Contents

Preface v

I Riemann–Hilbert problems 1

1 Classical applications of Riemann–Hilbert problems 3
- 1.1 Error function . . . 5
- 1.2 Elliptic integrals . . . 7
- 1.3 Airy function . . . 9
- 1.4 Monodromy . . . 11
- 1.5 Jacobi operators and orthogonal polynomials . . . 14
- 1.6 Spectral analysis of Schrödinger operators . . . 17

2 Riemann–Hilbert Problems 23
- 2.1 Precise statement of an RH problem . . . 23
- 2.2 Hölder theory of Cauchy integrals . . . 25
- 2.3 The solution of scalar Riemann–Hilbert problems . . . 34
- 2.4 The solution of some matrix Riemann–Hilbert problems . . . 40
- 2.5 Hardy spaces and Cauchy integrals . . . 45
- 2.6 Sobolev spaces . . . 57
- 2.7 Singular integral equations . . . 62
- 2.8 Additional considerations . . . 78

3 Inverse Scattering and NSD 87
- 3.1 The inverse scattering transform . . . 87
- 3.2 Nonlinear steepest descent . . . 97

II Numerical Solution of Riemann–Hilbert problems 107

4 Approximating Functions 109
- 4.1 The discrete Fourier transform . . . 109
- 4.2 Chebyshev series . . . 115
- 4.3 Mapped series . . . 118
- 4.4 Vanishing bases . . . 121

5 Numerical Cauchy Transforms 125
- 5.1 Convergence of approximation of Cauchy transforms . . . 126
- 5.2 The unit circle . . . 128

5.3 Case study: computing erfc 130
5.4 The unit interval and square root singularities 131
5.5 Case study: calculating elliptic integrals 136
5.6 Smooth functions on the unit interval 137
5.7 Approximation of Cauchy transforms near endpoint singularities . . 145

6 Numerical Solution of RH Problems 157
6.1 Projection methods 158
6.2 Collocation method for RH problems 163
6.3 Case study: Airy equation 169
6.4 Case study: Monodromy of a ODE with three singular points 172

7 Uniform Approximation Theory 175
7.1 A numerical Riemann–Hilbert framework 177
7.2 Uniform approximation 183
7.3 A collocation method realization 189

III The Computation of Nonlinear Special Functions 193

8 The KdV and mKdV Equations 195
8.1 The modified Korteweg–de Vries equation 205
8.2 The Korteweg–de Vries equation 211
8.3 Uniform approximation 231

9 The Focusing and Defocusing NLS equations 235
9.1 Integrability and Riemann–Hilbert problems 236
9.2 Numerical direct scattering 238
9.3 Numerical inverse scattering 240
9.4 Robin boundary conditions 246
9.5 Singular solutions 251
9.6 Uniform approximation 252

10 The Painlevé II Transcendents 257
10.1 Positive x, $s_2 = 0$ and $0 \leq 1 - s_1 s_3 \leq 1$ 260
10.2 Negative x, $s_2 = 0$ and $1 - s_1 s_3 > 0$ 261
10.3 Negative x, $s_2 = 0$ and $s_1 s_3 = 1$ 267
10.4 Numerical results 269

11 The Finite-Genus Solutions of the Korteweg–de Vries Equation 273
11.1 Riemann surfaces 274
11.2 The finite-genus solutions of the KdV equation 278
11.3 From a Riemann surface of genus g to the cut plane 282
11.4 Regularization 283
11.5 A Riemann–Hilbert problem with smooth solutions 288
11.6 Numerical computation 295
11.7 Analysis of the deformed and regularized RH problem 305
11.8 Uniform approximation 308

12 The Dressing Method and Nonlinear Superposition 311
12.1 A numerical dressing method for the KdV equation 312

12.2 A numerical dressing method for the defocusing NLS equation 324

IV Appendices 331

A Function Spaces and Functional Analysis 333
A.1 Banach spaces 333
A.2 Linear operators 340
A.3 Matrix-valued functions 342

B Fourier and Chebyshev Series 345
B.1 Fourier series 345
B.2 Chebyshev series 349

C Complex Analysis 359
C.1 Inferred analyticity 359

D Rational Approximation 363
D.1 Bounded contours 363
D.2 Lipschitz graphs 364

E Additional KdV Results 373
E.1 Comparison with existing numerical methods 373
E.2 The KdV g-function 375

Bibliography 379

Notation and Abbreviations 385

Index 391

Preface

This book grew out of the collaboration of the authors, which began in the Spring of 2010, and the first author's PhD dissertation. The second author developed much of the theory in Part II during his Junior Research Fellowship at St. John's College in Oxford, applying it to the Painlevé II equation in the non-asymptotic regime. The authors, together with Bernard Deconinck, then developed the methodology presented in Chapter 8 on the Korteweg-de Vries (KdV) equation. The accuracy that is observed begged for an explanation, leading to the framework in Chapter 7. Around the same time, the approaches for the nonlinear Schrödinger (NLS) equations, the Painlevé II transcendents and the finite-genus solutions of the KdV equation were developed. These applications along with the original KdV methodology make up Part III. Motivated by the difficulty in finding a comprehesive, beginning graduate-level reference for Riemann–Hilbert problems (RH problems) that included the L^2 theory of singular integrals, the first author compiled much of Part I during his PhD studies at the University of Washington.

Central to the philosophy of this book is the question: what does it mean to "solve" an equation? The most basic answer is to establish *existence* — the equation has at least one solution — and *uniqueness* — the equation has one and only one solution. A more concrete answer is that an equation is solved if its solution can be evaluated, typically by approximation in a reliable and efficient way. This can be accomplished via asymptotics: the solution to the equation is given by an approximation that improves with accuracy in certain parameter regimes. Otherwise, the solution can be evaluated by numerics: a sequence of approximations that converge to the true solution. In the case of linear partial differential equations (PDEs), standard solutions are given as integral representations obtained via, say, Fourier series or Green's functions. Integral representations are preferable to the original PDE because they satisfy all the properties of a "solution" to the equation:

1. *Existence* and *uniqueness* generally follow directly from the well-understood integration theory.

2. *Asymptotics* are achievable via the *method of stationary phase* or the *method of steepest descent*.

3. *Numerics* for integrals has a well-developed theory, and the integral representations can be readily evaluated via quadrature.

In place of integral representations, fundamental integrable nonlinear ordinary differential equations (ODEs) and PDEs have a RH problem representation. RH problems are boundary-value problems for piecewise (or sectionally) analytic functions in the complex plane. Our goal is to solve these integrable nonlinear ODEs and PDEs by utilizing their RH problem representations in a manner analogous to integral representations. In some cases, we use RH problems to establish existence and uniqueness, as well as derive asymptotics. But most importantly, we want to be able to accurately evaluate the solutions inside and outside of asymptotic regimes with a unified numerical approach.

The stringent requirements we put into our definition of a "solution" forces all solutions we find to be in a very particular category: *nonlinear special functions*. A *special function* is shorthand for a mathematical function which arises in many physical, biological or computational settings or in a variety of mathematical settings. A nonlinear special function is a special function arising from a fundamentally nonlinear setting. For centuries, mathematicians have been studying special functions. An important feature that separates special functions from other elementary functions is that they generically take a *transcendental*[1] form.

The catenary, discovered by Leibniz, Huygens and Bernoulli in the 1600s, describes the shape of a freely-hanging rope in terms of the hyperbolic cousin of the cosine function. The study of special transcendental functions continued with the discovery of the Airy and Bessel functions which share similar but more complicated series representations when compared to the hyperbolic cosine function. These series representations are oft derived using a differential equation that is satisfied by the given function. Such a derivation succeeds in many cases when the differential equation is linear.

The 19th century was a golden age for special function theory. Techniques from the field of complex analysis were invoked to study the so-called elliptic functions. These functions are of a fundamentally nonlinear nature: elliptic functions are solutions of nonlinear differential equations. The early twentieth century marked the work of Paul Painlevé and his collaborators in identifying the so-called *Painlevé transcendents*. The Painlevé transcendents are solutions of nonlinear differential equations that possess important properties in the complex plane. Independent of their mathematical properties, which are described at length in [52], the Painlevé transcendents have found use in the asymptotic study of water wave models [4, 40, 35] and in statistical mechanics [111].

Through the study of RH problems, we discuss classical special functions (the Airy function, elliptic functions, the error function), canonical nonlinear special functions (elliptic functions, the Painlevé II transcendents) and some non-canonical special functions (solutions of integrable nonlinear PDES) which we advocate for inclusion in the pantheon of nonlinear special functions based on the structure we describe.

We now present the layout of the book to guide the reader. A comprehensive table of notations is given just prior to the index and optional sections are marked with an asterisk.

