Exercise 1

Consider the $\lambda$-dependent linear system

$$(I - \lambda^{-1}H_{100})x = b, \quad \lambda > 0,$$

where $H_{100}$ is the $100 \times 100$ Hilbert matrix (see page 58 of the text). In MATLAB `hilb(100)` will return this matrix. For large enough $\lambda$ it can be shown that

$$\|x_n - x\|_2 \to 0 \text{ as } n \to \infty \text{ where } x_n = b + \sum_{i=1}^{n}(\lambda^{-1}H_{100})^ib.$$

We can approximate the solution $x$ by $x_n$ and this type of behavior is called \textit{convergence in norm}.

1. With $\lambda = 5$ and

$$b = \left[ \begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array} \right]^T,$$

evaluate $\|x - x_{10}\|_2$ (in MATLAB `norm(x)` returns the 2-norm of $x$).

2. How large does $n$ need to be so that $\|x - x_n\|_2 < 10^{-10}$?

3. If $\lambda$ is too small $x_n$ will not converge in norm. By restricting $\lambda$ to the integers, find the smallest value of $\lambda$ such that $x_n$ still converges in norm. Note: to investigate this you’ll need to vary $n$.

Please upload the main algorithm needed to compute $x_n$ for this problem. You don’t need to include every detail in your uploaded code.

The following exercises should be done by hand, showing all steps.

Exercise 2

Olver & Shakiban—2.5.14
Exercise 3
Olver & Shakiban—2.5.21b

Exercise 4
Olver & Shakiban—2.5.30

Exercise 5
Olver & Shakiban—3.1.7

Exercise 6
Olver & Shakiban—3.2.6

Exercise 7
Olver & Shakiban—3.3.11