FINITE REPRESENTABILITY OF HOMOGENEOUS HILBERTIAN OPERATOR SPACES IN SPACES WITH FEW COMPLETELY BOUNDED MAPS

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Abstract. For every homogeneous Hilbertian operator space $H$, we construct a Hilbertian operator space $X$ such that every infinite dimensional subquotient $Y$ of $X$ is completely indecomposable, and fails the Operator Approximation Property, yet $H$ is completely finitely representable in $Y$. If $H$ satisfies certain conditions, we also prove that every completely bounded map on such $Y$ is a compact perturbation of a scalar.

1. Introduction and the main result

In [GM], T. Gowers and B. Maurey gave the first example of a hereditarily indecomposable Banach space $Z$ (recall that an infinite dimensional space $Z$ is called hereditarily indecomposable if it is not isomorphic to a direct sum of two infinite dimensional Banach spaces). Since then, a variety of hereditarily indecomposable Banach spaces were constructed. An overview of the current state of affairs is given in [M].

A non-commutative counterpart of this space was obtained by E. Ricard and the author in [OR]. There, we gave an example of an operator space $X$, isometric to $\ell_2$ (as a Banach space), such that an operator $T : Y \to X$ ($Y$ being a subspace of $X$) is completely bounded if and only if $T = \lambda J_Y + S$, where $J_Y$ is the natural embedding, $\lambda \in \mathbb{C}$, and $S$ is a Hilbert-Schmidt map. In particular, $X$ is completely hereditarily indecomposable – that is, no infinite dimensional subspace $Y \hookrightarrow X$ is completely isomorphic to an $\ell_\infty$ sum of two infinite dimensional operator spaces. Moreover, $X$ fails the Operator Approximation Property (see below for the definition). For any $n$-dimensional subspace $Y \hookrightarrow X$, there exists a unitary $U : Y \to Y$ s.t. $\|U\|_{cb} \geq \sqrt{n}/16$.

Our present goal is to construct completely hereditarily indecomposable operator spaces with “some structure” – that is, spaces which are saturated with “nice” finite dimensional subspaces. More precisely, for any homogeneous Hilbertian operator space $H$, we construct a Hilbertian operator space $X$ such that:

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• For any infinite dimensional subspace \( Y \) of a quotient of \( X \), \( n \in \mathbb{N} \), and \( \varepsilon > 0 \), there exists a subspace \( F \hookrightarrow Y \) which is \((1+\varepsilon)\)-completely isomorphic to an \( n \)-dimensional subspace of \( H \).

• Any \( Y \) as above is completely hereditarily indecomposable, and fails the Operator Approximation Property.

If \( H \) satisfies certain conditions, then, in addition, any c.b. map on \( Y \) is a compact perturbation of a scalar.

Below we recall some facts and definitions concerning operator spaces. For more information, the reader is referred to [ER], [Pa], or [Pi].

We say that an operator space is \( c \)-Hilbertian if its underlying Banach space is \( c \)-isomorphic to a Hilbert space. \( X \) is \( c \)-homogeneous if \( \|T\|_{cb} \leq c \|T\| \) for any \( T \in B(X) \). An infinite dimensional operator space \( X \) is called completely indecomposable if it is not completely isomorphic to an \( \ell_\infty \) direct sum of two infinite dimensional operator spaces (equivalently, any c.b. projection on \( X \) has finite dimensional kernel, or finite dimensional range).

We use the term subquotient to mean a subspace of a quotient.

An operator space \( X \) is said to have the Operator Approximation Property (OAP, for short) if, for any \( x \in K \otimes X \) and \( \varepsilon > 0 \), there exists a finite rank map \( T : X \rightarrow X \) s.t. \( \|(I_K \otimes T)x - x\| < \varepsilon \) (here \( K \) is the space of compact operators on \( \ell_2 \), and \( \otimes \) denotes the minimal (injective) tensor product). \( X \) has the Compact Operator Approximation Property (COAP) if, for any \( x \in K \otimes X \) and \( \varepsilon > 0 \), there exists a compact map \( T : X \rightarrow X \) s.t. \( \|(I_K \otimes T)x - x\| < \varepsilon \). More details about the OAP, as well as several equivalent reformulations of this property, can be found in Chapter 11 of [ER].

The complete Banach-Mazur distance between the operator spaces \( X \) and \( Y \) is defined as

\[
d_{cb}(X, Y) = \inf \{ \|T\|_{cb} \|T^{-1}\|_{cb} \mid T \in CB(X, Y) \}.
\]

We say that an operator space \( Y \) is \( c \)-completely finitely representable in \( X \) if for any finite dimensional subspace \( Z \hookrightarrow Y \) there exists \( W \hookrightarrow X \) s.t. \( d_{cb}(W, Z) \leq c \). \( Y \) is called \( c \)-completely complementably finitely representable in \( X \) if for any finite dimensional subspace \( Z \hookrightarrow Y \) there exists a projection \( P \in CB(X) \) s.t. \( \|P\|_{cb} \leq c \), and \( d_{cb}(P(X), Z) \leq c \).

If \( H \) is a 1-homogeneous 1-Hilbertian operator space, we denote by \( H_n \) the \( n \)-dimensional operator space, completely isometric to (any) \( n \)-dimensional subspace of \( H \). We say that \( H \) has property \( (P) \) if there exists a sequence \( (m(n)) \subset \mathbb{N} \) s.t.

\[
\lim_{n \to \infty} \frac{1}{n} \|id : \text{MIN}_{m(n)}(R_n + C_n) \rightarrow H_n\|_{cb} = 0.
\]
Here, \( id \) is the formal identity map between \( n \)-dimensional Hilbert spaces, and the space \( \text{MIN}_k(X) \) (\( X \) being an operator space) is such that

\[
\|x\|_{K \otimes \text{MIN}_k(X)} = \sup\{\|I_K \otimes u(x)\|_{K \otimes M_k} \mid u \in CB(X, M_k), \|u\|_{cb} \leq 1\},
\]

where, as usual, \( M_k \) stands for the pace of \( k \times k \) matrices. The reader is referred to [OR] for more information about \( \text{MIN}_k \). For future reference, we need to consider a special case of the functor \( \text{MIN}_k \) – namely, \( \text{MIN}_1 \) (denoted by \( \text{MIN} \) for the sake of brevity). If \( X \) is a Banach or operator space, and \( x \in K \otimes X \), then

\[
\|x\|_{K \otimes \text{MIN}(X)} = \sup\{\|I_K \otimes f(x)\|_K \mid f \in X^*, \|f\|_{cb} \leq 1\}.
\]

In other words, if \( a_1, \ldots, a_n \in K \), and \( x_1, \ldots, x_n \in X \), then

\[
\|\sum a_i \otimes x_i\|_{K \otimes \text{MIN}(X)} = \sup\{\|\sum f(x_i)a_i\|_K \mid f \in X^*, \|f\|_{cb} \leq 1\}.
\]