- Part I contains an introduction to the applied theory of RH problems. Chapter 1 contains a survey of applications where RH problems arise. Then Chapter 2 contains the classical development of the theory of Cauchy transforms of Hölder continuous functions. This theory is used to explicitly solve many scalar RH problems. Lebesgue and Sobolev spaces are used to develop the theory of singular integral equations to deal with the matrix, or non-commutative, case. Some of these results are new while many others are compiled from a multitude of references. Finally, the method of nonlinear steepest descent developed by P. Deift and X. Zhou is reviewed in a simplified form in Chapter 3. On first reading, many of the proofs in this part can be omitted.

[1]In this context transcendental means that the function cannot be expressed as a finite number of algebraic steps, including rational powers, applied to a variable or variables [62].

- Part II contains a detailed development of the numerical methodology used to approximate the solutions of RH problems. While there is certainly some dependence of Part II on Part I, for the more numerically inclined, it can be read mostly independent because the dependencies are made clear. Many aspects of computational/numerical complex analysis are discussed in detail in Chapter 4 including convergence of trigonometric, Laurent and Chebyshev interpolation. The computation of Cauchy transforms is treated in Chapter 5. This numerical approach to Cauchy transforms is utilized in Chapter 6 to construct a collocation method for solving RH problems numerically. Finally, a uniform approximation theory that allows one to estimate the derivatives of solutions of RH problems is presented in Chapter 7.

- Part III contains the applications of the theory of Part II to specific integrable equations. Each of the chapters in Part III depends significantly on the material in Part I, specifically, Chapter 3 and on nearly all of Part II. Part III is written in a way that is appropriate for a reader interested only in numerical results wishing to understand the scope and applications of the methodology. As mentioned above, the applications are to the KdV equation (Chapter 8), the NLS equations (Chapter 9), the Painlevé II transcendents (Chapter 10), the finite-genus solutions of the KdV equation (Chapter 11) and the so-called dressing method (Chapter 12) applied to both the KdV and NLS equations to nonlinearly superimpose rapidly decaying solutions and periodic and quasi-periodic finite-genus solutions.

We would like to acknowledge the important contributions to this work of the first author's PhD advisor, Bernard Deconinck. We would also like to acknowledge the encouragement and input of Percy Deift, Ioana Dumitriu, Anne Greenbaum, Katie Oliveras, Bob O'Malley, Randy LeVeque, Natalie Sheils, Olga Trichtchenko, Ben Segal, Chris Swierczewski, and Vishal Vasan.

We sincerely hope the reader finds this to be a valuable resource.

Thomas Trogdon and Sheehan Olver

Part I

Riemann–Hilbert problems

Chapter 1

Classical applications of Riemann–Hilbert problems

A fundamental theme of complex analysis is that an analytic function can be uniquely expressed in terms of its behavior at the boundary of its domain of analyticity. A basic example of this phenomena is the partial fraction expansion of a rational function, which states that any rational function can be expressed as a sum of (finite) Laurent series around its poles. Consider the case of a rational function with only simple poles that is bounded at ∞. We can reinterpret the partial fraction result as a solution to the following problem:

Problem 1.0.1. *Given distinct poles* $\{z_1, \ldots, z_m\}$, *a normalization point* $z_0 \in \mathbb{C} \cup \{\infty\}$, *a normalization constant* $c_0 \in \mathbb{C}$, *and residues* $\{A_1, \ldots, A_m\}$, *find* $r : \mathbb{C} \setminus \{z_1, \ldots, z_m\} \to \mathbb{C}$ *that satisfies the following:*

1. $r(z)$ *is analytic off* $\{z_1, \ldots, z_m\}$,
2. $r(z)$ *is bounded at* ∞ *and has at most simple pole singularities throughout the complex plane:*
$$\lim_{\tau \to z} |(\tau - z) r(z)| < \infty \text{ for all } z \in \mathbb{C},$$
3. $r(z_0) = c_0$, *and*
4. $r(z)$ *satisfies the residue conditions*
$$\operatorname{Res}_{z=z_k} r(z) = A_k \quad \text{for} \quad k = 1, \ldots, m.$$

The conditions of this problem are sufficient to uniquely determine r: if there existed another solution $r^\star$, then their difference is bounded and analytic (by Theorem C.3), vanishes at z_0 and thence is zero by Liouville's theorem (Theorem C.2).

If an analytic function is not rational, then it necessarily has more exotic singularities; typically, branch cuts and essential singularities. However, in many instances we can build up analogues of Problem 1.0.1 using *Riemann–Hilbert (RH) problems* to uniquely determine a function by its behavior at such singularities. Loosely speaking, an RH problem consist of finding a *sectionally analytic function* in terms of its behavior at the boundary of regions of analyticity. For z in some contour Γ, we use the notation $\Phi^+(s)$ and $\Phi^-(s)$ to refer to the boundary values that a function $\Phi(z)$ takes as z approaches s from the left and right of Γ, respectively. The prototypical RH problem takes the

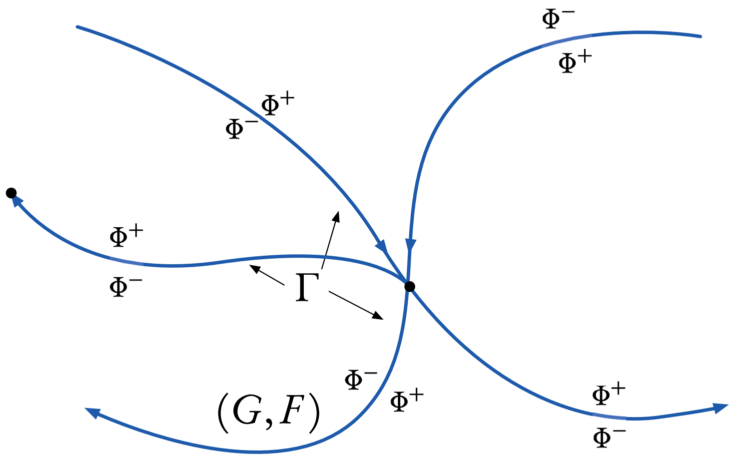

Figure 1.1. *A generic oriented jump contour* Γ *with the pair of jump functions* (G,F)*. When* $F=0$ *we specify the jump with* G *alone. The black dots signify (finite) endpoints of the contours.*

following form[2]:

Problem 1.0.2. *Given a contour* Γ*, a normalization point* $z_0 \in \mathbb{C}$*, a normalization constant* $C_0 \in \mathbb{C}^{m\times n}$*, and jump functions* $G:\Gamma \to \mathbb{C}^{m\times m}$ *(also called the* jump matrix*) and* $F:\Gamma\to\mathbb{C}^{m\times n}$*, find* $\Phi:\mathbb{C}\setminus\Gamma\to\mathbb{C}^{m\times n}$ *that satisfies the following:*

1. $\Phi(z)$ *is analytic off* Γ *(in each component of* $\mathbb{C}\setminus\Gamma$*),*
2. $\Phi(z)$ *is bounded at* ∞ *and has weaker than pole singularities throughout the complex plane,*
3. $\Phi(z_0) = C_0$*, and*
4. $\Phi(z)$ *satisfies the* jump condition

$$\Phi^+(s) = \Phi^-(s)G(s) + F(s) \quad \textit{for} \quad s \in \Gamma.$$

See Figure 1.1 for a schematic of a generic RH problem.

Remark 1.0.1. *When* $F(s)$ *is zero, we use the notation* $[G;\Gamma]$ *to refer to this RH problem, with appropriate additional regularity properties, see Section 2.7.*

Remark 1.0.2. *When the RH problem is scalar (*$m=n=1$*), we use lower case variables:* e.g., ϕ *satisfies the jump condition*

$$\phi^+(s) = \phi^-(s)g(s) + f(s).$$

In this chapter, we invoke the philosophy of A. R. Its [64] and demonstrate how

[2]The definition of a RH problem is made precise in Definition 2.38.

several classical problems can be reduced to RH problems. These fall under three fundamental categories:

1. Integral representations: we obtain simple, constant coefficient scalar RH problems for error functions and elliptic integrals.
2. Differential equations: we obtain matrix RH problems that encode the Airy function, and the solution to inverse monodromy problems.
3. Spectral analysis of operators: we construct matrix RH problems that encode the potential of a Schrödinger operator and the entries of a Jacobi operator in terms of their spectral data.

Unlike the partial fraction case, existence and uniqueness of such problems is a delicate question, which we defer until Chapter 2. However, for the purposes of this chapter a simple rule of thumb suffices: if G and F are bounded and piecewise continuous, and the winding number[3] of $\det G(s)$ is zero, we expect the RH problem to be solvable even if the solution is not unique[4].

