p d\mu < \epsilon_n$, for some positive $p < \infty$.
Show that the sequence $\{f_n\}$ converges to f a.e. on D.

Problem 2.3.13. Suppose that g and $\{f_n\}$ are in $L^2[0, 1]$, with $\int_0^1 |f_n(x)^2| dx \leq M < \infty$ for all values of n. Suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ for each x.
(a) Show that $f \in L^2[0, 1]$.
(b) Prove that $\int_E f_n(x) dx \to \int_E f(x) dx$ for each measurable subset E of $[0, 1]$.

(c) Prove that $\int_0^1 f_n(x)g(x)dx \to \int_0^1 f(x)g(x)dx$.

Problem 2.3.14. Let $(X, \mathfrak{A}, \mu)$ be a measure space. Let $\{f_n\}$ and f be real-valued $\mathfrak{A}$-measurable functions on a set $D \in \mathfrak{A}$.
(a) Define convergence of f_n to f in measure on D.
(b) Suppose f_n converges to two functions f and g in measure on D. Show that $f = g$ a.e. on D.

Problem 2.3.15. Let f be a real-valued Lebesgue measurable function on $[0, \infty)$ such that
1. f is Lebesgue integrable on every finite subinterval of $[0, \infty)$.
2. $\lim_{x\to\infty} f(x) = c \in \mathbb{R}$
Show that $\lim_{a\to\infty} \frac{1}{a} \int_{[0,a]} f dm_L = c$.

Problem 2.3.16. Suppose $f_n(x) \in C^1([0, 1])$ with $f_n(0) = 1$, is a sequence of functions such that

$$\sup_n \int_0^1 |f_n'(x)|^2 dx \leq 1$$

Show that there is a subsequence of $\{f_n\}$ which converges uniformly on $[0, 1]$. (Hint: First show that $\{f_n(x)\}$ are equicontinuous). Hint from Will: After following the first hint above, use Ascoli's theorem.

Problem 2.3.17. If $f_n(x) \geq 0$ and $f_n(x) \to f(x)$ in measure then $\int f(x)dx \leq \liminf \int f_n(x)dx$. (Remark: this is not just Fatou's Lemma).

Problem 2.3.18. Let $(X, \mathfrak{A}, \mu)$ be a finite measure space. Let $\{f_n\}$ be an arbitrary sequence of real-valued $\mathfrak{A}$-measurable functions on X. Show that for every $\epsilon > 0$ there exists $E \subset \mathfrak{A}$ with $\mu(E) < \epsilon$ and a sequence of positive real numbers $\{a_n\}$ such that

$$\lim_{n\to\infty} a_n f_n(x) = 0 \text{ for } x \in X \setminus E$$

Problem 2.3.19. Let $(X, \mathfrak{A}, \mu)$ be a finite measure space. Let f and $\{f_n\}$ be real-valued $\mathfrak{A}$-measurable functions on X and $f, f_n \in L^2(X, \mathfrak{A}, \mu)$. Suppose
1. $\lim_{n\to\infty} f_n = f$, μ-a.e.
2. $\|f_n\|_2 \leq C$ for all $n \in \mathbb{N}$.
Show that $\lim_{n\to\infty} \|f_n - f\|_1 = 0$

Problem 2.3.20. Suppose $m(X) < \infty$ and f_n converges to f in measure on X and g_n converges to g in measure on X. Prove that $f_n g_n$ converges to fg in measure on X. (Note from Timmy: Give a counter-example to show that this fails if $m(X) = \infty$)

Problem 2.3.21. Suppose that $f_n(x)$ is a sequence of measurable functions such that $f_n(x)$ converge to $f(x)$ a.e. If for each $\epsilon > 0$, there is $C < \infty$ such that $\int_{|f_n(x)|>C} |f_n(x)|dx < \epsilon$. Show that $f(x)$ is integrable on $[0, 2]$.
Note from Will: That's exactly how this problem appeared on the Winter 2000 qual. Note that C does not depend on n. To see this, find an example where C depends on n and on ϵ, such that the statement of the problem fails to hold.

Problem 2.3.22. Show that the Lebesgue Dominated Convergence Theorem holds if almost everywhere convergence is replaced by convergence in measure. (Hint from Will: First show that a.e. convergence can be replaced by convergence in measure in Fatou's lemma).

Problem 2.3.23. Let (X, Σ, μ) be a finite measure space, and let $\mathcal{F}$ denote the set of measurable functions ,where we identify functions that are equal μ-a.e. Define

$$\delta(f, g) = \int_X \frac{|f-g|}{1+|f-g|} d\mu$$

for $f, g \in \mathcal{F}$. Prove that δ is a metric on the set $\mathcal{F}$, and that $(\mathcal{F}, \delta)$ is a complete metric space. Prove also that convergence is equivalent to convergence in this metric space.