Note that, for any 1-homogeneous 1-Hilbertian space \( H \), \( \|id : \text{MIN}(\ell_2^n) \to H_n\|_{cb} \geq \|id : \text{MIN}_{m,n}(R_n + C_n) \to H_n\|_{cb} \), hence \( H \) has property (P) whenever \( \lim \sup_n \|id : \text{MIN}(\ell_2^n) \to H_n\|_{cb} / n = 0 \). In particular (by Chapter 10 of [Pi]), the spaces \( OH, R + C \), and \( R \cap C \) have (P). To describe another large class of spaces possessing (P), recall that an operator space \( X \) is exact if there exists \( C > 0 \) such that for any finite dimensional subspace \( E \hookrightarrow X \) there exists \( F \hookrightarrow M_N \) s.t. \( d_{cb}(E, F) \leq C \). The infimum of all such constants \( C \) is called the exactness constant of \( X \), and denoted by \( \text{ex}(X) \). Observe that \( H \) has property (P) if \( \lim_{n \to \infty} \text{ex}(H_n) / \sqrt{n} = 0 \). Indeed, by Smith’s Lemma (Proposition 8.11 of [Pa]), there exists a sequence of positive integers \( r(1) < r(2) < \ldots \) s.t., for every operator space \( X \), and every \( v \in CB(X, H_n) \),

\[
\|v : X \to H_n\|_{cb} \leq 2\text{ex}(H_n)\|I_{M_{r(n)}} \otimes v : M_{r(n)} \otimes X \to M_{r(n)} \otimes H_n\|
\]

(we could have used \( 1 + \varepsilon \) instead of 2). Then, by [OR],

\[
(2\text{ex}(H_n))^{-1}\|id : \text{MIN}_{r(n)}(R_n + C_n) \to H_n\|_{cb} \leq \|I_{M_{r(n)}} \otimes id : M_{r(n)} \otimes \text{MIN}_{r(n)}(R_n + C_n) \to M_{r(n)} \otimes H_n\| = \|I_{M_{r(n)}} \otimes id : M_{r(n)} \otimes (R_n + C_n) \to M_{r(n)} \otimes H_n\| \leq \|id : R_n + C_n \to H_n\|_{cb}.
\]

However, by Theorem 10.6 of [Pi],

\[
\|id : R_n + C_n \to H_n\|_{cb} \leq \|id : R_n + C_n \to \text{MAX}(\ell_2^n)\|_{cb} = \sqrt{n}.
\]

This establishes property (P).

The main result of this paper is

**Theorem 1.1.** Suppose \( H \) is a separable 1-homogeneous 1-Hilbertian operator space. Then there exists a separable 1-Hilbertian operator space \( X \) such that for every infinite dimensional subspace \( Y \) of \( X \) we have:

1. For any \( \varepsilon > 0 \), \( H \) is \( (1 + \varepsilon) \)-completely complementably finitely representable in \( Y \).
2. \( Y \) is completely indecomposable.
(3) $Y$ fails the Compact Operator Approximation Property.

(4) If $H$ has property $(P)$, then every completely bounded map on $Y$ is a compact perturbation of a scalar.

Clearly, the COAP implies the OAP. By Chapter 11 of [ER], the OAP passes from an operator space to its predual. Therefore, dualizing the space $X$ constructed in Theorem 1.1, we conclude:

**Corollary 1.2.** Suppose $H$ is a separable 1-homogeneous 1-Hilbertian operator space, whose dual $H^*$ has property $(P)$. Then there exists a separable 1-Hilbertian operator space $X$ such that for every infinite dimensional subquotient $Y$ of $X$ we have:

1. For any $\varepsilon > 0$, $H$ is $(1 + \varepsilon)$-completely complementably finitely representable in $Y$.
2. $Y$ is completely indecomposable.
3. $Y$ fails the Operator Approximation Property.
4. Every completely bounded map on $Y$ is a compact perturbation of a scalar.

In Section 2, we present a modification of the construction of asymptotic sets on the unit sphere of $\ell_2$ (initially due to E. Odell and T. Schlumprecht [OS1]). In Section 3, we use these asymptotic sets to construct the space $X$ from Theorem 1.1. Furthermore, we establish that all infinite dimensional subquotients of $X$ are completely indecomposable, and $H$ is completely complementably finitely representable in all such subquotients. In Section 4 we prove that all infinite-dimensional subquotients of $X$ fail the OAP. Finally, in Section 5 we show that any c.b. map on an infinite dimensional subquotient of $X$ is a compact perturbation of a scalar multiple of the identity, provided $H$ has property $(P)$.

2. **Asymptotic sets in $\ell_2$**

First we recall some Banach space notions, to be used in this and subsequent sections. All spaces are presumed to be infinite dimensional, unless stated otherwise. For a space $X$, $B_X = \{ x \in X \mid \| x \| \leq 1 \}$ and $S_X = \{ x \in X \mid \| x \| = 1 \}$ stand for the unit ball and the unit sphere of $X$, respectively.

We say that a sequence $(\delta_i)_{i=1}^\infty$ is a basis in a Banach space $X$ if for every $x \in X$ there exists a unique sequence of scalars $(a_i)$ s.t. $x = \sum_{i=1}^\infty a_i \delta_i$. Equivalently (see e.g. Proposition 1.a.3 of [LT]), the projections $P_n \in B(X)$, defined via $P_n(\sum_{i=1}^\infty a_i \delta_i) = \sum_{i=1}^n a_i \delta_i$, are well defined, and $\sup_n \| P_n \| < \infty$. If $E$ is a finite subset of $\mathbb{N}$, we write $E(\sum_{i=1}^\infty a_i \delta_i) = \sum_{i \in E} a_i \delta_i$. The support of $a = \sum_{i=1}^\infty a_i \delta_i$ (denoted by $\text{supp} a$) is the set of $i \in \mathbb{N}$ for which $a_i \neq 0$.

If $E$ and $F$ are finite subsets of $\mathbb{N}$, we write $E < F$ if $\max E < \min F$. If a Banach space $X$ has a basis $(\delta_i)_{i \in \mathbb{N}}$, we write $a < b$ ($a, b \in X$) if $\text{supp} a < \text{supp} b$. 
The basis \((\delta_i)_{i=1}^\infty\) is called 1-subsymmetric if \(\|\sum_i a_i \delta_i\| = \|\sum_i \omega_i a_i \delta_i\|\) for any finite sequence \((a_i)\), any \((\omega_i)\) with \(|\omega_i| = 1\), and any increasing sequence \(n_1 < n_2 < \ldots\) (sometimes, the term “1-unconditional 1-subsymmetric” is used to describe bases with this property).

For \(S_1, S_2 \subset X\), we set \(\text{dist}(S_1, S_2) = \inf\{\|x_1 - x_2\| \mid x_1 \in S_1, x_2 \in S_2\}\).

