Function theory using the prime function

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Exercises: The following exercises all pertain to the unit disc $|z| < 1$. Its prime function is $\omega(z,a)$ and

$$\tilde{K}(z,a) = a\frac{\partial}{\partial a}\log\omega(z,a).$$

1. **(Mapping properties of inversion)** Consider the inversion

$$w = \frac{1}{z}. \tag{1}$$

Show that, when viewed as a conformal map, the circle

$$|z-\delta|^2 = q^2 \tag{2}$$

is transplanted to the circle

$$|w-\Delta|^2 = Q^2 \tag{3}$$

where

$$\Delta = \frac{\overline{\delta}}{|\delta|^2 - q^2}, \qquad Q = \frac{q}{||\delta|^2 - q^2|}. \tag{4}$$

2. **(Möbius maps as conformal maps)** Consider the ratio of two prime functions as a conformal map:

$$w = A\ R(z;a,b), \qquad a,b,A \in \mathbb{C}. \tag{5}$$

When the prime function $\omega(z,a) = (z-a)$ this is a *Möbius map* for all choices of the parameters a,b and A. Show that (5) can be written as

$$w = A\ \frac{\omega(z,a)}{\omega(z,b)} = A + A\frac{\omega(b,a)}{\omega(z,b)}. \tag{6}$$

Hence argue that, geometrically, $z \mapsto w$ is equivalent to the composition of this sequence of three simpler maps

$$z \mapsto w_1 = z - b, \qquad w_1 \mapsto w_2 = \frac{1}{w_1}, \qquad w_2 \mapsto w = A + A\omega(b,a)w_2. \tag{7}$$

Hence show that a Möbius map transplants any circle in the z plane to a circle in the w plane.

3. **(Inverse of the Schwarz function of a circle)** Show that the inverse of the Schwarz function of a circle is its Schwarz conjugate function, i.e., show that

$$S^{-1}(z) = \overline{S}(z). \tag{8}$$

4. **(Reflection in C_0 as a Möbius map)** Consider the circle C inside the unit disc D_z defined by

$$|z - \delta|^2 = q^2 \tag{9}$$

for some $\delta \in \mathbb{C}$ and $q \in \mathbb{R}$ and let C' denote the image of C under reflection in the unit circle C_0.

(a) Let $S(z)$ denote the Schwarz function of the circle C and let $S_0(z)$ denote the Schwarz function of C_0. Show that the Möbius map defined by

$$\theta(z) \equiv \overline{S}(S_0(z)) \tag{10}$$

is the analytic function the transplants each point on C' to its corresponding point on C obtained by reflection of C' in C_0.

(b) Verify that

$$\theta^{-1}(z) = \frac{1}{\overline{\theta}(1/z)} \tag{11}$$

and that this map takes points on C to its corresponding point on C' obtained by reflection of C in C_0.

5. **(Map an eccentric to a concentric annulus)** It is required to map the eccentric annulus with boundaries

$$|z| = 1, \qquad |z - \delta| = q, \tag{12}$$

where $\delta, q > 0$ are positive real parameters, to a concentric annulus $\rho < |w| < 1$ for some real positive ρ. Show that the required map is

$$w = -\frac{z - a}{|a|(z - a^{-1})}, \tag{13}$$

where

$$a = \frac{1 + \delta^2 - q^2 - [(1 + \delta^2 - q^2)^2 - 4\delta^2]^{1/2}}{2\delta}, \tag{14}$$

and

$$\rho = \frac{1 - \delta^2 + q^2 - [(1 - \delta^2 + q^2)^2 - 4q^2]^{1/2}}{2q}, \tag{15}$$

and where the minus sign in (13) is chosen to ensure that $z = 1$ maps to $w = 1$.