1.1 ▪ Error function: from integral representation to Riemann–Hilbert problem

The *complementary error function* is defined by

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^\infty \mathrm{e}^{-s^2} \mathrm{d}s.$$

This is an entire function in z, and behaves like a smoothed step function on the real axis, with $\operatorname{erfc}(\infty) = 0$ and

$$\operatorname{erfc}(-\infty) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^\infty \mathrm{e}^{-x^2} \mathrm{d}x = 2.$$

The error function is important in statistics, the cumulative density function (CDF) of a Gaussian random variable is given by $1 - \frac{1}{2}\operatorname{erfc} x$, and it is a critical tool in describing asymptotics of solutions to differential equations near essential singularities [84].

One can certainly calculate error functions via quadrature: for a quadrature rule

$$\int_0^\infty f(x)\mathrm{d}x \approx \sum_{k=1}^n w_k f(x_k),$$

approximate

$$\operatorname{erfc} z \approx \sum_{k=1}^n w_k \mathrm{e}^{-(x_k+z)^2}.$$

However, the integrand becomes increasingly oscillatory as the imaginary part of z becomes large, so that an increasing number of quadrature points are required to resolve the oscillations.

[3]The winding number is the number of times an image of a function wraps around the origin. Technically speaking, a zero winding number in this context is not sufficient for existence and uniqueness but we defer these details to Chapter 2, see also Section 8.2.4.

[4]Typically, the solution can be made unique by imposing additional conditions.

An effective alternative to quadrature rules applied to the integral representation is to apply numerical methods to a RH problem. To this end, we reduce the integral representation to a simple RH problem. The first stage is to manipulate $\operatorname{erfc} z$ so that it has nice behavior at $+\infty$. We can determine its asymptotic behavior via integration by parts:

$$\int_z^\infty \mathrm{e}^{-s^2}\mathrm{d}s = -\int_z^\infty \left(\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{e}^{-s^2}}{2s}\right] + \frac{\mathrm{e}^{-s^2}}{2s^2}\right)\mathrm{d}s$$
$$= \frac{\mathrm{e}^{-z^2}}{2z} - \int_z^\infty \frac{\mathrm{e}^{-s^2}}{2s^2}\mathrm{d}s.$$

The second integral satisfies, for $\operatorname{Re} z \geq 0$ and $|z| > 1$,

$$\left|\mathrm{e}^{z^2}\int_z^\infty \frac{\mathrm{e}^{-s^2}}{2s^2}\mathrm{d}s\right| = \left|\int_0^\infty \frac{\mathrm{e}^{-2zx-x^2}}{2(x+z)^2}\mathrm{d}x\right| \leq \frac{1}{2|z|^2}\int_0^\infty \mathrm{e}^{-x^2}\mathrm{d}x = \frac{1}{4|z|^2}.$$

It follows that $\mathrm{e}^{z^2}\operatorname{erfc} z$ is bounded in the right half plane (boundedness for $|z| \leq 1$ follows from analyticity), and decays as $z \to +\infty$. Similarly, we can construct a companion function by integrating from $-\infty$ and normalizing: we have

$$\mathrm{e}^{z^2}(2 - \operatorname{erfc} z) = \frac{2}{\sqrt{\pi}}\mathrm{e}^{z^2}\int_{-\infty}^z \mathrm{e}^{-s^2}\mathrm{d}s$$

is bounded in the left half plane, and decays as $z \to -\infty$.

Combining these two functions, we can construct a sectionally analytic function

$$\psi(z) = \begin{cases} \mathrm{e}^{z^2}(2 - \operatorname{erfc} z), & \text{if } \operatorname{Re} z < 0, \\ -\mathrm{e}^{z^2}\operatorname{erfc} z, & \text{if } \operatorname{Re} z > 0. \end{cases}$$

This has a discontinuity along the imaginary axis, and we can encode the jump along this discontinuity via the following scalar RH problem:

Problem 1.1.1. *Find $\psi : \mathbb{C} \setminus \mathrm{i}\mathbb{R} \to \mathbb{C}$ that satisfies the following:*

1. *$\psi(z)$ is analytic off the imaginary axis,*
2. *$\psi(z)$ is bounded throughout the complex plane,*
3. *$\psi(z)$ decays at ∞, and*
4. *for y on the branch cut along the imaginary axis (oriented from $-\mathrm{i}\infty$ to $\mathrm{i}\infty$), it satisfies the jump*
$$\psi^+(y) - \psi^-(y) = \frac{2}{\sqrt{\pi}}\mathrm{e}^{y^2}\int_{-\infty}^\infty \mathrm{e}^{-x^2}\mathrm{d}x = 2\mathrm{e}^{y^2},$$
where $\psi^+(y) = \lim_{\epsilon\downarrow 0}\psi(y-\epsilon)$ and $\psi^-(y) = \lim_{\epsilon\downarrow 0}\psi(y+\epsilon)$ are the limits from the left and right, respectively.

Figure 1.2 depicts the jump contours and jump functions, Figure 1.3 depicts the solution.

This RH problem has a unique solution (Problem 2.3.1) and solving the RH problem *globally throughout the complex plane* can be accomplished via a single fast Fourier transform (Section 5.3).

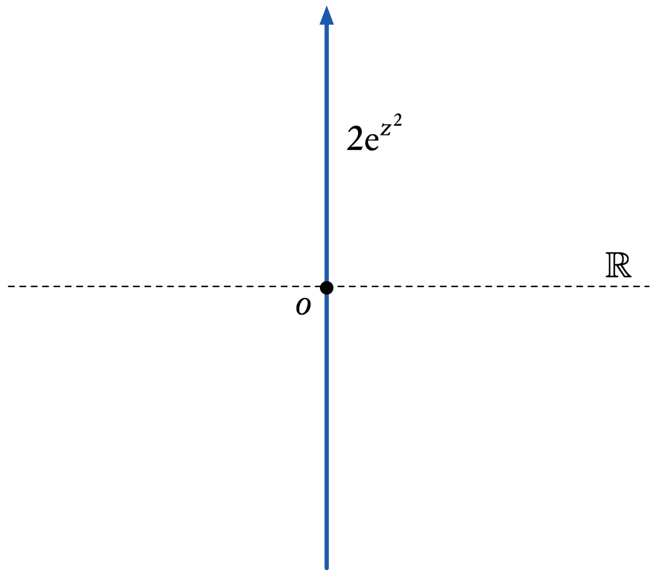

Figure 1.2. *The jump contour and jump function* $G(z) = 2e^{z^2}$ *for the complementary error function.*

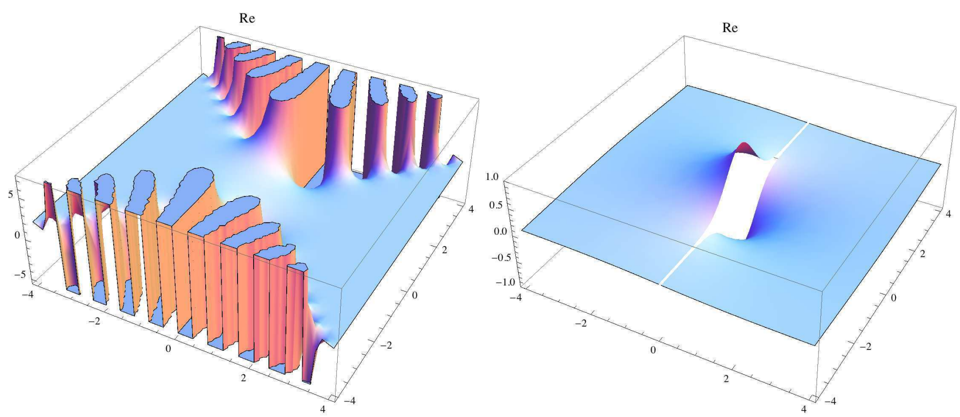

Figure 1.3. *The real part of* erfc z *(left) and the real part of* ψ *(right).*

1.2 ▪ Elliptic integrals

Define the *elliptic integral of the first kind*[5] by

$$g(z;a) = \int_0^z \frac{1}{\sqrt{1-z^2}\sqrt{a^2-z^2}} \mathrm{d}z,$$

for real $a > 1$. Here the integration path can be taken to be a straight line from the origin to z, which introduces branch cuts along $[1,\infty)$ and $(-\infty,-1]$.

[5]The usual convention is to define Legendre's elliptic integral of the first kind $F(\phi, m) \triangleq m^{-1} g(\sin\phi; m^{-1})$. However, the definition as g leads naturally to a RH problem.

Elliptic integrals arose to describe arclengths of ellipses. They form a fundamental tool in building functions on Riemann surfaces, along with their functional inverses the *Jacobi elliptic functions*. Like error functions, calculating elliptic integrals with quadrature has several issues, with the additional difficulty introduced by the singularities of the integrand at ± 1 and $\pm a$. Rephrasing g as a solution to a RH problem allows for the resolution of these difficulties.