A set \(A \subset X\) is called asymptotic if, for every infinite dimensional \(Y \subset X\), \(\text{dist}(A, Y) = 0\). If \((\delta_i)_{i\in\mathbb{N}}\) is a 1-subsymmetric basis for \(X\), we say that \(A \subset X\) is spreading (unconditional) if, for any \(\sum_{i=1}^{\infty} a_i \delta_i \in A\), we have \(\sum_{i=1}^{\infty} a_i \delta_i \in A\) for any \(n_1 < n_2 < \ldots\) (resp. \(\sum_{i=1}^{\infty} \omega_i a_i \delta_i \in A\) for any \(|\omega_i| = 1\)).

The idea of constructing a sequence of asymptotic sets, satisfying certain conditions, was used by E. Odell and T. Schlumprecht in [OS1] in order to prove that \(\ell_p\) is distortable for \(1 < p < \infty\). Below we prove a sharper version of one of their results.

**Theorem 2.1.** Suppose \(\varepsilon_1 > \varepsilon_2 > \ldots\) is a sequence of positive numbers, and \((K_i)_{i=1}^{\infty}\) is a sequence of positive integers. Then there exists a sequence of asymptotic spreading unconditional sets \(A_1, A_2, \ldots\), consisting of unit vectors in \(\ell_2\) with finite support, such that

\[
\sum_{k=1}^{K_n} |\langle a, b_k \rangle|^2 < \varepsilon_m^2
\]

whenever \(m < n\), \(a \in A_m\), \(b_1, \ldots, b_{K_n} \in A_n\), and \(b_1 < \ldots < b_{K_n}\).

The Schlumprecht space \(S\) is essential for proving this theorem. Recall (see [GM, OS1, OS2, S]) that \(S\) has a 1-subsymmetric basis \((\delta_i)_{i=1}^{\infty}\), and

\[
\|\sum_i a_i \delta_i\| = \sup\{\sup_i |a_i|, \sup_{n\geq 2, E_1 < \ldots < E_n} \frac{1}{\phi(n)} \sum_{j=1}^{n} \|\sum_i a_i \delta_i\|\}
\]

(here \(\phi(t) = \log(t + 1)\)). Using the ideas of [OS1], we first present “nice” sets in \(S\) and its dual.

**Lemma 2.2.** Suppose \(\sigma_1 > \sigma_2 > \ldots\) is a sequence of positive numbers, and \((K_i)_{i=1}^\infty\) is a sequence of positive integers. Then there exist spreading unconditional sets \(B_1, B_2, \ldots \subset S\) and \(B'_1, B'_2, \ldots \subset B_{S'}\), consisting of vectors with finite support, such that:

1. \(B_n\) is asymptotic for every \(n\).
2. \(|\langle a, Eb \rangle| < \sigma_{\min\{m,n\}}\) if \(a \in B_n\), \(b \in B_m^\star\), and \(E \subset \mathbb{N}\).
3. For every \(a \in B_m\) there exists \(b \in B_m^\star\) satisfying \(|\langle a, b \rangle| > 1 - \sigma_m\).
4. Suppose \(m < n\), \(a \in B_m\), \(b_1, \ldots, b_{K_n} \in B_n^\star\), \(b_1 < \ldots < b_{K_n}\), and \(E_1 < \ldots < E_{K_n}\). Then \(\sum_{k=1}^{K_n} |\langle a, E_k b_k \rangle| < 2\sigma_m\).

**Sketch of the proof.** We rely on the construction from Section 2 of [GM] (summarized in [OS1] as Lemma 3.3). There, T. Gowers and B. Maurey show the existence of
a rapidly increasing sequence $p_k \nearrow \infty$, and a rapidly decreasing sequence $\sigma'_k \searrow 0$, with the following property: for $n \in \mathbb{N}$, define
\[
B^*_n = \left\{ \frac{1}{\phi(p_n)} \sum_{j=1}^{p_n} b_j \left| b_j \in S^*, \|b_j\| = 1, b_1 < \ldots < b_{p_n} \right\} \subseteq B_{S^*},
\]
and let $B_n$ be the set of all $(\sum_{i=1}^{p_n} x_i)/\|\sum_{i=1}^{p_n} x_i\| \in S_S$, with constant $1 + \sigma'_n$ (we do not reproduce the definition of RIS, as it is quite cumbersome, and is not really necessary here; suffices to say that above, $x_1 < x_2 < \ldots < x_{p_n}$). Then the sets $B_n$ and $B^*_n$ are unconditional and spreading, and the statements (1), (2), and (3) of the lemma hold. It remains to prove (4).

By passing to a subsequence, we can assume that $\phi(K_n p_n) < 2\phi(p_n)$ for every $n$ (recall that $\phi(t) = \log(t + 1)$). Suppose $m$, $n$, $a$, and $(b_k)_{k=1}^{K_n}$ are as in (4). The sets $B_m$ and $B^*_m$ are unconditional, hence it suffices to prove (4) when all the entries of $a$ and $(b_k)$ are non-negative, and $E_k = \text{supp } b_k$ for each $k$. In this situation, we have to show that $\langle a, \sum_{k=1}^{K_n} b_k \rangle < 2\sigma_m$. By construction,
\[
b_k = \frac{1}{\phi(p_n)} \sum_{j=1}^{p_n} b_{jk},
\]
where $b_{jk} \in B_{S^*}$ ($1 \leq j \leq p_n$) are such that $b_{1k} < \ldots < b_{p_n k}$. By passing from $b_{jk}$ to $E_k b_{jk}$ if necessary, we can assume that $\text{supp } b_{jk} \subseteq \text{supp } b_k$ for each $j$, hence
\[
b_{11} < b_{21} < \ldots < b_{p_n 1} < b_{12} < \ldots < b_{p_n K_n}.
\]
Let
\[
\tilde{b} = \frac{1}{\phi(p_n K_n)} \sum_{k=1}^{K_n} \sum_{j=1}^{p_n} b_{jk} = \frac{1}{\phi(p_n K_n)} \sum_{k=1}^{K_n} b_k.
\]
By (2.2), $\|\tilde{b}\| \leq 1$, hence $\|\sum_{k=1}^{K_n} b_k\| \leq \phi(p_n K_n)/\phi(p_n) < 2$. Moreover, $a = \alpha \sum_{s=1}^{p_n} a_s$, where $\|a_s\| = 1$ for each $s$, $a_1 < a_2 < \ldots < a_{p_n}$, and $\alpha = \|\sum_{s=1}^{p_n} a_s\|$. By (2.2), $\alpha \leq \phi(p_m)/p_m$. By Lemma 5 of [GM] (and by the choice of sequences $(p_n)$ and $(\sigma'_n)$), $\langle a, \tilde{b} \rangle \leq 2\alpha < \sigma_m$. Thus, $\langle a, \sum_{k=1}^{K_n} b_k \rangle < 2\sigma_m$, as desired. $\blacksquare$

**Proof of Theorem 2.1.** Below we view elements of $S$, $S^*$, and $\ell_2$ as sequences (via the expansions with respect to the canonical bases of these spaces). Operations of multiplication etc. are defined pointwise.