6. **(Symmetric triply connected domain)** Consider the domain D_z comprising the unit disc with two smaller circular discs with boundaries C_1 and C_2 excised. Let δ_1 and δ_2 be the centres of C_1 and C_2 and let q_1 and q_2 be their radii and suppose that

$$\delta_1 = -\delta_2 = \delta \in \mathbb{R}, \qquad q_1 = q_2 = q, \qquad 0 < q < \delta < 1. \tag{16}$$

Thus the domain has reflectional symmetry about both the real and imaginary axes. Use the result of Exercise 5 to show that, under the disc automorphism

$$z \mapsto f(z) = -\frac{z-a}{|a|(z-1/\bar{a})} \tag{17}$$

with

$$a = \frac{(1+\delta^2-q^2) - [(1+\delta^2-q^2)^2 - 4\delta^2]^{1/2}}{2\delta} \tag{18}$$

that D_z is mapped to the circular domain $\tilde{D}_z$ for which the two interior circles $\tilde{C}_1$ and $\tilde{C}_2$ are

$$|z| = \rho, \qquad |z-D| = Q, \tag{19}$$

where

$$\rho = \frac{1-\delta^2+q^2 - [(1-\delta^2+q^2)^2 - 4q^2]^{1/2}}{2q}, \tag{20}$$

and

$$D = \frac{1}{2}\left[f(-\delta+q) + f(-\delta-q)\right], \qquad Q = \frac{1}{2}\left[f(-\delta+q) - f(-\delta-q)\right]. \tag{21}$$

7. **(Two views of a slit map)** Consider a slit along the real axis in a complex w plane between $[r, 1]$ for $r > 0$. We will construct two different, but equivalent, representations of the conformal mapping from the unit z disc D_z to the region exterior to this slit.

There are three real degrees of freedom in the mapping theorem, deriving from the three real degrees of freedom in the automorphisms of the disc. We will fix two of these by insisting that $z = b \in \mathbb{R}$ is the preimage of infinity; we fix the remaining real degree of freedom by insisting that $z = 1$ is the preimage of the end of the slit at $w = 1$.

(a) One approach is to construct this map as an instance of radial slit map. Show that the required mapping is

$$w = f_1(z) = \frac{\omega(1,b)\omega(1,1/b)\omega(z,a)\omega(z,1/a)}{\omega(1,a)\omega(1,1/a)\omega(z,b)\omega(z,1/b)} \tag{22}$$

where $a \in \mathbb{R}$ determines the position of the endpoint r.

(b) This same mapping can be constructed as an instance of a parallel slit mapping. Show that the required map is

$$w = f_2(z) = \frac{\tilde{K}(z,b) - \tilde{K}(z,1/b) - (\tilde{K}(a,b) - \tilde{K}(a,1/b))}{\tilde{K}(1,b) - \tilde{K}(1,1/b) - (\tilde{K}(a,b) - \tilde{K}(a,1/b))}. \tag{23}$$

Remark: The function $f_2(z)$ must be identical with $f_1(z)$. Indeed, $f_2(z)$ is simply the partial fraction decomposition of the rational function $f_1(z)$.

(c) Show that in the limit $b \to 0$, both (22) and (23) become

$$\frac{z + 1/z - (a + 1/a)}{2 - (a + 1/a)}. \tag{24}$$

This is the well-known *Joukowski map*.

8. Show that the unbounded circular slit mapping

$$w = -\frac{\omega(z,a)\omega(z,1/a)}{\omega(z,\overline{a})\omega(z,1/\overline{a})}, \qquad a = \mathrm{i}r \tag{25}$$

maps the unit circle $|z| = 1$ to a circular slit on the unit w circle with

$$\pi - \phi < \arg[w] < \pi + \phi \tag{26}$$

where

$$r = \tan\left(\frac{\phi}{4}\right). \tag{27}$$

9. **(Möbius invariance of cross-ratio)** Show that the cross ratio function $p(z, w, a, b)$ is invariant under the transformations

$$z \mapsto M(z), \qquad w \mapsto M(w), \qquad a \mapsto M(a), \qquad b \mapsto M(b), \tag{28}$$

where M is any Möbius map.

10. **(Boundary normal derivative of the Green's function)** Let $G_0(z, a)$ denote the Green's function of the disc, as introduced in the lectures, and let $\mathcal{G}_0(z, a)$ be its analytic extension. For points z on the unit circle C_0 verify the formula

$$\frac{\partial G_0(z,a)}{\partial n_z} ds_z = -d\mathcal{G}_0(z,a), \tag{29}$$

where n_z denotes the outward normal to D_z and ds_z denotes an element of arclength.