Problem 2.3.24. Suppose that $\{f_n\}$ is a sequence of real-valued Lebesgue measurable functions such that

$$\int_0^1 |f_n(x) - x| dx \to 0$$

when $n \to \infty$. Is it true that $f_n(x)$ converges to x for almost every x in $(0, 1)$? Why?

Problem 2.3.25. Let $(X, \mathcal{M}, \mu)$ be a measure space.
(a) Suppose $\mu(X) < \infty$. If f and f_n are $\mathcal{M}$-measurable functions with $f_n \to f$ a.e., prove that there exist sets $H, E_k \in \mathcal{M}$ such that $X = H \cup (\cup_{k=1}^{\infty} E_k)$, $\mu(H) = 0$, and $f_n \to f$ uniformly on each E_k.
(b) Is the result of (a) true if $(X, \mathcal{M}, \mu)$ is σ-finite?

Problem 2.3.26. Consider the real valued function $f(x, t)$, where $x \in E \subset \mathbb{R}^n$ and $t \in I$ where I is an interval in the real line. Suppose
1. $f(x, \cdot)$ is integrable over I for all $x \in E$.
2. There exists an integrable function $g(t)$ on I such that $|f(x, t)| \leq g(t)$ for all $x \in E$ and $t \in I$.
3. For some $x_0 \in E$ the function $f(\cdot, t)$ is continuous at x_0 for all $t \in I$.
Then show that the function $F(x) = \int_I f(x, t) dt$ is continuous at x_0.

Problem 2.3.27. (a) Let $\{c_{n,i} : n \in \mathbb{N}, i \in \mathbb{N}\}$ be an array of nonnegative extended real numbers. Show that

$$\liminf_{n\to\infty} \sum_{i\in\mathbb{N}} c_{n,i} \geq \sum_{i\in\mathbb{N}} \liminf_{n\to\infty} c_{n,i}$$

(b) Show that if $\{c_{n,i}\}$ is an increasing sequence for each $i \in \mathbb{N}$, then

$$\lim_{n\to\infty} \sum_{i\in\mathbb{N}} c_{n,i} = \sum_{i\in\mathbb{N}} \lim_{n\to\infty} c_{n,i}$$

Problem 2.3.28. Let $f_n : [0, \infty) \to [0, \infty)$ be Lebesgue measurable functions such that $f_1 \geq f_2 \geq \cdots \geq 0$. Suppose $f_n(x) \to f(x)$ as $n \to \infty$, for a.e. $x \in [0, 1]$.
(a) Assume in addition that $f_1 \in L^1([0, 1])$. Prove that $\lim_{n\to\infty} \int_0^1 f_n dx = \int_0^1 f dx$.
(b) Show that this is not necessarily true if $f_1 \notin L^1([0, 1])$.

Problem 2.3.29. Let $\{q_k\}$ be all the rational numbers in $[0,1]$. Show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2\sqrt{|x-q_k|}} \text{converges a.e. in}[0,1]$$

Problem 2.3.30. Evaluate the following
a) $\lim_{n\to\infty} \int_0^\infty e^{-\frac{x^2+3x}{n^2}-2x} dx$
b) $\lim_{n\to\infty} \int_0^\pi x^2 \cos \frac{3x}{n} dx$

Problem 2.3.31. Find a sequence of pointwise convergent measurable functions $f_n : [0,1] \to \mathbb{R}$ such that f_n does not converge uniformly on any set X with $m(X) = 1$.

Problem 2.3.32. Does there exist a sequence of functions $f_n \in C([0,1])$ such that

$$\lim_{n\to+\infty} f_n(x) = 0$$

for all $x \in [0,1]$ and

$$\lim_{n\to+\infty} \max_{x\in[0,1]} |f_n(x)| \neq 0?$$

Either construct an example or prove such a sequence does not exist.

Problem 2.3.33. Suppose that $f \in L^1(\mathbb{R})$. Show that for any $\alpha > 0$

$$\lim_{n\to+\infty} n^{-\alpha} f(nx) = 0, \text{ for a.e } x$$

Problem 2.3.34. Suppose that $\{a_n\}$ is a sequence of nonnegative numbers and $\{E_n\}$ a sequence of measurable subsets of $[0,1]$. Assume that there exists $\delta > 0$ such that $m(E_n) \geq \delta$ and

$$g(x) = \sum_{n=1}^{\infty} a_n \chi_{E_n} < \infty \text{ for a.e } x \in [0,1].$$

Show that

$$\sum_{n=1}^{\infty} a_n < \infty$$

Problem 2.3.35. Let f be a nonnegative measurable function on $\mathbb{R}$. Prove that if $\sum_{n=-\infty}^{\infty} f(x+n)$ is integrable, then $f = 0$ a.e.