Suppose $A_1, A_1^*, B_2, B_2^*, \ldots$ are as in the previous lemma, with $2\sigma_k/(1 - \sigma_k) < \varepsilon_k$. Define $A_k$ as the set of vectors $x \in \ell_2$ for which $|x|^2 = ab/(a, b)$, with $a \in B_k$, $b \in B_k^*$, $a, b \geq 0$, and $(a, b) > 1 - \sigma_k$. It follows from [OS1] that the sets $A_k$ are asymptotic, spreading, and unconditional. To show (2.1), suppose $m < n$, and consider non-negative $x, y_1, \ldots, y_{K_n} \in \ell_2$ s.t. $x^2 = ab$ and $y_k^2 = a_k b_k$ with $a \in B_m$, $b \in B_m^*$, $a_k \in B_n$, $b_k \in B_n^*$ (for $1 \leq k \leq K_n$), and $y_1 < y_2 < \ldots < y_{K_n}$. Let $E_k = \text{supp } y_k$. By
Cauchy-Schwartz Inequality,
\[ \sum_k \langle x, y_k \rangle^2 = \sum_k \langle \sqrt{a_k} \sqrt{b_k} E_k a_k E_k b_k \rangle^2 \leq \sum_k \langle a, E_k b_k \rangle \langle a, E_k b \rangle. \]
By the previous lemma, \( \sum_k \langle a, E_k b_k \rangle < 2\sigma_m \) and \( \langle a, E_k b \rangle < \sigma_m \). Therefore,
\[ \sum_k \left( \frac{x}{\|x\|} \cdot \frac{y_k}{\|y_k\|} \right)^2 \leq \frac{2\sigma_m^2}{(1 - \sigma_m)^2}. \]
This establishes (2.1).

3. Construction and basic properties of \( X \)

Construct a sequence of sets \( A_n \) as in Theorem 2.1, with \( \varepsilon_n = 239^{-n} \) and \( K_n = 10^n \). Let \( (\delta_i)_{i=1}^N \) and \( (\delta_i)_{i=1}^\infty \) be the canonical bases in \( \ell_2^N \) and \( \ell_2 \), respectively.

Denote by \( U \) the set of operators \( U : \ell_2 \to \ell_{2^n} \) (n even) of the form
\[ U \xi = \sum_{j=1}^{K_n} \langle \xi, f_j \rangle \delta_j \] with \( f_1, \ldots, f_{K_n} \in A_n \), \( f_1 < \ldots < f_{K_n} \), or
\[ U \xi = \frac{1}{\sqrt{2}} \sum_{j=1}^{K_n} \langle \xi, f_{j+i} + \varepsilon f_j \rangle \delta_j \] with \( f_1 < \ldots < f_{2K_n} \), \( \varepsilon = \pm 1 \), and either \( f_1, \ldots, f_{2K_n} \in A_n \), or \( f_1, \ldots, f_{K_n} \in A_n \), \( f_{K_n+1}, \ldots, f_{2K_n} \in A_{n+2} \) (in both cases, \( \xi \in \ell_2 \)). Let \( (U_i) \) be a countable dense subset in \( U \) (that is, for every \( U \in U \) and every \( \varepsilon > 0 \) there exists \( i \in \mathbb{N} \) s.t. the range spaces of \( U \) and \( U_i \) coincide, and \( \| U - U_i \|_1 < \varepsilon \).

Denote by \( \mathcal{W} \) the set of operators \( W \in B(\ell_2) \) s.t. \( W \xi = \sum_{j=1}^{K_n} \langle \xi, f_j \rangle \delta_j \) for \( \xi \in \ell_2 \), where \( n \in \mathbb{N} \) is odd, and \( f_1 < \ldots < f_{K_n} \) belong to \( A_n \).

Following [OR], fix a sequence \( s_0 < s_1 < \ldots \) (increasing “sufficiently fast”), and define spaces \( E_i = \text{MIN}_{s_i}(\text{MAX}_{s_{i-1}}(R_{n_i} \cap C_{n_{i}})) \), for which:

1. \( n_i = 10^j \) for some \( j = j(i) \in \mathbb{N} \), and moreover, for each \( j \in \mathbb{N} \) the number \( 100^j \) occurs infinitely many times in the sequence \( (n_i) \).
2. For any operator \( u : E_i^* \to E_j \), we have \( \|u\|_1/5 \leq \|u\|_{cb} \leq \|u\|_1 \) if \( i = j \), \( \|u\|_{cb} = \|u\|_2 \) if \( i \neq j \).
3. If, in addition, \( H \) has property (P), then \( \lim_{j \to \infty} \gamma_j/100^j = 0 \), where
\[ \gamma_j = \|id : \text{MIN}_{s_{i-1}}(R_{100^j} + C_{100^j}) \to H_{100^j}\|_{cb}, \] and \( i \) is the smallest integer satisfying \( n_i = 100^j \) (or in other words, \( i = \min\{k | j = j(k)\} \)). Consequently, \( \|id : E_i^* \to H_{100^j}\|_{cb} \leq \gamma_j \) for any \( i \).

Define the operator space \( X \) by setting, for \( x \in K \otimes \ell_2 \),
\[ (3.1) \|x\|_{\kappa \otimes X} = \max \left\{ \|x\|_{\kappa \otimes \text{MIN}(\ell_2)}, \sup_{i \in \mathbb{N}} \| (I_{K \otimes U_i}) x \|_{\kappa \otimes E_i}, \sup_{W \in W} \| (I_{K \otimes W}) x \|_{\kappa \otimes H} \right\}. \]
(recall that, for \( x = \sum_i a_i \otimes \delta_i \in K \otimes \text{MIN}(\ell_2) \),
\[
\| x \|_{K \otimes \text{MIN}(\ell_2)} = \sup \{ \| \sum_i \alpha_i a_i \|_K \mid \sum_i |\alpha_i|^2 \leq 1 \}.
\]
It is easy to check that \( X \) satisfies Ruan’s axioms, hence it is an operator space.
Also, \( X \) is isometric to \( \ell_2 \). We shall show that it has all the desired properties. Start
by showing that elements of \( \mathcal{U} \) and \( \mathcal{W} \) “ignore” each other.