11. **(Prime function as a double limit)** Verify the identity

$$\lim_{\substack{z\to a\\ w\to b}} \left[-(z-a)(w-b)p(z,w,a,b)^{-1}\right] = \omega(a,b)^2. \tag{30}$$

12. **(Cross ratio from the Green's function)** In this exercise $\mathcal{G}_0(z,a)$ is again the analytic extension of the Green's function of the disc. Consider the function $\Pi_{a,b}^{z,w}$ defined by the double integral

$$\Pi_{a,b}^{z,w} = 2\pi\mathrm{i} \int_w^z \int_b^a \frac{\partial^2 \mathcal{G}_0(z',a')}{\partial z' \partial a'} dz' da', \tag{31}$$

where the integration contour between w and z and that between b and a are chosen arbitrarily within the unit disc D_z.

(a) Show that

$$2\pi\mathrm{i}\frac{\partial^2 \mathcal{G}_0(z,a)}{\partial z \partial a} = \frac{1}{(z-a)^2}. \tag{32}$$

(b) Hence show that

$$\Pi_{a,b}^{z,w} = \log p(z,w,a,b) + 2\pi\mathrm{i}n \tag{33}$$

for some integer $n \in \mathbb{Z}$ where $p(z,w,a,b)$ is the cross ratio.

(c) Use your answer to Exercise 11 to show that

$$\lim_{\substack{z\to a\\ w\to b}} \left[-(z-a)(w-b)e^{-\Pi_{a,b}^{z,w}}\right] = \omega(a,b)^2. \tag{34}$$

Remark: Notice how (the square of) the prime function can be retrieved by this double limit starting with the Green's function of the disc.

Function theory in a concentric annulus

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Exercises: The following exercises all pertain to the concentric annulus $\rho < |z| < 1$. Its prime function is $\omega(z,a)$ and

$$\tilde{K}(z,a) = a\frac{\partial}{\partial a}\log\omega(z,a).$$

C_0 denotes $|z| = 1$, C_1 denotes $|z| = \rho$.

1. **(Two views of a 2-slit map)** Consider the conformal mapping to the unbounded domain in a w-plane exterior to two slits $[t,s]$ and $[r,1]$ on the real axis. Two of the three real degrees of freedom of the Riemann mapping theorem have been used to insist that the preimage circle C_1 of the second slit is concentric with C_0. The third and final degree of freedom can be used to insist that

$$f(1) = 1.$$

By the reflectional symmetry of the domain with respect to the real axis we expect the preimage of $w = \infty$ to be at some point $z = b \in \mathbb{R}$.

(a) We can construct this map as an instance of a radial slit mapping. Show that the relevant function is

$$w = f_1(z) = \frac{\omega(1,b)\omega(1,1/b)\omega(z,a)\omega(z,1/a)}{\omega(1,a)\omega(1,1/a)\omega(z,b)\omega(z,1/b)}.$$

This depends on three real parameters: ρ, b and a. These parameters determines the values of r, s and t, the end points of the slits.

(b) The same mapping can be derived as an instance of a parallel slit mapping. Show that the required map is

$$w = f_2(z) = \frac{\tilde{K}(z,b) - \tilde{K}(z,1/b) - (\tilde{K}(a,b) - \tilde{K}(a,1/b))}{\tilde{K}(1,b) - \tilde{K}(1,1/b) - (\tilde{K}(a,b) - \tilde{K}(a,1/b))}.$$

Remark: $f_1(z)$ must be identifical with $f_2(z)$: the latter can be thought of as

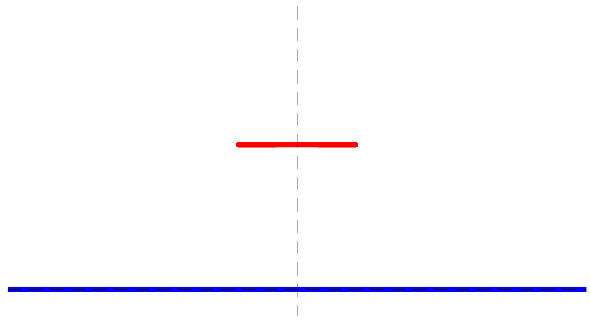


Figure 1: Microstrip above a wall (Exercise 2)

the "partial fraction" decomposition of the function $f_1(z)$ (although remember that we are no longer dealing with rational functions).