To determine a RH problem for g, we first construct a RH problem for the integrand

$$\psi(z) = \frac{1}{\sqrt{1-z^2}\sqrt{a^2-z^2}} = \frac{1}{\sqrt{1-z}\sqrt{1+z}\sqrt{a-z}\sqrt{a+z}}.$$

The square root (with standard choice of branch) satisfies a multiplicative jump for $x < 0$ of

$$\sqrt{x}^{+} = -\mathrm{i}\sqrt{|x|} = -\sqrt{x}^{-}.$$

This implies that $\psi^+(x) = \psi^-(x)$ for $x > a$ and $x < -a$, *i.e.*, $\psi(x)$ is continuous and hence analytic (by Theorem C.11) on $[a, \infty)$ and $(-\infty, -a]$. On the remaining branch cuts along $[-a, -1]$ and $[a, 1]$, it satisfies the jump

$$\psi_+(x) + \psi_-(x) = 0 \quad \text{for} \quad -a < x < -1 \quad \text{and} \quad 1 < x < a. \tag{1.1}$$

Returning to g, since the singularities of ψ are integrable, we have the limit of g from above and below for $x \geq 0$ given by

$$g_\pm(x) = \int_0^x \psi_\pm(x)\mathrm{d}x = \begin{cases} \int_0^x \psi(x)\mathrm{d}x, & \text{if } 0 \leq x \leq 1, \\ \int_0^1 \psi(x)\mathrm{d}x + \int_1^x \psi_\pm(x)\mathrm{d}x, & \text{if } 1 \leq x \leq a, \\ \int_0^1 \psi(x)\mathrm{d}x + \int_1^a \psi_\pm(x)\mathrm{d}x + \int_a^x \psi(x)\mathrm{d}x, & \text{if } a \leq x. \end{cases}$$

Define the *complete Elliptic integral*[6]

$$K(s) \triangleq \int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-s^2x^2}}\mathrm{d}x. \tag{1.2}$$

We use the cancellation of the integrand to determine, for $1 \leq x \leq a$,

$$g_+(x) + g_-(x) = \int_0^x \left[\psi_+(x) + \psi_-(x)\right]\mathrm{d}x = 2\int_0^1 \psi(x)\mathrm{d}x = 2\frac{K(a^{-1})}{a}.$$

Similarly, the analyticity of ψ between $[0,1]$ and $[1,\infty)$ tells us, for $x > a$,

$$g_+(x) - g_-(x) = \int_0^x \left[\psi_+(x) - \psi_-(x)\right]\mathrm{d}x = 2\int_1^a \psi_+(x)\mathrm{d}x = 2\mathrm{i}\frac{K\left(\sqrt{1-a^{-2}}\right)}{a},$$

where the final expression follows from applying the change of variables $x = \frac{1}{\sqrt{1-(1-a^{-2})t^2}}$ and simplifying. From the symmetry relationship $g(-z) = -g(z)$, we determine the equivalent jumps on the negative real axis. We thus arrive at a RH problem:

[6] We use the convention of [84, Chapter 19], which differs from MATHEMATICA's `EllipticK` routine: `EllitpicK`$(s^2) \equiv K(s)$.

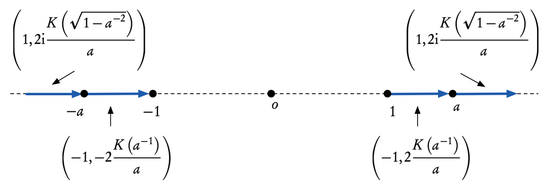

Figure 1.4. *The jump contours and jump functions for the Jacobi elliptic integral RH problem.*

Problem 1.2.1. *Find* g *that satisfies the following:*

1. $g(z)$ *is analytic off* $[1,\infty)\cup(-\infty,-1]$,
2. $g(z)$ *is bounded throughout the complex plane,*
3. $g(0)=0$, *and*
4. *on the branch cuts it satisfies the jumps*

$$\begin{aligned} g_+(x)+g_-(x) &= 2\frac{K(a^{-1})}{a} \quad \textit{for} \quad 1<x<a, \\ g_+(x)+g_-(x) &= -2\frac{K(a^{-1})}{a} \quad \textit{for} \quad -a<x<-1, \\ g_+(x)-g_-(x) &= 2\mathrm{i}\frac{K\left(\sqrt{1-a^{-2}}\right)}{a} \quad \textit{for} \quad a<x, \\ g_+(x)-g_-(x) &= 2\mathrm{i}\frac{K\left(\sqrt{1-a^{-2}}\right)}{a} \quad \textit{for} \quad x<-a. \end{aligned}$$

See Figure 1.4 for the jump contours and jump functions.

We solve this RH problem numerically, giving an approximation to the elliptic integral that is accurate throughout the complex plane, see Section 5.5.

1.3 ▪ Airy function: from differential equation to Riemann–Hilbert problem

RH problems can also be seen as a way to recover solutions to differential equations from their asymptotic behavior. The canonical example is the *Airy equation*

$$y''(z)=zy(z).$$

Liouville–Green approximation (or WKB approximation) informs us that along any given direction for which z approaches ∞, there are two linearly independent solutions that satisfy the asymptotic behavior

$$z^{-1/4}\mathrm{e}^{\pm\frac{2}{3}z^{3/2}}.$$

However, the well-known *Stokes' phenomenon* says that the asymptotic behavior of a single solution changes depending on the sector of the complex plane in which z approaches

∞.

A canonical particular solution is the *Airy function*, which satisfies

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} \mathrm{e}^{-\frac{2}{3}z^{3/2}} \quad \text{for} \quad -\frac{2\pi}{3} \leq \arg z \leq \frac{2\pi}{3}$$

uniformly[7]. By plugging into the ODE, we see that $\mathrm{Ai}(\omega z)$ and $\mathrm{Ai}(\omega^2 z)$ are also solutions to the Airy equation, for $\omega = \mathrm{e}^{-\frac{2\mathrm{i}\pi}{3}}$. We can deduce asymptotics of these from the asymptotics of the Airy function, with a bit of care taken due to the branch cut of the asymptotic formula:

$$\omega \mathrm{Ai}(\omega z) \sim -\frac{z^{-1/4}}{2\sqrt{\pi}} \begin{cases} \mathrm{i}\mathrm{e}^{\frac{2}{3}z^{3/2}}, & \text{if } 0 \leq \arg z \leq \pi, \\ \mathrm{e}^{-\frac{2}{3}z^{3/2}}, & \text{if } -\pi \leq \arg z \leq -\frac{2\pi}{3}, \end{cases}$$
$$\omega^2 \mathrm{Ai}(\omega^2 z) \sim \frac{z^{-1/4}}{2\sqrt{\pi}} \begin{cases} -\mathrm{e}^{-\frac{2}{3}z^{3/2}}, & \text{if } \frac{2\pi}{3} \leq \arg z \leq \pi, \\ \mathrm{i}\mathrm{e}^{\frac{2}{3}z^{3/2}}, & \text{if } -\pi \leq \arg z \leq 0. \end{cases}$$

We choose two linearly independent solutions in each sector

$$y(z) = \begin{cases} \left[-\omega \mathrm{Ai}(\omega z),\ -\mathrm{i}\omega^2 \mathrm{Ai}(\omega^2 z)\right], & \text{if } -\pi < \arg z < -\frac{2\pi}{3}, \\ \left[\mathrm{Ai}(z),\ -\mathrm{i}\omega^2 \mathrm{Ai}(\omega^2 z)\right], & \text{if } -\frac{2\pi}{3} < \arg z < 0, \\ \left[\mathrm{Ai}(z),\ \mathrm{i}\omega \mathrm{Ai}(\omega z)\right], & \text{if } 0 < \arg z < \frac{2\pi}{3}, \\ \left[-\omega^2 \mathrm{Ai}(\omega^2 z),\ \mathrm{i}\omega \mathrm{Ai}(\omega z)\right], & \text{if } \frac{2\pi}{3} < \arg z < \pi, \end{cases}$$

so that

$$y(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} \left[\mathrm{e}^{-\frac{2}{3}z^{3/2}},\ \mathrm{e}^{\frac{2}{3}z^{3/2}}\right]$$

throughout the complex plane.

We now determine the jumps of y using the symmetry relationship

$$\mathrm{Ai}(z) + \omega \mathrm{Ai}(\omega z) + \omega^2 \mathrm{Ai}(\omega^2 z) = 0.$$

This relationship holds true since $\mathrm{Ai}(z) + \omega \mathrm{Ai}(\omega z) + \omega^2 \mathrm{Ai}(\omega^2 z)$ also solves the Airy equation with zero initial conditions:

$$\begin{aligned} (1 + \omega + \omega^2)\mathrm{Ai}(0) &= \left(1 + 2\cos\frac{2\pi}{3}\right)\mathrm{Ai}(0) = 0 \\ (1 + \omega^2 + \omega^4)\mathrm{Ai}'(0) &= 0. \end{aligned}$$

[7]This asymptotic behavior can be extended to $|\arg z| \leq \pi - \delta$ for any δ, however, we only use the asymptotics in the stated sector.