**Lemma 3.1.** If \( U \in \mathcal{U} \) and \( W \in \mathcal{W} \), then \( \| UW^* \|_1 \leq 1 \).

**Proof.** It suffices to prove that \( \| UV \|_1 \leq 1/2 \) when \( U \in B(\ell_2, \ell_2^{K_n}) \) and \( V \in B(\ell_2, \ell_2) \)
are given by
\[
(3.2) \quad U \xi = \sum_{j=1}^{K_m} \langle \xi, g_j \rangle \delta_j, \quad \text{and} \quad V \delta_i = \begin{cases} f_i & i \leq K_n \\ 0 & i > K_n \end{cases},
\]
where \( f_1 < \ldots < f_{K_n} \) belong to \( A_n \), and \( g_1 < \ldots < g_{K_m} \) belong to \( A_\ell \), for \( \ell \geq m \),
and \( n \not\in \{m, \ell\} \). Indeed, the adjoint of any element of \( \mathcal{W} \) equals \( V \) as above, while
any element of \( \mathcal{U} \) either equals to a \( U \) of the above form, or can be represented as
\( (U_1 + U_2)/\sqrt{2} \), with \( U_1 \) and \( U_2 \) resembling \( U \) in (3.2). Note that, for \( U \) and \( V \) as in
(3.2),
\[
UV \delta_i = \begin{cases} \sum_{j=1}^{K_m} \langle f_i, g_j \rangle \delta_j & i \leq K_n \\ 0 & i > K_n \end{cases},
\]
and therefore,
\[
(3.3) \quad \| UV \|_2^2 = \sum_{i=1}^{K_n} \sum_{j=1}^{K_m} |\langle f_i, g_j \rangle|^2.
\]
To estimate \( \| UV \|_1 \), suppose first that \( n < \ell \). By construction of \( A_n \) and \( A_\ell \),
\[
\sum_{j=1}^{K_m} |\langle f_i, g_j \rangle|^2 < \varepsilon_n^2 \quad \text{for} \quad 1 \leq i \leq K_n.
\]
Therefore, by (3.3), \( \| UV \|_2^2 \leq K_n \varepsilon_n^2 \). Moreover, \( \text{rank} \ UV \leq \text{rank} \ U = K_n \), hence
\[
\| UV \|_1 \leq \sqrt{\text{rank} UV} \| UV \|_2 = K_n \varepsilon_n < \frac{1}{2},
\]
by our choice of \( K_n \) and \( \varepsilon_n \). If \( n > \ell \), we similarly obtain \( \| UV \|_1 \leq K_m \varepsilon_\ell \leq K_\ell \varepsilon_\ell < 1/2 \) (we use the fact that \( m \leq \ell \)). \( \blacksquare \)

We shall identify subquotients of \( X \) with subspaces of \( X \) (as linear spaces). More
precisely, suppose \( X'' \hookrightarrow X' \hookrightarrow X \). Then \( Y = X/X'' \) and \( Y' = X'/X'' \) are identified
with \( X \ominus X'' \) and \( X' \ominus X'' \), respectively.

**Proposition 3.2.** \( H \) is \((1 + \varepsilon)\)-completely complementably finitely representable in
any infinite dimensional subquotient of \( X \).
Proof. Fix an odd \( n \), and consider \( f_1, \ldots, f_{K_n} \in A_n \) such that \( f_1 < \ldots < f_{K_n} \). Denote by \( X_f \) the span of \( f_1, \ldots, f_{K_n} \) in \( X \). We shall show that \( X_f \) is completely contractively complemented in \( X \), and completely isometric to \( H_{K_n} \). Indeed, there exists \( W_0 \in W \) s.t. \( W_0 \xi = \sum_{j=1}^{K_n} \langle \xi, f_j \rangle \delta_j \) for \( \xi \in X \). By (3.1), \( \| W_0 \|_{cb} = 1 \).

Consider \( W_0^* \) as an operator \( V : H \to X \). Then

\[
\| V \|_{cb} = \max \left\{ \| V \|_{CB(H, \text{MIN}(\ell_2))}, \sup_{i \in \mathbb{N}} \| U_i V \|_{CB(H, E_i)}, \sup_{W \in W} \| WV \|_{CB(H)} \right\}.
\]

But \( \| V \|_{CB(H, \text{MIN}(\ell_2))} = \| V \| = 1, \| WV \|_{CB(H)} = \| WV \| \leq 1, \) and \( \| U_i V \|_{CB(H, E_i)} \leq \| U_i V \|_1 \leq 1 \) by Lemma 3.1. Thus, both \( W_0 \) and \( V \) are complete contractions, hence \( X_f \) is completely isometric to \( H_{K_n} \). Moreover, \( P = VW_0 \) is a completely contractive projection onto \( X_f \).

Now consider \( Y' = X'/X'' \) (with \( X'' \hookrightarrow X' \hookrightarrow X \)). By perturbing \( X' \) and \( X'' \) slightly, and identifying \( Y' \) with a subspace of \( X \) (as explained above), we can assume that \( Y' \cap A_n \) contains \( f_1 < \ldots < f_{K_n} \). Denote by \( Z \) the span of \( f_1, \ldots, f_{K_n} \) in \( Y' \). We claim that \( Z \) is completely isometric to \( H_{K_n} \), and completely contractively complemented in \( Y' \). Indeed, consider the orthogonal projection \( P \) from \( X \) onto \( Z \).

Above we have established that \( P \) is completely contractive as an operator on \( X \). Therefore, for any \( z \in K \otimes Z \),

\[
\| z \|_{K \otimes X'} \geq \inf \{ \| (I_K \otimes P)(z + x) \|_{K \otimes X'}, \| x \in K \otimes X'' \} = \| z \|_{K \otimes X'},
\]

since \( X'' \subset P \). Thus, \( Z \) is completely isometric to the span of \( f_1, f_2, \ldots, f_{K_n} \) in \( X' \), which, by the above, is completely isometric to \( H_{K_n} \). Moreover, \( P \) (viewed as an operator on \( Y' \)) is completely contractive.

The following result yields a useful lower estimate for c.b. norms of operators on \( X \) and its subquotients.

**Proposition 3.3.** Suppose \( X'' \hookrightarrow X' \hookrightarrow X \), and let \( Y \) and \( Y' \) are the quotient spaces \( X/X'' \) and \( X'/X'' \), respectively.

(a) Consider the operators \( T : Y' \to Y, U : Y \to \ell_2^{10^n}, \) and \( V : \ell_2^{10^n} \to Y' \), such that \( U, V^* \in U \). Then

\[
\| T \|_{cb} \geq \frac{\| UTV \|_1}{5 \max\{10^n, \| UV \|_1\}}.
\]

Consequently, \( \| T \|_{cb} \geq \| UTV \|_1/(5 \cdot 10^n) \) whenever \( U \) and \( V \) as above satisfy \( UV = 0 \).

(b) Suppose \( H \) has property \( (P) \), and consider the operators \( T : Y' \to Y, U : Y \to \ell_2^{10^n}, \) and \( V : \ell_2^{10^n} \to Y' \), such that \( U \in U \). Then

\[
\| T \|_{cb} \geq \frac{\| UTV \|_1}{5 \max\{10^n \| V \|, \gamma_n \| V \|, \| UV \|_1\}}.
\]

For the proof, we need the following two lemmas. Below, \( X'', X', X'', Y', \) and \( Y \) are as in the statement of Proposition 3.3.
Lemma 3.4. Suppose $P$ is the orthogonal projection from $X$ onto $Y'$, and $U_i : X \to E_i$ is as in the definition of $X$. Then $\|U_i|_{Y'}\|_{CB(Y',E_i)} \leq 1 + 2\|U_i - U_iP\|_1$.