Problem 2.3.36. For each $n \geq 2$, define $f_n : [0,1] \to \mathbb{R}$ as follows:

$$f_n(x) = \begin{cases} n^2 & \text{for } x \in [\frac{i}{n}, \frac{i}{n} + \frac{1}{n^3}] \text{ and } i = 0, 1, \dots, n-1 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Show that $\lim_{n\to+\infty} f_n = 0$ for a.e $x \in [0,1]$
(b) For $g \in C[0,1]$, what is $\lim_{n\to+\infty} \int_{[0,1]} f_n(x)g(x)dx$?
(c) Is your answer to part (b) still valid for all $g \in L^\infty(0,1)$? Justify

Problem 2.3.37. Suppose that $f_n \in L^4(X, \mu)$ with $\|f_n\|_4 \leq 1$ and $f_n \to 0$ μ-a.e on X. Show that for every $g \in L^{4/3}(X, \mu)$,

$$\int_X f_n(x)g(x)d\mu(x) \to 0$$

as $n \to \infty$

Problem 2.3.38. Either prove or give a counterexample: If a sequence of functions f_n on a measure space (X, μ) satisfies $\int_X |f_n| d\mu \leq \frac{1}{n^2}$, then $f_n \to 0$ μ-a.e.

Problem 2.3.39. For a nonnegative function $f \in L^1([0, 1])$, prove that

$$\lim_{n\to\infty} \int_0^1 \sqrt[n]{f(x)}dx = m(\{x : f(x) > 0\})$$

Problem 2.3.40. Compute

$$\lim_{n\to\infty} \int_0^\infty \frac{n \sin(x/n)}{x(1 + x^2)} dx.$$

Justify each step.

Problem 2.3.41. Give an example of a sequence of functions, f_n in $L^1[0, 1]$ such that $\lim_{n\to\infty} ||f_n||_1 = 0$ but f_n does not converge to 0 almost everywhere.

Problem 2.3.42. Show that if a sequence, f_k in $L^1[0, 1]$ satisfies $||f_k||_1 \leq 2^{-k}$ for $k \geq 1$, then $f_k \to 0$ a.e.

Problem 2.3.43. Let $f \in C[0, 1]$ be real-valued. Prove that there is a monotone increasing sequence of polynomials $\{p_n(x)\}_{n=1}^\infty$ converging uniformly on $[0, 1]$ to $f(x)$.

Problem 2.3.44. Suppose $f : [0, 1] \to \mathbb{R}$ continuous
(a) Prove that $\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0$
(b) Prove that $\lim_{n\to\infty} n \int_0^1 x^n f(x)dx = f(1)$

Problem 2.3.45. Suppose $g(x) = \lim_{n\to\infty} g_n(x)$ for $x \in [0, 1]$ where $\{g_n\}$ are positive and continuous on $[0, 1]$. Also suppose that $\int_0^1 g_n dx = 1$.
(a) Is it always true that $\int_0^1 g(x)dx \leq 1$?
(b) Is it always true that $\int_0^1 g(x)dx \geq 1$?

Problem 2.3.46. Let $f(x, y)$ be defined on the unit square $S = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$ and suppose that f has the following properties:
(i) For each fixed x, the function $f(x, y)$ is an integrable function of y on the unit interval.
(ii) The partial derivative $\frac{\partial f}{\partial x}$ exists at every point $(x, y) \in S$, and is a bounded function on S.
Show that:
(a) The partial derivative $\frac{\partial f}{\partial x}$ is a measurable function of y for each x.
(b)

$$\frac{d}{dx} \int_0^1 f(x, y)dy = \int_0^1 \frac{\partial f}{\partial x}(x, y)dy$$

Problem 2.3.47. Fatou's Lemma can be written as: Let f_n be a sequence of nonnegative measurable functions. Then

$$\int \liminf f_n dx \leq \liminf \int f_n dx.$$

Show that this version is equivalent to: If f_n is a sequence of nonnegative measurable functions, and $\lim f_n = f$ a.e, then

$$\int f dx \leq \liminf \int f_n dx.$$

Problem 2.3.48. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on X that converges pointwise on X to f, and f integrable over X. Show that if $\int_X f = \lim_{n\to\infty} \int_X f_n$, then $\int_E f = \lim_{n\to\infty} \int_E f_n$, for any measurable $E \subseteq X$.