It follows that y satisfies the following jumps:

$$y_+(x) = y_-(x) \begin{bmatrix} 1 & -\mathrm{i} \\ 0 & 1 \end{bmatrix} \quad \text{for} \quad x \in (0,\infty),$$
$$y_+(s) = y_-(s) \begin{bmatrix} 1 & 0 \\ -\mathrm{i} & 1 \end{bmatrix} \quad \text{for} \quad s \in (0, \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\infty),$$
$$y_+(s) = y_-(s) \begin{bmatrix} 1 & 0 \\ -\mathrm{i} & 1 \end{bmatrix} \quad \text{for} \quad s \in (0, \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\infty),$$
$$y_+(x) = y_-(x) \begin{bmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix} \quad \text{for} \quad x \in (-\infty, 0).$$

We now remove the asymptotic behavior

$$\phi(z) = 2\sqrt{\pi} z^{1/4} y(z) \begin{bmatrix} \mathrm{e}^{\frac{2}{3}z^{3/2}} & 0 \\ 0 & \mathrm{e}^{-\frac{2}{3}z^{3/2}} \end{bmatrix},$$

so that $\phi(z) \sim [1,\ 1]$ as $z \to \infty$. We thus obtain the following vector-valued RH problem:

Problem 1.3.1. *Find* $\phi(z) : \mathbb{C}\setminus\Gamma \to \mathbb{C}^{1\times 2}$, $\Gamma = [0,\infty) \cup (-\infty, 0] \cup [0, \mathrm{e}^{\frac{2\mathrm{i}\pi}{3}}\infty) \cup [0, \mathrm{e}^{-\frac{2\mathrm{i}\pi}{3}}\infty)$, *that satisfies the following:*

1. $\phi(z)$ *is analytic off* Γ,
2. $\phi(z)$ *has weaker than pole singularities throughout the complex plane,*
3. $\phi(z) \sim [1,\ 1]$ *at* ∞, *and*
4. *on* Γ, ϕ *satisfies the jumps*

$$\phi_+(x) = \phi_-(x) \begin{bmatrix} 1 & -\mathrm{i}\mathrm{e}^{-\frac{4}{3}x^{3/2}} \\ 0 & 1 \end{bmatrix} \quad \textit{for} \quad x \in (0,\infty),$$
$$\phi_+(s) = \phi_-(s) \begin{bmatrix} 1 & 0 \\ -\mathrm{i}\mathrm{e}^{\frac{4}{3}s^{3/2}} & 1 \end{bmatrix} \quad \textit{for} \quad s \in (0, \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\infty),$$
$$\phi_+(s) = \phi_-(s) \begin{bmatrix} 1 & 0 \\ -\mathrm{i}\mathrm{e}^{\frac{4}{3}s^{3/2}} & 1 \end{bmatrix} \quad \textit{for} \quad s \in (0, \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}\infty),$$
$$\phi_+(x) = \phi_-(x) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \textit{for} \quad x \in (-\infty, 0).$$

This vector-valued RH problem is solvable numerically, giving a numerical method for calculating the Airy function that is accurate throughout the complex plane, see Section 6.3.

1.4 ▪ Monodromy

Suppose we are given a second order *Fuchsian ODE*, *i.e.*,

$$Y'(z) = A(z)Y(z) = \sum_{k=1}^{r} \frac{A_k}{z - z_k} Y(z)$$

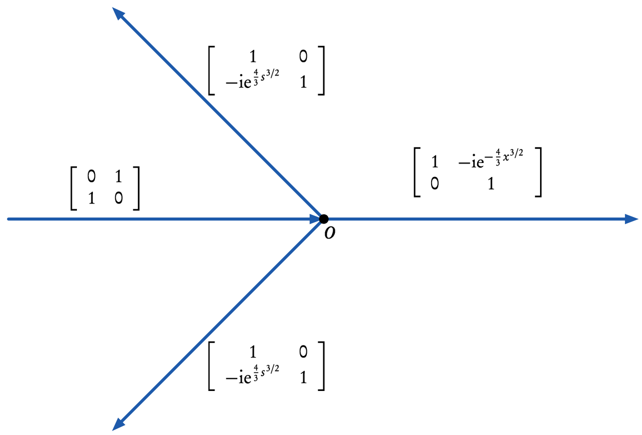

Figure 1.5. *The jump contour and jump functions for the Airy function RH problem.*

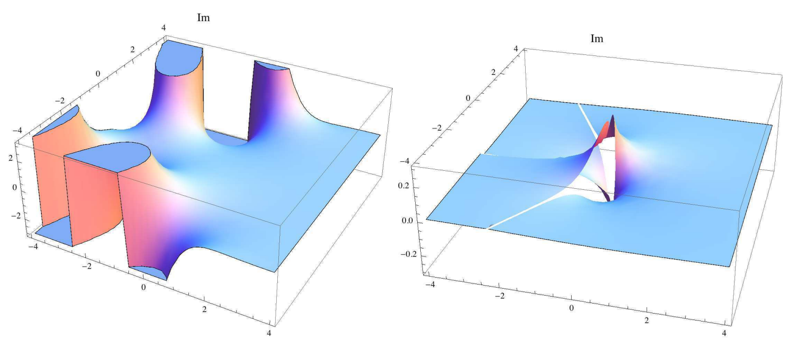

Figure 1.6. *The imaginary part of* Ai z *(left) and the imaginary part of* ϕ *(right).*

where z_k are distinct and $A_k \in \mathbb{C}^{2\times 2}$. As a concrete example, we consider the case of three singular points $z_1 = 1$, $z_2 = 2$ and $z_3 = 3$, and take $Y(0) = I$. By integrating the differential equation along a straight contour, we obtain a solution $Y(z)$ that is analytic off the contour $(1,\infty)$. For any x not equal to a singular point z_k, we define the limits from above and below:

$$Y_+(x) = \lim_{\epsilon \downarrow 0} Y(x + \mathrm{i}\epsilon) \qquad \text{and} \qquad Y_-(x) = \lim_{\epsilon \downarrow 0} Y(x - \mathrm{i}\epsilon).$$

These functions can also be considered as solutions to the differential equation obtained by integrating along arcs that avoid the singular points.

Note that $Y_+(x)$ and $Y_-(x)$ satisfy the same ODE as Y, hence we know that the columns of $Y_+(x)$ are a linear combination of the columns of $Y_-(x)$. In other words, there is a *monodromy matrix* $M_1 \in \mathbb{C}^{2\times 2}$ so that $Y_+(x) = Y_-(x)M_1$ for any $x \in (1,2)$. By analytic continuation, this relationship is satisfied for all $x \in (1,2)$. Similarly, for all $x \in (2,3)$ we have $Y_+(x) = Y_-(x)M_2$ and for all $x \in (3,\infty)$ we have $Y_+(x) = Y_-(x)M_3$.

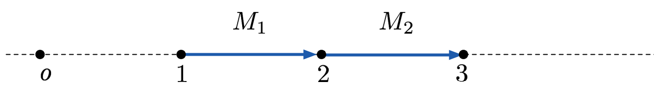

Figure 1.7. *The jump contour and jump functions for the inverse monodromy RH problem.*

Consider the special case where

$$\sum_{k=1}^{r} A_k = 0.$$

We find that that $Y(z)$ is analytic at ∞, as verified by doing a change of variables $U(z) = Y(1/z)$ so that

$$\begin{aligned} U'(z) &= -\frac{1}{z^2} Y'(1/z) = -\frac{1}{z^2} \sum_{k=1}^{r} \frac{A_k}{z^{-1} - z_k} Y(1/z) = \sum_{k=1}^{r} \frac{A_k z^{-1}}{1 - z_k z} U(z) \\ &= \sum_{k=1}^{r} A_k z^{-1}(1 + z_k z + \mathcal{O}(z^2)) U(z) = \sum_{k=1}^{r} A_k (z_k + \mathcal{O}(z)) U(z) \end{aligned}$$

has a normal point at the origin. For our special case of three singular points, this implies that $Y_+(x) = Y_-(x)$ for all $x \in (3, \infty)$, hence $M_3 = I$.