Proof. Observe first that
\[
\|U_iP\|_{CB(X,E_i)} \leq 1 + \|U_i - U_iP\|_{CB(X,E_i)} \leq 1 + \|U_i - U_iP\|_1.
\]
Moreover, $\|U_iP\|_{CB(X,E_i)} \geq \|U_iP\|_{Y'}\|_{CB(Y',E_i)}$. Indeed, suppose $y \in M_n \otimes Y'$ satisfies $\|y\|_{M_n \otimes Y'} < 1$. Then there exists $x \in M_n \otimes X$ such that $\|x\|_{M_n \otimes X} < 1$, and $I_{M_n} \otimes P(x) = y$. We conclude that
\[
\|I_{M_n} \otimes U_iP(y)\|_{M_n \otimes E_i} = \|I_{M_n} \otimes U_iP(x)\|_{M_n \otimes E_i} < \|U_iP\|_{CB(X,E_i)}.
\]
To finish the proof, note that $\|U_i|_{Y'}\|_{CB(Y',E_i)} \leq \|U_iP\|_{Y'}\|_{CB(Y',E_i)} + \|U_i - U_iP\|_1$. □

Lemma 3.5. Suppose $V$ as an operator from $E_i^* \to Y'$. Then
\[
\|V\|_{CB(E_i^*,Y')} \leq \max \left\{ \|U_iV\|_1, \|V\|_2, \sup_{W \in W} \|WV\|_{cb} \right\}.
\]
Consequently:

1. If $V^* \in \mathcal{U}$, then $\|V\|_{CB(E_i^*,Y')} \leq \max \{\|U_iV\|_1, \|V\|_2\}$.

2. If $H$ has property $(P)$ and $n_i = 100^k$, then
\[
\|V\|_{CB(E_i^*,Y')} \leq \max \left\{ \|U_iV\|_1, \max\{\sqrt{n_i}; \gamma_k\}\|V\| \right\}.
\]

Proof. Let $q : X' \to Y'$ is the complete quotient map. By (3.1),
\[
\|V\|_{CB(E_i^*,Y')} = \|qV\|_{CB(E_i^*,Y')} \leq \|V\|_{CB(E_i^*,X)} = \max \left\{ \|V\|_{CB(E_i^*,\text{MIN}(\ell_2))}, \sup_{j \in \mathbb{N}} \|U_jV\|_{CB(E_i^*,E_j)}, \sup_{W \in W} \|WV\|_{CB(E_i^*,H)} \right\}.
\]
However, $\|V\|_{CB(E_i^*,\text{MIN}(\ell_2))} = \|V\|$, $\|U_iV\|_{cb} = \|U_iV\|_1$, while $\|U_jV\|_{cb} = \|U_jV\|_2 \leq \|V\|_2$ for $j \neq i$. If $V^* \in \mathcal{U}$, then, by Lemma 3.1, $\|WV\|_{cb} \leq \|WV\|_1 \leq 1$. If $H$ has property (P) and $n_i = 100^k$, then $\|WV\|_{cb} \leq \gamma_k\|V\|$.

Proof of Proposition 3.3. We observe that, for any $i \in \mathbb{N}$,
\[
\|T\|_{cb} \geq \frac{\|U_iTV\|_{CB(E_i^*,E_i)}}{\|U_i|_{CB(Y,E)}\|V\|_{CB(E_i^*,Y')}} \geq \frac{\|U_iTV\|_1}{5\|U_i|_{CB(Y,E)}\|V\|_{CB(E_i^*,Y')}}.
\]
Approximating $U$ with operators $U_i$, and using estimates for $\|U_i\|_{cb}$ and $\|V\|_{cb}$ obtained in Lemmas 3.4 and 3.5, we achieve the result. □

Corollary 3.6. Any infinite dimensional subquotient of $X$ is completely indecomposable.

Proof. Suppose $P$ is a projection on $Y' = X'/X''$ (here, $X'' \hookrightarrow X' \hookrightarrow X$), and both the range and the kernel of $P$ are infinite dimensional. The sets $A_n$ involved in the construction of $X$ are asymptotic, and therefore, by a small perturbation
argument, we can assume that for any even \( n \) there exist \( f_1, \ldots, f_{2K_n} \in A_n \cap Y' \) s.t. \( f_1 < \ldots < f_{2K_n} \), and

\[
Pf_j = \begin{cases} f_j & j \leq K_n \\ 0 & j > K_n \end{cases}.
\]

Consider the operators \( U, V \in B(X, \ell_2^{K_n}) \), defined by

\[
U\xi = \frac{1}{\sqrt{2}} \sum_{s=1}^{K_n} \langle \eta, f_{s+K_n} - f_s \rangle \delta_s, \quad V\xi = \frac{1}{\sqrt{2}} \sum_{s=1}^{K_n} \langle \eta, f_{s+K_n} + f_s \rangle \delta_s \quad (\xi \in \ell_2).
\]

Then \( U, V \in \mathcal{U} \), and \( UV^* = 0 \). Therefore, by Proposition 3.3,

\[
\|P\|_{cb} \geq \frac{\|UPV^*\|_1}{5 \cdot 10^{n/2}} = \frac{10^n/2}{5 \cdot 10^{n/2}} = 10^{n/2-1}.
\]

The even integer \( n \) can be arbitrarily large, hence \( P \) is not completely bounded. \( \blacksquare \)

## 4. Subquotients of \( X \) Fail the OAP

As in the previous section, we assume that \( X'' \hookrightarrow X' \hookrightarrow X \), and \( Y' = X'/X'' \) is infinite dimensional. We establish

**Theorem 4.1.** \( Y' \) fails the Compact Operator Approximation Property.

Our main tool is

**Lemma 4.2.** Suppose \( Z \) is an operator space with the Compact Operator Approximation Property, \( (Z_i)_{i=0}^\infty \) a sequence of finite dimensional subspaces of \( Z \), \( (F_i)_{i=1}^\infty \) a sequence of 1-exact operator spaces, and the function \( f : \mathbb{N} \to (2, \infty) \) is such that \( \lim_{n \to \infty} f(n) = \infty \). Then there exists a compact operator \( \psi : Z \to Z \) such that \( \psi|_{Z_0} = I_{Z_0} \), and \( \|u_i\psi|_{Z_i}\|_{cb} \leq f(i\|u_i\|_{cb}) \) for any \( i \in \mathbb{N} \) and \( u_i : Z \to F_i \).

We omit the proof, as it is identical to the proof of Lemma 6.1 of [OR].