Problem 2.3.49. (a) Suppose f_n is a sequence of nonnegative measurable functions on E and that it converges pointwise to f on E, and that $f_n \leq f$ on E for all n. Show that $\int_E f_n \to \int_E f$.
(b) Let f be a nonnegative measurable function on $\mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_{-n}^{n} f = \int_{\mathbb{R}} f$$

Problem 2.3.50. Evaluate $\displaystyle\lim_{n\to\infty} \int_0^{1/n} \frac{n}{1 + n^2x^2 + n^6x^8} dx$

Problem 2.3.51. Suppose f and g are real-valued on $\mathbb{R}$, such that f is μ_L-integrable, and g belongs to $C_0(\mathbb{R})$ (compact support). For $c > 0$ define $g_c(t) = g(ct)$. Prove that
a)$\lim_{c\to\infty} \int_{\mathbb{R}} f g_c d\mu_L = 0$
b) $\lim_{c\to 0} \int_{\mathbb{R}} f g_c d\mu_L = g(0) \int_{\mathbb{R}} f d\mu_L$

Problem 2.3.52. Prove or give a counterexample: Suppose X is a finite measure space. Let $\{f_n\}$ be a sequence of measurable functions on X that converges in measure on X to real-valued function f. Then for each $\epsilon > 0$, there is a closed set F contained in X for which $f_n \to f$ uniformly on F and $m(X \setminus F) < \epsilon$.

Problem 2.3.53. Let E be of finite measure. Suppose the sequence of functions $\{f_n\}$ is uniformly integrable over E. If f_n converges to f in measure on E, then f is integrable over E and $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Problem 2.3.54. Let μ be the Lebesgue measure on $\mathbb{R}$. Let $f_n = -\frac{1}{n}\chi_{[0,n]}$. Prove that $f_n \to f = 0$ a.e., but

$$\liminf_{n\to\infty} \int_R f_n d\mu < \int_{\mathbb{R}} f d\mu$$

Does this contradict Fatou's Lemma?

2.4 Uniform Cont./ Bounded Var./ Abs. Cont.

Problem 2.4.1. a) Let f be absolutely continuous on $[\epsilon, 1]$ for each $\epsilon > 0$. Let f be continuous at 0 and of bounded variation on $[0, 1]$. Prove that f is absolutely continuous on $[0, 1]$.
(b) Give an example of continuous function of unbounded variation on $[2, 3]$.

Problem 2.4.2. Let $(\mathbb{R}, \mathcal{U}, \mu)$ be the Lebesgue measure space, and consider an extended real-valued (U)-measurable function f on $\mathbb{R}$. Let $B_r(x) = \{y \in \mathbb{R} : |y - x| < r\}$, with $r > 0$ fixed. Define the function g on $\mathbb{R}$ via

$$g(x) = \int_{B_r(x)} f(y)d\mu(y)$$

for $x \in \mathbb{R}$.
(a) Suppose f is locally μ-integrable on $\mathbb{R}$. Show that g is a real-valued, continuous function on $\mathbb{R}$.

(b) Show that if f is μ-integrable on $\mathbb{R}$, then g is uniformly continuous on $\mathbb{R}$.

Problem 2.4.3. Suppose that $f \in L^1[0, 1]$. Let $F(x) = \int_0^x f(t)dt$. Let ϕ be a Lipschitz function. Show that there exists $g \in L^1[0, 1]$ such that $\phi(F(x)) = \int_0^x g(t)dt$.

Problem 2.4.4. Let g be an absolutely continuous monotone function on $[0, 1]$. Prove that, if $E \subset [0, 1]$ is a set of Lebesgue measure zero, then the set $g(E)$ is also of Lebesgue measure zero.

Problem 2.4.5. Let $\{f_n\}$ be a sequence of real-valued functions and f be a real-valued function on $[a, b]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \in [a, b]$. Show that

$$V_a^b(f) \leq \liminf_{n\to\infty} V_a^b(f_n)$$

Problem 2.4.6. Let f be a real-valued, continuous function of bounded variation on $[a, b]$. Suppose f is absolutely continuous on $[a + \eta, b]$ for every $\eta \in (0, b - a)$. Show that f is absolutely continuous on $[a, b]$.

Problem 2.4.7. Let f be a real-valued function on (a, b) such that the derivative f' exists and $|f'(x)| \leq M$ for $x \in (a, b)$ for some $M \geq 0$. Let μ_L^* be the Lebesgue outer measure on $\mathbb{R}$. Show that for every subset E of (a, b) we have

$$\mu_L^*(f(E)) \leq M\mu_L^*(E)$$

Problem 2.4.8. Suppose μ is the Lebesgue measure on $\mathbb{R}$, E is a Lebesgue measurable subset of $\mathbb{R}$, and let

$$F = \left\{x \in E : \lim_{\epsilon\to 0} \frac{\mu(E \cap [x, x + \epsilon])}{\epsilon} = 1\right\}$$

Show that $\mu(E \setminus F) = 0$

Problem 2.4.9. Suppose $f \in L^1(-\infty, \infty)$, and let

$$g(x) = \int_{x-1}^{x+1} f(t)dt$$

(a) Prove that $\int_{-\infty}^{\infty} |g(t)|dt \leq 2\int_{-\infty}^{\infty} |f(t)|dt$
(b) Prove that g is absolutely continuous (that is, its restriction on any closed subinterval of $\mathbb{R}$ is absolutely continuous).