The map from $\{A_k\}$ to the monodromy matrices $\{M_k\}$ is known as the *Riemann–Hilbert correspondence*. We now consider the inverse map: recovering $\{A_k\}$ from $\{M_k\}$. Observe that the solution Y can be described by a RH problem given in terms of the monodromy matrices:

Problem 1.4.1. *Find* $Y(z) : \mathbb{C} \setminus [1,3] \to \mathbb{C}^{2\times 2}$ *that satisfies the following:*

1. $Y(z)$ *is analytic off* $[1,3]$,
2. $Y(z)$ *has weaker than pole singularities throughout the complex plane,*
3. $Y(0) = I$, *and*
4. Y *satisfies the jump*

$$\begin{aligned} Y_+(x) &= Y_-(x) M_1 \quad \textit{for} \quad x \in (1,2), \\ Y_+(x) &= Y_-(x) M_2 \quad \textit{for} \quad x \in (2,3). \end{aligned}$$

Hilbert's twenty-first problem essentially posed the question whether one can uniquely recover $\{A_k\}$ from the monodromy matrices $\{M_k\}$. If we can solve this RH problem, then we know that

$$A(z) = Y'(z) Y(z)^{-1},$$

and if this is rational then we can recover A_k from the Laurent series of A around each singular point. We solve this problem numerically in Section 6.4

Remark 1.4.1. *Whether the RH problem is always solvable with $A(z)$ rational is a delicate question. For the case of three or fewer singular points, the problem is solvable in terms of hypergeometric functions. The case of four or more singular points is substantially more difficult to solve directly and requires Painlevé transcendents, see [52, pp.80]. In the general case, it was originally answered in the affirmative by Plemelj [94], however, there was a flaw in the argument restricting its applicability to the case where at least one M_k is diagonalizable. Counter examples were found by Bolibrukh [15], see also [6]. On the other hand, setting up and numerically solving an RH problem is straightforward regardless of the number of singular points.*

1.5 ▪ Jacobi operators and orthogonal polynomials

RH problems can be used for *inverse spectral problems*. Our first example is the family of *Jacobi operators*:

$$J = \begin{bmatrix} a_0 & \sqrt{b_1} & & & \\ \sqrt{b_1} & a_1 & \sqrt{b_2} & & \\ & \sqrt{b_2} & a_2 & \sqrt{b_3} & \\ & & \ddots & \ddots & \ddots \\ & & & & \end{bmatrix} \tag{1.3}$$

We assume that J is endowed with a domain space so that it is self-adjoint with respect to the ℓ^2 inner product[8]. Self-adjointness ensures that $(J-z)^{-1}$ is bounded for all z off the real axis, and the spectral theorem guarantees the existence of a *spectral measure* μ for J, a probability measure that satisfies

$$\mathbf{e}_0^\top (J-z)^{-1}\mathbf{e}_0 = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{x-z} \quad \text{for} \quad z \notin \operatorname{supp}\mu \equiv \sigma(J)$$

where $\mathbf{e}_0 = [1,0,0,\ldots]^\top$ and $\sigma(J)$ denotes the spectrum of J. The *spectral map* is the map from the operator J to its spectral measure μ.

We now consider the *inverse spectral map*: recovering the operator J given its spectral measure. The key to this inverse spectral problem are the polynomials orthogonal with respect to the weight $\mathrm{d}\mu(x)$; *i.e.*, orthogonal with respect to the inner product

$$\langle f(x), g(x)\rangle_\mu = \int_{\mathbb{R}} f(x)g(x)\mathrm{d}\mu(x).$$

We see below that the entries of the operator J are embedded in the three-term recurrence relationship that the orthogonal polynomials satisfy.

Applying the Gram–Schmidt procedure to the sequence $\{1, x, x^2, \ldots\}$ produces a se-

[8] If J is bounded on ℓ^2, then J is self-adjoint on ℓ^2. In the unbounded case, J can be viewed as the self-adjoint extension of J_0, the restriction of the Jacobi operator to acting on the space of vectors with a finite number of non-zero entries.

quence of (monic) orthogonal polynomials[9]

$$\begin{aligned}\pi_{-1}(x) &= 0,\\ \pi_0(x) &= 1,\\ \pi_{n+1}(x) &= (x-\alpha_n)\pi_n(x) - \beta_n \pi_{n-1}(x).\end{aligned}$$

Here the coefficients α_n, β_n are those in the usual *three-term recurrence relation*, given by

$$\alpha_n = \langle x\pi_n(x), \pi_n(x)\rangle_\mu \gamma_n, \text{ and } \beta_n = \langle x\pi_n(x), \pi_{n-1}(x)\rangle_\mu \gamma_{n-1}$$

for

$$\gamma_n = \langle \pi_n(x), \pi_n(x)\rangle_\mu^{-1}.$$

It is easy to see such coefficients exist using $\langle f(x), x g(x)\rangle_\mu = \langle x f(x), g(x)\rangle_\mu$. A remarkable fact, which we do not demonstrate here (see [28, p. 31]) is that $\alpha_n = a_n$ and $\beta_n = b_n$ for a large class of Jacobi operators.

Another remarkable fact is that these coefficients can be expressed in terms of a solution of a RH problem. Let $w(x)\mathrm{d}x = \mathrm{d}\mu(x)$, assuming the spectral measure has a continuous density. Then, consider the function

$$\begin{aligned}\Psi_n(z) &= \begin{bmatrix} \pi_n(z) & \mathscr{C}_\mathbb{R}[\pi_n w](z) \\ -2\pi \mathrm{i}\gamma_{n-1}\pi_{n-1}(z) & -2\pi\mathrm{i}\gamma_{n-1}\mathscr{C}_\mathbb{R}[\pi_{n-1}w](z)\end{bmatrix},\\ \mathscr{C}_\mathbb{R} f(z) &= \frac{1}{2\pi \mathrm{i}}\int_\mathbb{R} \frac{f(x)}{x-z}\mathrm{d}x,\end{aligned}$$

i.e., $\mathscr{C}_\mathbb{R} f(z)$ is the *Cauchy integral* of f. For f with sufficient smoothness, the Cauchy integral satisfies $\mathscr{C}_\mathbb{R}^+ f(x) - \mathscr{C}_\mathbb{R}^- f(x) = f(x)$ for all $x \in \mathbb{R}$, where $\mathscr{C}_\mathbb{R}^\pm f(x)$ is the limit of $\mathscr{C}_\mathbb{R} f(z)$ from above and below, as described in Lemma 2.7. It follows immediately that $\Psi_n(z)$ satisfies the following jump on the real axis:

$$\Psi_n^+(x) = \Psi_n^-(x)\begin{bmatrix} 1 & w(x)\\ 0 & 1\end{bmatrix} \quad \text{for} \quad x \in \mathbb{R}.$$

We can further use orthogonality of π_k with every lower degree polynomial to determine the asymptotic behavior as $z \to \infty$

$$\begin{aligned}\mathscr{C}_\mathbb{R}[\pi_k w](z) &= \frac{1}{2\pi\mathrm{i}}\int_\mathbb{R}\frac{\pi_k(x)}{x-z}\mathrm{d}\mu(x) = -\frac{1}{2\pi\mathrm{i}z}\int_\mathbb{R}\pi_k(x)\left(1+\frac{x}{z}+\frac{x^2}{z^2}+\cdots\right)\mathrm{d}\mu(x)\\ &= -\frac{1}{2\pi\mathrm{i}z^{k+1}}\int_\mathbb{R}\pi_k(x)x^k\left(1+\frac{x}{z}+\frac{x^2}{z^2}+\cdots\right)\mathrm{d}\mu(x)\\ &\sim -\frac{\int_\mathbb{R}\pi_k(x)x^k\mathrm{d}\mu(x)}{2\pi\mathrm{i}z^{k+1}} = -\frac{\langle \pi_k, \pi_k\rangle_\mu}{2\pi\mathrm{i}z^{k+1}} = -\frac{1}{2\pi\mathrm{i}\gamma_k z^{k+1}}.\end{aligned} \tag{1.4}$$

We thus find that $\Psi_n(z)$ has the following asymptotic behavior:

[9] It can be shown that $\mathbf{e}_0^\top J^k \mathbf{e}_0 = \int_\mathbb{R} x^k \mathrm{d}\mu(x)$, implying that all moments are finite and guaranteeing the validity of the Gram–Schmidt procedure.

$$\Psi_n(z)\begin{bmatrix} z^{-n} & 0 \\ 0 & z^n \end{bmatrix} = I + \mathcal{O}(z^{-1}) \text{ as } |z| \to \infty.$$

We can now state the inverse spectral problem in terms of the solution of a RH problem.