2. **(Microstrip geometry)** Consider the map

$$w = \mathrm{i}(\tilde{K}(z,1) - \tilde{K}(-1,1)).$$

(a) Show that it transplants the concentric annulus $\rho < |z| < 1$ to the unbounded region in the upper half w plane exterior to a slit of finite length in the upper half plane and parallel to the real axis. (Hint: think about the parallel slit maps as $a \to 1$).

(b) Using the Schwarz reflection principle show that this same conformal mapping transplants the extended annulus

$$\rho < |z| < 1/\rho$$

to the unbounded region exterior to two slits of equal length that are reflections of each other about the real w axis.[1]

3. **(Inclined flat plate above an infinite wall)** Let $0 < \theta < \pi$. Show that the Cayley-type map

$$w = e^{-\mathrm{i}\theta} \frac{\omega(z, -e^{2\mathrm{i}\theta})}{\omega(z, -1)}$$

transplants the concentric annulus $\rho < |z| < 1$ to the unbounded region in the upper half w plane exterior to a slit of finite length inclined at angle θ to the positive real axis. [2]

[1]The map in this question is essentially the one used by M. V. Schneider, Microstrip lines for microwave integrated circuits, *Bell Syst. Tech. J.*, **48**, (1969)

[2]This mapping was first used in D.G.C & J.S. Marshall, The motion of a point vortex through gaps in walls", *J. Fluid Mech.*, **551**, (2006)

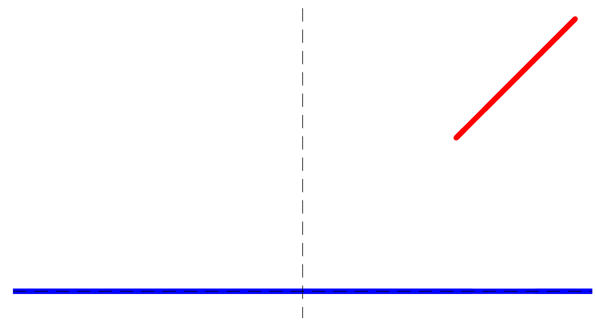


Figure 2: Inclined flat plate above a wall (Exercise 3)

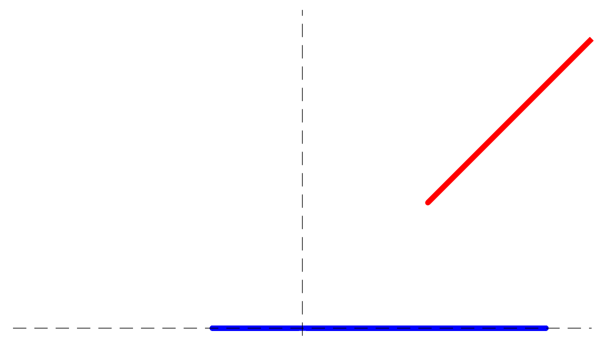


Figure 3: Two flat plates at an arbitrary angle (Exercise 4)

4. **(Two inclined plates)** Consider the map from $\rho < |z| < 1$ given by

$$w = e^{-\mathrm{i}\theta} \frac{\omega(z,1)\omega(z,e^{2\mathrm{i}\theta})}{\omega(z,\sqrt{\rho})\omega(z,1/\sqrt{\rho})}.$$

Show that this maps to the unbounded region exterior to a finite slit along the real w axis and covering the origin $w = 0$ and another slit oriented at angle θ.

5. **(Weis-Fogh mechanism)** In a two-dimensional model of the Weis-Fogh mechanism of insect wings in flight, two detached flat plates of equal length inclined at a fixed angle 2θ with $0 < \theta < \pi$ to each other move in opposite directions with their midpoints at a fixed y position but moving at constant speed ± 1 in the x direction. By considering a radial slit map show that the conformal mapping from a concentric annulus $\rho < |z| < 1$ to the unbounded region exterior to *both* flat plates is of the functional form

$$w = \mathrm{i}e^{-3\mathrm{i}\theta} A \frac{\omega(z,\sqrt{\rho}e^{2\mathrm{i}\theta})\omega(z,e^{2\mathrm{i}\theta}/\sqrt{\rho})}{\omega(z,\sqrt{\rho})\omega(z,1/\sqrt{\rho})} - \mathrm{i}d$$