Problem 2.4.10. Prove that the discontinuity points of a real-valued function f are at most countable if
(a) f is a decreasing function on $[0,1]$.
(b) f is a function of bounded variation on $[0,1]$.

Problem 2.4.11. Let $C([0,1],\rho)$ be a space of continuous functions on $[0,1]$ with a uniform metric ρ. Let $F : C([0,1]) \to C([0,1])$ be given by

$$\text{(a) } (Ff)(t) = \int_0^t e^{tu} \sin(f(u))du$$

$$\text{(b) } (Ff)(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 f\left(\frac{tu}{n}\right) du$$

Prove that F is uniformly continuous.

Problem 2.4.12. (a) Find $Var_0^5(\cos(x))$.
(b) Give an example of a function f such that $Var_2^4(f) = 0$.
(c) Give an example of a function f such that $Var_2^4(f) = 2$.

Problem 2.4.13. Let f be absolutely continuous on $[0,1]$ and $P(x)$ a polynomial. Show that $P(f(x))$ is absolutely continuous on $[0,1]$.

Problem 2.4.14. Suppose that f is continuous and of bounded variation on $[0,1]$. Prove that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right) \right|^2 = 0$$

Problem 2.4.15. Suppose f is Lipschitz continuous in $[0,1]$, that is $|f(x) - f(y)| \leq L|x-y|$ for some constant L. Show that
(a) $m(f(E)) = 0$ if $m(E) = 0$
(b) If E is measurable, then $f(E)$ is also measurable.

Problem 2.4.16. Assume that f is absolutely continuous, and that f and f' are both in $L^1(\mathbb{R})$. Let

$$g(x) = \sum_{k=0}^{\infty} |f(x+k)|.$$

Show that $g \in L^\infty(I)$ for any bounded interval I. (Hint: first prove for intervals with length $\leq 1/2$)

2.5 L^p/L^∞ Spaces

Problem 2.5.1. Let X be a measure space and let μ be a positive Borel measure over X, with $\mu(X) < \infty$. Let $f \in L^\infty(X, \mu)$. For each positive integer n, we let

$$\alpha_n = \int_X |f(x)|^n d\mu(x)$$

(a) Show that $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty$ if $\|f\|_\infty > 0$.
(b) Use part (a) to prove that $\lim_{n\to\infty} (\alpha_n)^{1/n} = \|f\|_\infty$.

Problem 2.5.2. Let $(X, \mathfrak{U}, \mu)$ be a measure space. Let $f \in L^p(X, \mathfrak{U}, \mu) \cap L^q(X, \mathfrak{U}, \mu)$ with $1 \leq p < q < \infty$. Show that $f \in L^r(X, \mathfrak{U}, \mu)$ for all $p \leq r \leq q$.

Problem 2.5.3. Let $f \in L^p([0, 10])$, $p \geq 1$. (a) Prove that $\lim_{t\to 1^+} (t-1)^{\frac{1}{p}-1} \int_1^t f(s)ds = 0$
(b) Suppose $\int_0^\infty x^{-2}|f|^5 dx < \infty$. Prove that $\lim_{t\downarrow 0} t^{-6/5} \int_0^t f(x)dx = 0$.

Problem 2.5.4. Suppose that $\int_0^1 x^{-1}|f|^3 dx < \infty$. Show that $\lim_{t\to 0} t^{-1} \int_0^t f(x)dx = 0$

Problem 2.5.5. Let $1 \leq p < q < \infty$. Which of the following statements are true and which are false? Justify.
1. $L^p(\mathbb{R}, \mathfrak{M}_L, \mu_L) \subset L^q(\mathbb{R}, \mathfrak{M}_L, \mu_L)$
2. $L^q(\mathbb{R}, \mathfrak{M}_L, \mu_L) \subset L^p(\mathbb{R}, \mathfrak{M}_L, \mu_L)$
3. $L^p([2, 5], \mathfrak{M}_L, \mu_L) \subset L^q([2, 5], \mathfrak{M}_L, \mu_L)$
4. $L^q([2, 5], \mathfrak{M}_L, \mu_L) \subset L^p([2, 5], \mathfrak{M}_L, \mu_L)$

Problem 2.5.6. Let $f \in L^{\frac{3}{2}}([0, 5], \mathfrak{M}_L, \mu_L)$. Prove that

$$\lim_{t\downarrow 0} \frac{1}{t^{\frac{1}{3}}} \int_0^t f(s)ds = 0$$

Problem 2.5.7. Let $p, q \in [1, \infty)$.
(a) Show that $L^p(0, \infty)$ is not a subset of $L^q(0, \infty)$ when $p \neq q$.
(b) If $f \in L^p(\mathbb{R})$ and f is uniformly continuous on $\mathbb{R}$, show that $\lim_{|x|\to\infty} f(x) = 0$. Is the result still true if f is only assumed to be continuous?