**Proof of Theorem 4.1.** By a small perturbation argument, we may assume that \( Y' \) contains vectors \( f_{ij} \) \((j \in \mathbb{N}, 1 \leq i \leq K_{2j})\) with finite support such that \( f_{ij} \in A_{2j} \), and \( f_{ij} < f_{ik} \) if \( j < \ell \), or \( j = \ell \) and \( i < k \). For every \( j \in \mathbb{N} \), \( 1 \leq m \leq 100 \), and \( \varepsilon = \pm 1 \), define operators \( A_{j,m,\varepsilon} : Y' \to \ell_2^{K_{2j}} \) and \( B_{j,m,\varepsilon} : \ell_2^{K_{2j}} \to Y' \) by setting \( m' = K_{2j}(m - 1) \),

\[
B_{j,m,\varepsilon} \delta_{ij} = \frac{1}{\sqrt{2}} (f_{ij} - \varepsilon f_{m'i+j+1,j+1}) \quad \text{for} \ 1 \leq i \leq 100^j
\]

\((\delta_{ij})_{i=1}^{K_{2j}}\) is the canonical basis of \( \ell_2^{K_{2j}} \), and

\[
A_{j,m,\varepsilon} \xi = \frac{1}{\sqrt{2}} \sum_{i=1}^{100^j} \langle \xi, f_{ij} + \varepsilon f_{m'i+j+1,j+1} \rangle \delta_i \quad \text{for} \ \xi \in Y'.
\]

We can assume that, for every triple \((j, m, \varepsilon)\) as above, there exists \( s = s(j, m, \varepsilon) \in \mathbb{N} \) for which \( \dim E_s = K_{2j} \), and \( U_s = A_{j,m,\varepsilon} \) (here, we identify \( E_s \) with \( \ell_2^{K_{2j}} \)).
Suppose, for the sake of contradiction, that $Y'$ has the COAP. By Lemma 4.2, there exists a compact operator $\psi : Y' \to Y'$ such that $\psi f_{i,3} = f_{i,3}$ for $1 \leq i \leq 100^3$, and

$$\|A_{j,m,\varepsilon}\|_{cb} \leq j\|A_{j,m,\varepsilon}\|_{cb} \|B_{j,m,\varepsilon}\|_{cb} \text{ for } j \geq 3, \ 1 \leq m \leq 100, \ \varepsilon = \pm 1,$$

with $A_{j,m,\varepsilon}$ and $B_{j,m,\varepsilon}$ viewed as elements of $CB(Y', E_{s(j,m,\varepsilon)})$ and $CB(E_{s(j,m,\varepsilon)}^*, Y')$, respectively. However, $\|A_{j,m,\varepsilon}\|_{cb} \leq 1$, and $\|B_{j,m,\varepsilon}\|_{cb} \leq \sqrt{K_2} = 10^j$ (by Lemma 3.4 and Lemma 3.5, respectively). Thus, we have

$$\|A_{j,m,\varepsilon}\|_{CB(E_{s(j,m,\varepsilon)}^*, E_{s(j,m,\varepsilon)})} \leq j \cdot 10^j$$

for any appropriate triple $(j, m, \varepsilon)$. By the basic properties of spaces $E_i$, we have

$$\text{Re} \left( \text{tr} (A_{j,m,\varepsilon} \psi B_{j,m,\varepsilon}) \right) \leq \|A_{j,m,\varepsilon}\|_{cb} \|B_{j,m,\varepsilon}\|_{cb} \leq 5j \cdot 10^j.$$

An easy computation shows that

$$\text{tr} (A_{j,m,\varepsilon} \psi B_{j,m,\varepsilon}) = \frac{1}{2} \sum_{i=1}^{K_{2j}} (\psi(f_{ij} - \varepsilon f_m + f_{ij} + \varepsilon f_m)),$$

Therefore,

$$\text{Re} \left( \text{tr} (A_{j,m,1} \psi B_{j,m,1} + A_{j,m,-1} \psi B_{j,m,-1}) \right) = \text{Re} \left( \sum_{i=1}^{K_{2j}} (\langle \psi(f_{ij}), f_{ij} \rangle - \langle \psi(f_{m} + f_{ij}), f_{m} \rangle) \right) \leq 10^{j+1}.$$

Consequently,

$$\text{Re} \left( \sum_{i=1}^{K_{2j}} \langle \psi(f_{m} + f_{ij}), f_{m} \rangle \right) \geq \text{Re} \left( \sum_{i=1}^{K_{2j}} \langle \psi(f_{ij}), f_{ij} \rangle \right) - 2 \cdot 10^{j+1}.$$

Summing over all values of $m$ ($1 \leq m \leq 100$), we obtain

$$(4.1) \quad S_{j+1} \geq 100(S_j - 2 \cdot 10^{j+1}),$$

where $S_j = \text{Re} \sum_{i=1}^{100^3} \langle \psi(f_{ij}), f_{ij} \rangle$. This allows us to show by induction that

$$(4.2) \quad S_j > \frac{j + 1}{2j} 100^j > \frac{100^j}{2}$$

whenever $j \geq 3$. Indeed, $\psi(f_{i,3}) = f_{i,3}$ for $1 \leq i \leq 100^3$, hence $S_3 = 100^3$. Assuming (4.2) holds for some $j \geq 3$, observe that

$$\frac{2 \cdot 10^{j+1}}{S_j} < 10^{2-j} \frac{j + 2}{j + 2},$$

hence, by (4.1),

$$S_{j+1} \geq 100S_j \left(1 - \frac{2 \cdot 10^{j+1}}{S_j} \right) > \frac{j + 1}{2j} 100^j \left(1 - \frac{1}{j + 2} \right) = \frac{j + 1}{2(j + 1)} 100^j.$$

This proves (4.2) for $j + 1$. 

On the other hand, $\psi$ is compact, hence $\max_{1 \leq i \leq K_2} \|\psi(f_{ij})\| < 1/2$ when $j$ is sufficiently large. For such $j$, $S_j < 100^j/2$. This contradicts (4.2).

As a corollary, we prove:

**Corollary 4.3.** In the above notation, the spaces $Y'$ and $Y''$ are not exact.

For the proof, we need the following proposition (it may be known to specialists).

**Lemma 4.4.** Suppose $Z$ is an infinite-dimensional $\lambda$-exact operator space. Then $Z$ contains a $C$-completely basic sequence for any $C > \lambda$.

**Proof.** We select a $C$-completely basic sequence $(z_i) \subset Z$ inductively. More precisely, we select linearly independent vectors $z_1, z_2, \ldots \in Z$, finite codimensional subspaces $\ldots \hookrightarrow Z_2 \hookrightarrow Z_1 \hookrightarrow Z$, and finite rank projections $P_n \in CB(Z_n)$ such that, for any $n$, $z_1, \ldots, z_n \in Z_n$, ran $P_n = \text{span}[z_1, \ldots, z_n]$, $\|P_n\|_{\text{cb}} < C$, and $P_n z_n = 0$ whenever $m < n$ (then the operators $P_n|_{\text{span}[z_k]|_{k \in N}}$ play the role of basis projections).