where A and d are real parameters. Give a prescription for the determination

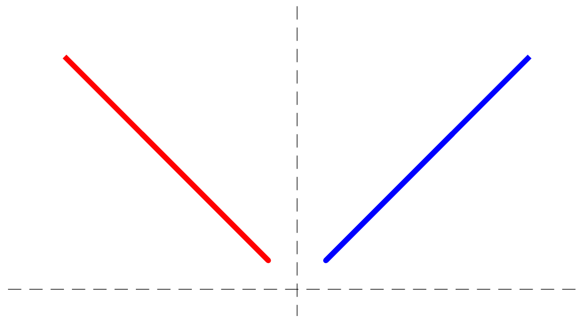


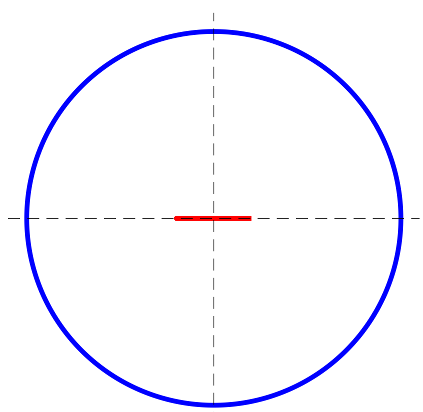
Figure 4: Two plate model of Weis-Fogh mechanism (Exercise 5)

of the three real parameters A, d and ρ to ensure the plates have unit length and have midpoints at $(\pm X, Y)$ for given X and Y [3]

6. **(Disc with a slit)** Show that the map

$$w = \frac{\omega(z, -1) + \omega(z, +1)}{\omega(z, -1) - \omega(z, +1)}$$

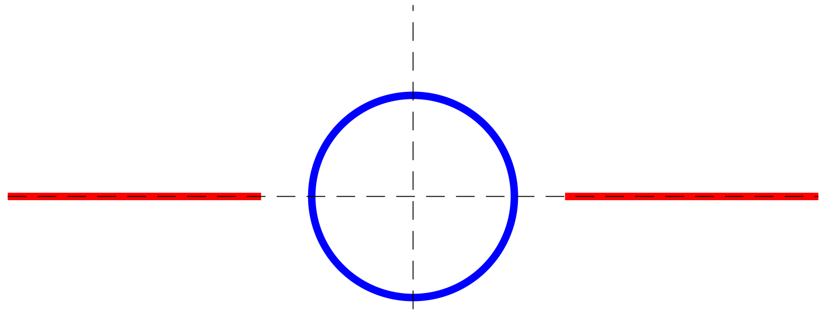
transplants the annulus $\rho < |z| < 1$ to a the unit disc in the w plane exterior to a slit $[-r, r]$ along the real w axis.



Disc with a slit (Exercise 6)

7. **(Cylinder blocking a gap in a wall)** Use the result in Exercise 6 to find the conformal map from the annulus $\rho < |z| < 1$ to the unbounded region exterior to an infinite wall $-\infty < w < -1, 1 < w < \infty$ and exterior to a circular cylinder sitting in the middle of the gap between $-1 < w < 1$.

[3]This mapping was first used in D.G.C, "The spreading phase in Lighthill's model of the Weis-Fogh lift mechanism", *J. Fluid Mech.*, **641**, (2009).



Cylinder blocking a gap in a wall (Exercise 7)

8. **(Another view of a 2-slit map)** Combine the classical Joukowski map

$$\frac{1}{2}\left(\frac{1}{z}+z\right)$$

with the result in Exercise 6 to show that a map from the annulus $\rho < |z| < 1$ to the unbounded region exterior to two slits on the real axis is

$$w = \frac{\omega(z,-1)^2 + \omega(z,+1)^2}{\omega(z,-1)^2 - \omega(z,+1)^2}.$$

Verify that the image of $|z| = 1$ is the interval $[-1, 1]$.