Problem 1.5.1. *Find* $\Psi \equiv \Psi_n : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ *that satisfies the following:*

1. $\Psi(z)$ *is analytic off* $\mathbb{R}$,
2. $\Psi(z)$ *has the asymptotic behavior*
$$\Psi(z)\begin{bmatrix} z^{-n} & 0 \\ 0 & z^n \end{bmatrix} = I + \mathcal{O}(z^{-1}) \quad \text{as} \quad |z| \to \infty, \quad \text{and}$$
3. $\Psi(z)$ *satisfies the jump*
$$\Psi^+(x) = \Psi^-(x)\begin{bmatrix} 1 & w(x) \\ 0 & 1 \end{bmatrix}, \quad x \in \mathbb{R}.$$

With some technical assumptions on $w(z)$, this RH problem is uniquely solvable and we can recover the Jacobi operator J which has spectral measure $\mathrm{d}\mu(x) = w(x)\mathrm{d}x$. This is accomplished by expanding the solution near infinity

$$\Psi(z) = \left(I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \mathcal{O}(z^{-3})\right)\begin{bmatrix} z^n & 0 \\ 0 & z^{-n} \end{bmatrix}.$$

The terms in this expansion can be determined from (1.4), from which we compute

$$(Y_1)_{12}(Y_1)_{21} = \frac{\gamma_{n-1}}{\gamma_n} = \langle \pi_n, \pi_n \rangle_\mu \gamma_{n-1} = \left\langle \pi_n, x\pi_{n-1} \right\rangle_\mu \gamma_{n-1} = \beta_n,$$

and

$$\begin{aligned} \frac{(Y_2)_{12}}{(Y_1)_{12}} - (Y_1)_{22} &= \gamma_n \left\langle x^{n+1}, \pi_n \right\rangle_\mu - \gamma_{n-1} \left\langle x^n, \pi_{n-1} \right\rangle_\mu \\ &= \gamma_n \left\langle x^n, \pi_{n+1} + \alpha_n \pi_n + \beta_n \pi_{n-1} \right\rangle_\mu - \gamma_{n-1} \left\langle x^n, \pi_{n-1} \right\rangle_\mu \\ &= \alpha_n + \gamma_n \beta_n \left\langle x^n, \pi_{n-1} \right\rangle_\mu - \gamma_{n-1} \left\langle x^n, \pi_{n-1} \right\rangle_\mu = \alpha_n. \end{aligned}$$

While this RH problem has differing asymptotic conditions than our model RH problem (Problem 1.0.2), it can be reduced to the required form using the so-called equilibrium measure arising in potential theory, see [28].

Remark 1.5.1. *The special case of solving this RH problem when* $n = 0$ *is considered in Example 2.17.*

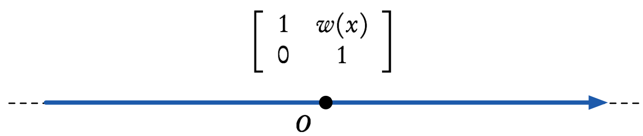

Figure 1.8. *The jump contour and jump functions for the orthogonal polynomial RH problem.*

Remark 1.5.2. *The numerical Riemann–Hilbert approach can be adapted to this setting to compute this inverse spectral map in specific cases [91, 108].*

1.6 ▪ Spectral analysis of Schrödinger operators

Our second example of an inverse spectral problem is for the time-independent *Schrödinger operator*

$$Lu = -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - V(x)u \tag{1.5}$$

where V is, for simplicity, smooth with exponential decay[10]. One initially considers L as acting on smooth, rapidly decaying functions on $\mathbb{R}$. With an appropriate extension, as in the case of Jacobi operators, L becomes self-adjoint: one replaces strong with weak differentiation and has L act on the Sobolev space $H^2(\mathbb{R})$. This particular example is of great importance for integrable systems.

We see that, similar to the spectral map for Jacobi operators, there exists a spectral map from the Schrödinger operator to *spectral data* or *scattering data*. RH problems can then be used for the *inverse spectral map*, recovering the potential V from the operators spectral data. Details on the spectrum of Schrödinger operators can be found in [67, Chapt. 5] (see also [32]).

The determination of the spectral data starts with the free Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2$, whose (essential) spectrum is the positive real axis $[0,\infty)$[11]. The operator L is a relatively compact perturbation of $-\mathrm{d}^2/\mathrm{d}^2x$, hence the spectrum of L must only differ from $-\mathrm{d}^2/\mathrm{d}^2x$ by the possible addition of discrete eigenvalues lying in $(-\infty,0)$. The assumptions made on V are sufficient to ensure that there are only a finite number of eigenvalues $\{\lambda_j\}_{j=1}^n$ [32].

The associated spectral data is defined on the spectrum, by considering the solutions to equation $Lu = \lambda u$[12]. The solution space of the free equation is spanned by $\mathrm{e}^{\pm \mathrm{i}sx}$, $s^2 = \lambda$. Thus, because V decays rapidly, we expect the solutions of (1.5) to be approximately

[10]Assuming that $|V(x)|$ is integrable and its first moment is finite is enough for most of the results we state here [32]

[11]This can be verified by considering the Fourier transform of the operator.

[12]Generally speaking, to rigorously establish the facts we state here one proceeds with the analysis of an integral equation [2, Chapter 2] (see also Chapter 3 for a detailed analysis of similar integral equations).

equal to a linear combination of these free solutions asymptotically. Define four solutions of $Lu = \lambda u$, for $s \in \mathbb{R}$ by

$$\begin{aligned}
\psi_{\mathrm{p}}(x;s) &\sim \mathrm{e}^{+\mathrm{i}sx}(1+o(1)), \text{ as } x \to +\infty,\\
\phi_{\mathrm{p}}(x;s) &\sim \mathrm{e}^{+\mathrm{i}sx}(1+o(1)), \text{ as } x \to -\infty,\\
\psi_{\mathrm{m}}(x;s) &\sim \mathrm{e}^{-\mathrm{i}sx}(1+o(1)), \text{ as } x \to +\infty,\\
\phi_{\mathrm{m}}(x;s) &\sim \mathrm{e}^{-\mathrm{i}sx}(1+o(1)), \text{ as } x \to -\infty.
\end{aligned}$$

As there can only be two linearly independent solutions of a second order differential equation, there must exist an x-independent *scattering matrix* $S(s)$ such that

$$\begin{bmatrix} \psi_{\mathrm{p}}(x;s) & \psi'_{\mathrm{p}}(x;s) \\ \psi_{\mathrm{m}}(x;s) & \psi'_{\mathrm{m}}(x;s) \end{bmatrix} = S(s) \begin{bmatrix} \phi_{\mathrm{p}}(x;s) & \phi'_{\mathrm{p}}(x;s) \\ \phi_{\mathrm{m}}(x;s) & \phi'_{\mathrm{m}}(x;s) \end{bmatrix}. \tag{1.6}$$

Because V is real, we have two further properties, similar to the symmetries of the Fourier transform of a real-valued function, that will prove useful. Complex conjugacy commutes with L and hence we have a conjugacy relationship between the solutions:

$$\psi_{\mathrm{p}}(x;s) = \overline{\psi_{\mathrm{m}}(x;s)} \qquad \text{and} \qquad \phi_{\mathrm{p}}(x;s) = \overline{\phi_{\mathrm{m}}(x;s)}.$$

Furthermore, by direct substitution into the differential equation we have the symmetry relationship with respect to negating the spectral variable:

$$\psi_{\mathrm{p}}(x;s) = \psi_{\mathrm{m}}(x;-s) \qquad \text{and} \qquad \phi_{\mathrm{p}}(x;s) = \phi_{\mathrm{m}}(x;-s).$$

The scattering matrix encodes the spectral data associated with the continuous spectrum of L, in particular, we have the *reflection coefficient* $\rho(z) \triangleq b(z)/a(z)$. The determinants of these matrices can be expressed in terms of the *Wronskian* $W(f,g) = fg' - gf'$, which we use to simplify the definition of S. Indeed, *Abel's formula* indicates that the Wronskian of any two solutions must be independent of x. From the asymptotic behavior of the solutions, assuming the expansion can be differentiated, one can deduce $W(\psi_{\mathrm{p}}, \psi_{\mathrm{m}}) = -2\mathrm{i}s = W(\phi_{\mathrm{p}}, \phi_{\mathrm{m}})$. Then $S(s)$ is expressed as

$$S(s) = \begin{bmatrix} A(s) & B(s) \\ b(s) & a(s) \end{bmatrix} = -\frac{1}{2\mathrm{i}s} \begin{bmatrix} \psi_{\mathrm{p}}(x;s) & \psi'_{\mathrm{p}}(x;s) \\ \psi_{\mathrm{m}}(x;s) & \psi'_{\mathrm{m}}(x;s) \end{bmatrix} \begin{bmatrix} \phi'_{\mathrm{m}}(x;s) & -\phi'_{\mathrm{p}}(x;s) \\ -\phi_{\mathrm{m}}(x;s) & \phi_{\mathrm{p}}(x;s) \end{bmatrix}, \tag{1.7}$$

so that each element is again a Wronskian divided by $-2\mathrm{i}s$. The conjugacy relationship of the solutions ensures that $A(s) = \overline{a}(s)$ and $B(s) = \overline{b}(s)$, therefore $\overline{\rho}(s) = \frac{B(s)}{A(s)}$.