Problem 2.5.8. Let $g(x)$ be measurable and suppose $\int_a^b f(x)g(x)dx$ is finite for any $f(x) \in L^2$. Prove that $g(x) \in L^2$. (Note from Timmy: This is kind of an unfair question in that you have to use the uniform boundedness property from Functional Analysis, what they probably forgot to say was that instead of finite, that the integral of fg is uniformly bounded.)

Problem 2.5.9. If $f(x) \in L^p \cap L^\infty$ for some $p < \infty$, show that
(a) $f(x) \in L^q$ for all $q > p$.
(b) $\|f\|_\infty = \lim_{q\to\infty} \|f\|_{L^q}$

Problem 2.5.10. If $f_n(x), f(x) \in L^2$, and $f_n(x) \to f(x)$ a.e., then $\|f_n - f\|_{L^2} \to 0$ if and only if $\|f_n\|_{L^2} \to \|f\|_{L^2}$.
Remark: You can actually prove the more general case, replacing 2 with p.

Problem 2.5.11. Let f be a Lebesgue measurable function on $[0,1]$ and let $0 < f(x) < \infty$ for $x \in [0,1]$. Show that

$$\left\{\int_{[0,1]} f d\mu\right\}\left\{\int_{[0,1]} \frac{1}{f}\right\} \geq 1.$$

Justify each step.

Problem 2.5.12. Consider $L^p([0,1])$. Prove that $\|f\|_p$ is increasing in p for any bounded measurable function f. Prove that $\|f\|_p \to \|f\|_\infty$ when $p \to \infty$.

Problem 2.5.13. Suppose $1 \leq p \leq \infty$, $f \in L^1(\mathbb{R}, dx)$ and $g \in L^p(\mathbb{R}, dx)$. Define the function $f * g$ by $(f * g)(x) = \int f(x-y)g(y)dy$, whenever it makes sense. Show that $f * g$ is defined almost everywhere, and that $f * g \in L^p(\mathbb{R}, dx)$ with

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p$$

Problem 2.5.14. Let $f_n \to f$ in L^p, with $1 \leq p < \infty$, and let g_n be a sequence of measurable functions such that $|g_n| \leq M < \infty$ for all n and $g_n \to g$ a.e.. Prove that $g_n f_n \to gf$ in L^p.

Problem 2.5.15. Let E be a measurable subset of the real line. Prove that $L^\infty(E)$ is complete.

Problem 2.5.16. Let $(X, \mathfrak{A}, \mu)$ be a measure space and let f be an extended real-valued $\mathfrak{A}$-measurable function on X such that $\int_X |f|^p d\mu < \infty$ for some $p \in (0, \infty)$. Show that

$$\lim_{\lambda \to \infty} \lambda^p \mu \{x \in X : |f(x)| \geq \lambda\} = 0$$

Problem 2.5.17. Show that $L^3([-1,\infty)) \supset L^2([-1,\infty)) \cap L^5([-1,\infty))$

Problem 2.5.18. Suppose that $f \in L^p([0,1])$ for some $p > 2$. Prove that $g(x) = f(x^2) \in L^1([0,1])$.

Problem 2.5.19. Suppose f is a Borel measurable function and $\int_0^1 x^{-1/2}|f(x)|^3 dx < \infty$. Show that $\lim_{t \to 0} t^{-5/6} \int_0^t f(x)dx = 0$

Problem 2.5.20. If (X, μ) is a finite measure space, $1 < p_1 < p_2 < \infty$, $f \geq 0$, and

$$\sup_{\lambda > 0} \lambda^{p_2} \mu(x : f(x) > \lambda) < \infty$$

prove that $f \in L^{p_1}$

Problem 2.5.21. Let $0 < r < p < q \leq \infty$. For $f \geq 0$ in $L^p(\mathbb{R})$, show that $f = g + h$ where $g \in L^r(\mathbb{R})$ and $h \in L^q(\mathbb{R})$. Furthermore, show that given $b > 0$, prove that g, h can be chosen so that $\|g\|_r^r \leq b^{r-p}\|f\|_p^p$ and $\|h\|_q^q \leq b^{q-p}\|f\|_p^p$.

Problem 2.5.22. Let A be the collection of functions $f \in L^1(X,\mu)$ such that $||f||_1 = 1$ and $\int_X f d\mu = 0$. Prove that for every $g \in L^\infty(X,\mu)$,

$$\sup_{f\in A} \int_X fgd\mu = \frac{1}{2}(ess\sup g - ess\inf g)$$

Problem 2.5.23. Let $f(x)$ be continuous function on $[0,1]$ with a continuous derivative $f'(x)$. Given $\epsilon > 0$, prove that there is a polynomial $p(x)$ so that

$$||f(x) - p(x)||_\infty + ||f'(x) - p'(x)||_\infty < \epsilon$$

Problem 2.5.24. Let X be a σ-finite measure space, and $f_n : X \to \mathbb{R}$ a sequence of measurable function on it. Suppose $f_n \to 0$ in L^2 and L^4. Give a proof or counterexample to the following:
(a) $f_n \to 0$ in L^1
(b) $f_n \to 0$ in L^3
(c) $f_n \to 0$ in L^5