Function theory in a triply connected domain

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Exercises: The following exercises all pertain to a triply connected circular domain D_z comprising the interior of the unit z disc with two excised circular discs with centres $\{\delta_1, \delta_2\}$ and radii $\{q_1, q_2\}$. The prime function is $\omega(z,a)$ and

$$\tilde{K}(z,a) = a\frac{\partial}{\partial a}\log\omega(z,a).$$

C_0 denotes $|z| = 1$, C_1 denotes $|z| = \rho$.

1. **(Two views of a 3-slit map)** Consider the conformal mapping from the triply connected circular domain D_z to the unbounded domain in a w-plane exterior to three slits $[v,u]$, $[t,s]$ and $[r,1]$ on the real axis. Two of the three real degrees of freedom of the Riemann mapping theorem have been used to insist that the preimage circle C_1 of the second slit is concentric with C_0 (i.e. $\delta_1 = 0$). We use up the final degree of freedom, to insist that

$$f(1) = 1.$$

By the reflectional symmetry of the domain with respect to the real axis we expect the preimage of $w = \infty$ to be at some point $z = b \in \mathbb{R}$; we also expect the centre of C_2 to be on the real axis, i.e. $\delta_2 \in \mathbb{R}$.

(a) We can construct this map as an instance of a radial slit mapping. Show that the relevant function is

$$w = f_1(z) = \frac{\omega(1,b)\omega(1,1/b)\omega(z,a)\omega(z,1/a)}{\omega(1,a)\omega(1,1/a)\omega(z,b)\omega(z,1/b)}$$

which depends on three real parameters: ρ, b and a. These parameters determines the values of r, s and t, the end points of the slits.

(b) The same mapping can be constructed as an instance of a parallel slit mapping. Show that the required map is

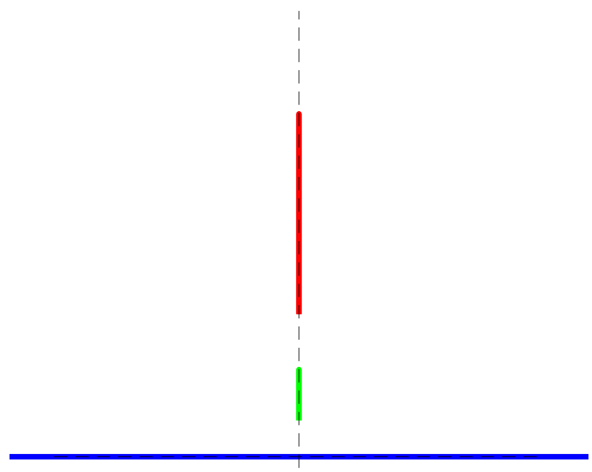
$$w = f_2(z) = \frac{\tilde{K}(z,b) - \tilde{K}(z,1/b) - (\tilde{K}(a,b) - \tilde{K}(a,1/b))}{\tilde{K}(1,b) - \tilde{K}(1,1/b) - (\tilde{K}(a,b) - \tilde{K}(a,1/b))}.$$

The function $f_2(z)$ must be identifical with $f_1(z)$. Indeed, $f_2(z)$ is simply the partial fraction decomposition of the rational function $f_1(z)$.

2. **(Two flat plates perpendicular to a wall)** Suppose that both C_1 and C_2 are centred on the real z axis so that $\delta_1, \delta_2 \in \mathbb{R}$. Show that the Cayley-type mapping

$$w = -\mathrm{i}\,\frac{\omega(z, +1)}{\omega(z, -1)}$$

transplants D_z to the upper half w-plane exterior to two slits of finite length sitting on the positive imaginary w axis as shown in Figure 2



Two slits above wall (Exercise 2)

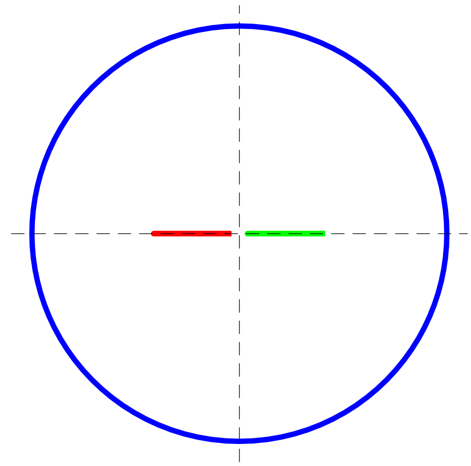
3. **(Disc with two slits)** Suppose again that both C_1 and C_2 are centred on the real z axis with

$$\delta_1 = -\delta_2 = \delta \in \mathbb{R}, \qquad q_1 = q_2 = q, \qquad 0 < q < d < 1.$$

Show that the map

$$w = \frac{\omega(z, -1) + \omega(z, +1)}{\omega(z, -1) - \omega(z, +1)}$$

transplants D_z to the unit disc in the w plane exterior to two slits $[-s, -r]$ and $[r, s]$ along the real w axis. Find expressions for r and s in terms of q and d.