First pick an arbitrary non-zero $z_1 \in Z$. By Hahn-Banach Theorem, there exists a contractive projection $P_1$ onto $E_1 = \text{span}[z_1]$. Moreover, $P_1$ has rank 1, hence it is completely contractive. Let $Z_1 = Z$.

Now suppose we have selected $z_1, \ldots, z_n$, $Z_1, \ldots, Z_n$, and $P_1, \ldots, P_n$, as above. Pick an arbitrary non-zero $z_{n+1} \in Z_n \cap (\cap_{m=1}^n \text{ker } P_m)$. Let $E = \text{span}[z_1, \ldots, z_{n+1}]$. Find $F \hookrightarrow M_N$ and $u : E \to F$ s.t. $\|u\|_{\text{cb}} = 1$, $\|u^{-1}\|_{\text{cb}} < C$. By Arveson-Wittstock-Stinespring-Paulsen extension theorem, there exists $\tilde{u} : Z_n \to M_N$ s.t. $\tilde{u}|_E = u$, and $\|\tilde{u}\|_{\text{cb}} = 1$. Let $Z_{n+1} = \text{span}[E, \text{ker } \tilde{u}] \hookrightarrow Z_n$, and note that dim $Z_n/\text{ker } \tilde{u} \leq \dim M_N < \infty$, hence dim $Z_n/Z_{n+1} < \infty$. Furthermore, $\tilde{u}(Z_{n+1}) \subset F$. It is easy to see that $P_{n+1} = u^{-1}\tilde{u}|_{Z_{n+1}}$ is a projection from $Z_{n+1}$ onto $\text{span}[z_1, \ldots, z_{n+1}]$, with $\|P_{n+1}\|_{\text{cb}} < C$. Moreover, $P_m z_{n+1} = 0$ for $m \leq n$.

5. Completely bounded maps on subquotients of $X$

In this section, we assume that $H$ has property (P), $X'' \hookrightarrow X' \hookrightarrow X$, $Y = X/X''$, and $Y' = X'/X''$ is infinite dimensional. We denote by $J_Y$, the natural embedding of $Y'$ into $Y$. We show:

**Theorem 5.1.** Any completely bounded operator $S : Y' \to Y$ is of the form $S = cJ_Y + S'$, where $c \in C$ and $S'$ is compact.

For the proof, we need the following proposition (it may be known to specialists).
Proposition 5.2. Suppose $Z'$ is a subspace of a Hilbert space $Z$, and $T \in B(Z', Z)$. Then either $T$ is a compact perturbation of a scalar multiple of $J$ (the natural embedding of $Z'$ into $Z$), or there exist mutually orthogonal projections of infinite rank $P \in B(Z')$, $Q \in B(Z)$ such that $QT|_{\text{ran} P} \in B(\text{ran} P, \text{ran} Q)$ is invertible.

Proof. First denote by $Q_0$ the orthogonal projection in $B(Z)$ whose kernel equals $Z'$. If there are no infinite rank projections $P$ and $Q$ s.t. $\text{ran} Q \subset \text{ran} Q_0$ and $QT|_{\text{ran} P}$ is invertible, then $Q_0 T$ is compact. This reduces the problem to the case of $Z' = Z$.

We denote by $\mathcal{K}(H)$ the space of compact operators on $H$. We shall show that, if $c = \text{dist}(T, CI_Z + \mathcal{K}(Z)) > 0$, then there exist mutually orthogonal projections $P$ and $Q$ of infinite rank s.t. $QT|_{\text{ran} P} \in B(\text{ran} P, \text{ran} Q)$ is invertible.

Note that $\text{dist}(RTR, CR + \mathcal{K}(ran R)) = c$ for any orthogonal projection $R \in B(Z)$ with finite dimensional kernel. By Theorem 9.12 of [D], for such an $R$ there exist mutually orthogonal norm 1 vectors $\xi(R), \eta(R) \in \text{ran} R$ s.t. $\langle T\xi(R), \eta(R) \rangle > c/3$. This allows us to construct inductively vectors $(\xi_n)_{n \in \mathbb{N}}$ and $(\eta_n)_{n \in \mathbb{N}}$ in $Z$, such that, for any $k, j$,

$$\langle \xi_k, \eta_j \rangle = 0, \quad \langle \xi_k, \xi_j \rangle = \langle \eta_k, \eta_j \rangle = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}, \quad \langle T\xi_k, \eta_j \rangle \begin{cases} > c/3 & k = j \\ = 0 & k \neq j \end{cases}.$$  

Indeed, let $R_1 = I_Z$, $\xi_1 = \xi(R_1)$, and $\eta_1 = \eta(R_1)$. Suppose $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ have already been selected in such a way that (5.1) holds whenever $j, k \leq n$. Let $R_{n+1}$ be the orthogonal projection whose kernel is spanned by $(\xi_i)_{i=1}^n$, $(\eta_i)_{i=1}^n$, $(T\xi_i)_{i=1}^n$, and $(T^*\eta_i)_{i=1}^n$. Let $\xi_{n+1} = \xi(R_{n+1})$, $\eta_{n+1} = \eta(R_{n+1})$, and observe that now (5.1) holds for all $j, k \leq n + 1$.

Denote by $Q$ and $P$ the orthogonal projections from $Z$ onto $\text{span}[\eta_n \mid n \in \mathbb{N}]$ and $\text{span}[\xi_n \mid n \in \mathbb{N}]$, respectively. By the above, $QT|_{\text{ran} P}$ is invertible. 

Proof of Theorem 5.1. Suppose $T : Y' \rightarrow Y$ is not a compact perturbation of $J_{Y'}$. We shall show $T$ is not completely bounded. By Proposition 5.2, there exist mutually orthogonal projections $P$ and $Q$ of infinite rank s.t. $\|QT\xi\| \geq \|\xi\|/C$ for any $\xi \in \text{ran} P$ ($C > 0$). By a small perturbation argument, assume the existence of $f_1 < \ldots < f_{K_n}$ in $\text{ran} Q \cap A_n$ ($n$ even). Consider $U \in \mathcal{U}$ which sends $f_j$ into $\delta_j$ ($1 \leq j \leq K_n$), and annihilates $\text{span}[f_1, \ldots, f_{K_n}]^\perp$. Define $V : \ell_2^{K_n} \rightarrow \text{ran} P \rightarrow Y'$ by setting $V\delta_j = (QT)^{-1}f_j$ (once again, $1 \leq j \leq K_n$). Then $\|V\| \leq C$, $UV = 0$, and $UTV$ is the identity on $\ell_2^{K_n}$. Applying Lemma 3.3, we conclude that

$$\|T\|_{cb} \geq \frac{100^n}{5C \max\{\gamma_n, 10^n\}}.$$  

$n$ can be chosen to be arbitrarily large, hence $T$ is not completely bounded. 

References