Problem 2.5.25. Let f be a nonnegative measurable function and $ess\sup f = M > 0$. Then, if $\mu(X) < \infty$, prove

$$\lim_{n\to\infty} \frac{\int f^{n+1}}{\int f^n} = M$$

Problem 2.5.26. Let $k + m = km$ and let f and g be nonnegative measurable functions. Show that if $0 < k < 1$ or $k < 0$ then

$$\int fg \geq (\int f^k d\mu)^{1/k} (\int g^m d\mu)^{1/m}$$

Problem 2.5.27. Let $0 < p < 1$ and $f \geq 0, g \geq 0, f, g \in L^p$. Show that

$$||f + g||_p \geq ||f||_p + ||g||_p$$

Problem 2.5.28. Suppose $1 \leq p \leq \infty$, $f \in L^p([0,1])$, and $h(t)$ is the Lebesgue measure of the set $\{x \in [0,1] : |f(x)| > t\}$ for $0 \leq t < \infty$. Show that $\int_0^\infty h(t)dt < \infty$ if $1 < p \leq \infty$. Is this still true for $p = 1$? Prove or find a counterexample.

2.6 Radon-Nikodym Theorem

Problem 2.6.1. Let ν be a finite Borel measure on the real line, and set $F(x) = \nu\{(-\infty, x]\}$. Prove that ν is absolutely continuous with respect to the Lebesgue measure μ if, and only if, F is an absolutely continuous function. In this case show that its Radon-Nikodym derivative is the derivative of F, that is: $\frac{d\nu}{d\mu} = F'$ almost everywhere.

Problem 2.6.2. Let (X, Σ, μ) be a σ-finite measure space, and let Σ_0 be a sub-σ-algebra of Σ. Given an integrable function f on (X, Σ, μ), show that there is a Σ_0-measurable function f_0, such that

$$\int_X fgd\mu = \int_X f_0 gd\mu$$

for every Σ_0-measurable function g such that fg is integrable. (Hint: Use the Radon-Nikodym Theorem).

Problem 2.6.3. Let f, g be extended real-valued, measurable functions over a σ-finite measure space $(X, \mathfrak{A}, \mu)$. Show that if $\int_E f d\mu = \int_E g d\mu$ for all $E \in \mathfrak{A}$, then $f = g$ on X μ-a.e.

Problem 2.6.4. Let $\mathcal{M}$ denote the Lebesgue measurable sets on the real line. Consider two measures on $\mathcal{M}$: Lebesgue measure m and the counting measure τ. Show that m is absolutely continuous with respect to τ, but that $\frac{dm}{d\tau}$ does not exist. Does this contradict the Radon-Nikodym theorem?

Problem 2.6.5. Prove the following statement. Suppose that F is a sub-σ-algebra of the Borel σ-algebra on the real line. If $f(x)$ and $g(x)$ are F-measurable and if

$$\int_A f dx = \int_A g dx \text{ for all } A \in F$$

then $f(x) = g(x)$ a.e.
Note from Will: This is precisely how the above problem appeared on the Winter 2000 real analysis qual. Note that the statement is in general false, as it appears above. One needs to add the assumption that F is σ-finite. So prove the following:
(a) σ-finiteness of F does NOT follow from it being a sub-σ-algebra of a σ-finite σ-algebra.
(b) Show that if σ-finiteness is not required, then the statement of the problem is in general false.
(c) Prove the statement of the problem with the additional assumption that F is σ-finite.

Problem 2.6.6. Let μ and ν be two measures on the same measurable space, such that μ is σ-finite and ν is absolutely continuous with respect to μ.
(a) If f is a nonnegative measurable function, show that

$$\int f d\nu = \int f \left[\frac{d\nu}{d\mu}\right] d\mu$$

(b) If f is a measurable function, prove that f is integrable with respect to ν if and only if $f\left[\frac{d\nu}{d\mu}\right]$ is integrable with respect to μ, and that in this case we also have equality as above.

Problem 2.6.7. Let $\lambda \ll \mu$ and $\mu \ll \lambda$ be σ-finite measures on $(X, \mathcal{X})$. What is the relationship between the Radon-Nikodym derivatives $\frac{d\mu}{d\lambda}$ and $\frac{d\lambda}{d\mu}$?