Disc with two slits (Exercise 3)

4. **(Cylinder between two slits)** Use the result in Exercise 3 to find the conformal map from D_z to the unbounded region exterior to a circular cylinder with two equal length slits $[-s, -r]$ and $[r, s]$ at either side of it.

5. **(Yet another view of a 3-slit map)** Combine the Joukowski map

$$\frac{1}{2}\left(\frac{1}{z} + z\right)$$

with the result in Exercise 3 to show that a map from the annulus $\rho < |z| < 1$ to the unbounded region exterior to three slits on the real axis is

$$w = \frac{\omega(z, -1)^2 + \omega(z, +1)^2}{\omega(z, -1)^2 - \omega(z, +1)^2}.$$

Verify that the image of C_0 is the interval $[-1, 1]$.

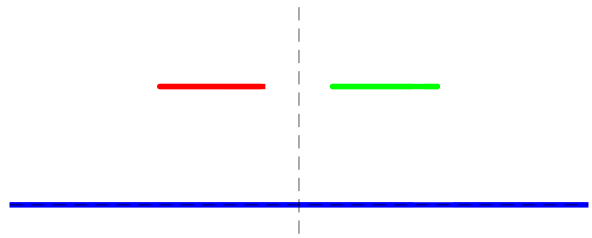
6. **(Two equal flat plates above and parallel to an infinite wall)** Suppose again that both C_1 and C_2 are centred on the real z axis with

$$\delta_1 = -\delta_2 = \delta \in \mathbb{R}, \qquad q_1 = q_2 = q, \qquad 0 < q < d < 1.$$

Show that the map

$$w = \mathrm{i}\left(\tilde{K}(z, \mathrm{i}) - \tilde{K}(-\mathrm{i}, \mathrm{i})\right),$$

transplants D_z to the upper half plane exterior to two equal length slits parallel to the real w axis as shown in Figure 6.



Two slits above wall and parallel to it (Exercise 6)

7. **(Vertical stack of three parallel plates)** Suppose again that D_z is such that both C_1 and C_2 are centred on the real z axis with

$$\delta_1 = -\delta_2 = \delta \in \mathbb{R}, \qquad q_1 = q_2 = q, \qquad 0 < q < d < 1.$$

Use the result on Exercise Sheet 1 to map D_z to the circular domain $\tilde{D}_z$ for which the two interior circles $\tilde{C}_1$ and $\tilde{C}_2$ are

$$|z| = \rho, \qquad |z - D| = Q,$$

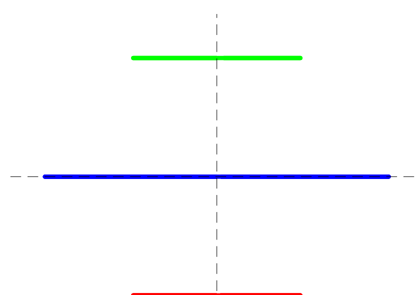
where D and Q are reported on Exercise Sheet 1. Show that

$$w = -\frac{\mathrm{i}}{a}\tilde{K}(z, -a) - \frac{\mathrm{i}}{a}\tilde{K}(z, -1/a),$$

where

$$a = \frac{(1 + \delta^2 - q^2) - [(1 + \delta^2 - q^2)^2 - 4\delta^2]^{1/2}}{2\delta}$$

maps $\tilde{D}_z$ to the unbounded region exterior to three flat plates parallel to the real w axis and stacked parallel to the imaginary w axis with the upper and lower slit being of identical length. Write down a condition on q and d such that all plates have the same length.



Three symmetric and vertically stacked slits (Exercise 7)