Problem 2.6.8. Assume that ν and μ are two finite measures on a measurable space $(X, \mathcal{M})$. Prove that

$$\nu \ll \mu \iff \lim_{n\to\infty} (\nu - n\mu)^+ = 0$$

Problem 2.6.9. Suppose that we have $\nu_1 \ll \mu_1$ and $\nu_2 \ll \mu_2$ for positive measures ν_i and μ_i on measurable spaces $(X_i, \mathcal{M}_i)$, $(i = 1, 2)$. Show that we have $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x, y) = \frac{d\nu_1}{d\mu_1}(x)\frac{d\nu_2}{d\mu_2}(y)$$

Problem 2.6.10. Suppose μ and ν are σ-finite measures on a measurable space $(X, \mathcal{A})$, such that $\nu \leq \mu$ and $\nu \ll \mu - \nu$. Prove that

$$\mu(\{x \in X : \frac{d\nu}{d\mu} = 1\}) = 0$$

2.7 Tonelli/Fubini Theorem

Problem 2.7.1. Let $f(x)$ be a locally integrable function on $(0, \infty)$. We define

$$F(x) = \frac{1}{x} \int_0^x f(t)dt$$

for $x > 0$. Prove that if $f(x) > 0$ on $(0, \infty)$ and $\int_0^\infty f(x)dx < \infty$, then $\int_0^\infty F(x)dx = \infty$.

Problem 2.7.2. Let $(X, \mathfrak{U}, \mu)$ and $(X, \mathfrak{V}, \nu)$ be the measure spaces given by

$$X = Y = [0, 1]$$

$$\mathfrak{U} = \mathfrak{V} = \mathfrak{B}_{[0,1]}$$

where $\mathfrak{B}_{[0,1]}$ is the Borel σ-algebra, and μ is the Lebesgue measure, ν is the counting measure. Consider the product measurable space $(X \times Y, \sigma(\mathfrak{U} \times \mathfrak{V}))$ and a subset in it defined by $E = \{(x, y) \in X \times Y : x = y\}$.

(a) Show that $E \in \sigma(\mathfrak{U} \times \mathfrak{V})$.
(b) Show that

$$\int_X \left[\int_Y \chi_E d\nu \right] d\mu \neq \int_Y \left[\int_X \chi_E d\mu \right] d\nu$$

(c) Why is Tonelli's Theorem not applicable?

Problem 2.7.3. Let $f_n(x) = \frac{n}{2}\chi_{[-\frac{1}{n}, \frac{1}{n}]}$. Prove that for $g \in L^1(\mathbb{R})$,

$$\int \left| \int f_n(y - x)g(x)dx - g(y) \right| dy \to 0 \text{ as } n \to \infty$$

Problem 2.7.4. Let f be a real valued measurable function on the finite measure space (X, Σ, μ). Prove that the function $F(x, y) = f(x) - 5f(y) + 4$ is measurable in the product measure space $(X \times X, \sigma(\Sigma \times \Sigma), \mu \times \mu)$, and that F is integrable if and only if f is integrable.

Problem 2.7.5. Let $f \in L^1(\mathbb{R}, \mathfrak{M}_L, \mu_L)$. With $h > 0$ fixed, define a function ϕ_h on $\mathbb{R}$ by setting

$$\phi_h(x) = \frac{1}{2h} \int_{[x-h, x+h]} f(t)d\mu_L(t) \text{ for } x \in \mathbb{R}$$

(a) Show that ϕ_h is $\mathfrak{M}_L$-measurable on $\mathbb{R}$.
(b) Show that ϕ_h is continuous on $\mathbb{R}$ (hint from Will: actually continuity will imply Borel measurability, hence measurability).
(c) Show that $\phi_h \in L^1(\mathbb{R}, \mathfrak{M}_L, \mu_L)$ and $\|\phi_h\|_1 \leq \|f\|_1$

Problem 2.7.6. Let $(X, \mathcal{M}, \mu)$ be a complete measure space and let f be a nonnegative integrable function on X. Let $b(t) = \mu\{x \in X : f(x) \geq t\}$. Show that

$$\int f d\mu = \int_0^\infty b(t)dt$$

Problem 2.7.7. If g is a Lebesgue measurable real function on $[0,1]$ such that the function $f(x,y) = 2g(x) - 3g(y)$ is Lebesgue integrable over the square $[0,1]^2$, show that g is Lebesgue integrable over $[0,1]$.

Problem 2.7.8. Let $A = \{(x,y) \in [0,1]^2 : x + y \notin \mathbb{Q}, xy \notin \mathbb{Q}\}$. Find $\int_A y^{-1/2} \sin x dm_2$.

Problem 2.7.9. Let f be Lebesgue integrable on $(0,1)$ For $0 < x < 1$ define

$$g(x) = \int_x^1 t^{-1} f(t)dt.$$

Prove that g is Lebesgue integrable on $(0,1)$ and that

$$\int_0^1 g(x)dx = \int_0^1 f(x)dx$$

Problem 2.7.10. Evaluate

$$\int_0^\infty \int_0^\infty x \exp(-x^2(1+y^2))dxdy$$

Justify each step.