We must also determine the spectral data associated with the discrete spectrum. Consider the $(2,2)$ element of $S(s)$:

$$a(s) = -\frac{W(\psi_{\mathrm{m}}, \phi_{\mathrm{p}})}{2\mathrm{i}s}.$$

Rephrasing these solutions to a second order differential equation in terms of solutions to Volterra integral equations, one shows that ψ_{p} and ϕ_{m} can be analytically continued in the s variable throughout the lower-half plane while ψ_{m} and ϕ_{p} can be analytically continued in the s variable throughout the upper-half plane. This latter fact indicates that $a(z)$ is analytic in the upper-half plane. Now, assume that for discrete values $z_1, \ldots, z_n$ in the upper-half plane, ψ_p is a multiple of ϕ_m, both of which must decay exponentially with respect to x in their respective directions of definition. The conclusion is that ψ_p must be an $L^2(\mathbb{R})$ eigenfunction of L and $z_j^2 = \lambda_j$ is an eigenvalue of L. Furthermore, $a(z)$ must vanish at each such $L^2(\mathbb{R})$ eigenvalue and these zeros can be shown to be simple [2, p. 78]. The discrete spectral data are then given by the *norming constants* $c_j = b(z_j)/a'(z_j)$.

Remark 1.6.1. *It is important to note that b is defined on $\mathbb{R}$. It may or may not have an analytic continuation into the upper-half plane. When we define $b(z_j)$ we are not referring to the function on the real axis or its analytic continuation. We are referring to to the proportionality constant: $\psi_\mathrm{p}(x;z_j) = b(z_j)\phi_\mathrm{m}(x;z_j)$. In this way, $b(z_j)$ could be viewed as abuse of notation because one could have $b = 0$ on $\mathbb{R}$ but $b(z_j) \neq 0$.*

We now consider the problem of recovering the potential V from the *spectral data* ρ, $\{c_j\}$ and $\{\lambda_j\}$. We accomplish this task through the analyticity properties of $\psi_{\mathrm{p/m}}(x;s)$ and $\phi_{\mathrm{p/m}}(x;s)$ with respect to the spectral variable s. Before deriving the correct construction, we investigate an approach that fails: rearranging (1.6) only in terms of the analyticity properties of the solutions. This gives us the jump

$$\begin{gathered}\begin{bmatrix} \psi_\mathrm{m}(x;s) & \phi_\mathrm{p}(x;s) \end{bmatrix} = \begin{bmatrix} \phi_\mathrm{m}(x;s) & \psi_\mathrm{p}(x;s) \end{bmatrix} \tilde{G}(s), \quad s \in \mathbb{R}, \\ \text{for} \quad \tilde{G}(s) = \begin{bmatrix} a(s) - \dfrac{B(s)b(s)}{A(s)} & -\dfrac{B(s)}{A(s)} \\ \dfrac{b(s)}{A(s)} & \dfrac{1}{A(s)} \end{bmatrix}\end{gathered} \tag{1.8}$$

The left-hand side of this equation is analytic in the upper-half plane while the vector on the right-hand side is analytic in the lower-half plane. For fixed x, it can be shown that

$$\begin{aligned} \left[\psi_\mathrm{m}(x;z),\ \phi_\mathrm{p}(x;z)\right] \begin{bmatrix} \mathrm{e}^{\mathrm{i}zx} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}zx} \end{bmatrix} &= [1,\ 1] + \mathcal{O}(1/z), \\ \left[\phi_\mathrm{m}(x;z),\ \psi_\mathrm{p}(x;z)\right] \begin{bmatrix} \mathrm{e}^{\mathrm{i}zx} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}zx} \end{bmatrix} &= [1,\ 1] + \mathcal{O}(1/z), \end{aligned}$$

as $|z| \to \infty$ in their respective domains of analyticity. This is close to a valid RH problem. The main issue here is that $\det \tilde{G}(s) = a(s)/A(s)$ does not generically have a zero winding number[13] which violates our rule of thumb, and therefore we do not expect the problem to have a (unique) solution.

Instead, consider

$$\left[\frac{\psi_\mathrm{m}(x;s)}{a(s)},\ \phi_\mathrm{p}(x;s)\right] = \left[\phi_\mathrm{m}(x;s),\ \frac{\psi_\mathrm{p}(x;s)}{A(s)}\right] G(s), \quad s \in \mathbb{R}, \tag{1.9}$$

and our task is to determine G. Straightforward algebra, combined with the definition of

[13]This can be demonstrated by considering that $a(s)$ and $A(s)$ are analytic functions in the upper- and lower-half planes respectively. Furthermore, $a(s)$ has zeros at $\{z_j\}$ and $A(s)$ at $\{\overline{z}_j\}$. Both functions have poles at $s = 0$ and this must be taken into account when computing the winding number with the argument principle.

ρ, demonstrates

$$G(s)=\begin{bmatrix} 1-\dfrac{b(s)}{a(s)}\dfrac{B(s)}{A(s)} & -\dfrac{B(s)}{A(s)} \\ \dfrac{b(s)}{a(s)} & 1 \end{bmatrix} = \begin{bmatrix} 1-\rho(s)\overline{\rho}(s) & -\overline{\rho}(s) \\ \rho(s) & 1 \end{bmatrix},$$

which has unit determinant and therefore no winding number issues are present for $\det G(s)$. The division by $a(s)$ and $A(s)$ forces us to to consider a sectionally *meromorphic* vector-valued function, depending parametrically on x

$$\Phi(z)\equiv\Phi(x;z)\triangleq\begin{cases} \begin{bmatrix}\dfrac{\psi_{\mathrm{m}}(x;z)}{a(z)}, \ \phi_{\mathrm{p}}(x;z)\end{bmatrix}\begin{bmatrix} e^{izx} & 0 \\ 0 & e^{-izx}\end{bmatrix}, & \text{if } z\in\mathbb{C}^+, \\ \begin{bmatrix}\phi_{\mathrm{m}}(x;z), \ \dfrac{\psi_{\mathrm{p}}(x;z)}{A(z)}\end{bmatrix}\begin{bmatrix} e^{izx} & 0 \\ 0 & e^{-izx}\end{bmatrix}, & \text{if } z\in\mathbb{C}^-, \end{cases} \tag{1.10}$$

so that for $s\in\mathbb{R}$

$$\Phi^+(s)=\Phi^-(s)\begin{bmatrix} e^{-isx} & 0 \\ 0 & e^{isx}\end{bmatrix}G(s)\begin{bmatrix} e^{isx} & 0 \\ 0 & e^{-isx}\end{bmatrix}, \quad \Phi(\infty)=[1, \ 1]. \tag{1.11}$$

Because Φ has poles on the imaginary axis corresponding to $L^2(\mathbb{R})$ eigenvalues of L (the points where a vanishes) we must impose residue conditions; as we must dictate the behavior of Φ at *all* of its singularities. Given $z_j\in i\mathbb{R}^+$ corresponding to an eigenvalue ($z_j^2=\lambda_j$), we compute

$$\begin{aligned}\operatorname{Res}_{z=z_j}\Phi(z) &= \begin{bmatrix}\dfrac{\psi_{\mathrm{m}}(x;z_j)}{a'(z_j)}e^{iz_jx}, \ 0\end{bmatrix} \\ &= \begin{bmatrix}\dfrac{b(z_j)\phi_{\mathrm{p}}(x;z_j)}{a'(z_j)}e^{iz_jx}, \ 0\end{bmatrix} \\ &= \lim_{z\to z_j}\Phi(z)\begin{bmatrix} 0 & 0 \\ c_je^{2iz_jx} & 0\end{bmatrix}.\end{aligned} \tag{1.12}$$

where c_j are precisely the norming constants of the spectral data. Because $\psi_{\mathrm{p/m}}(x;-s)=\psi_{\mathrm{m/p}}(x;s)$ and $\phi_{\mathrm{p/m}}(x;-s)=\phi_{\mathrm{m/p}}(x;s)$, we have $a(-z)=A(z)$ and hence $\Phi(z)$ satisfies

$$\Phi(-z)=\Phi(z)\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}.$$

Residue conditions may be obtained in the lower-half plane from this relationship.

We arrive at the following inverse spectral problem[14].

Problem 1.6.1. *Find* $\Phi:\mathbb{C}\setminus\mathbb{R}\to\mathbb{C}^{1\times 2}$ *that satisfies the following:*

[14] In the language of integrable systems solving this RH problem is called *inverse scattering*

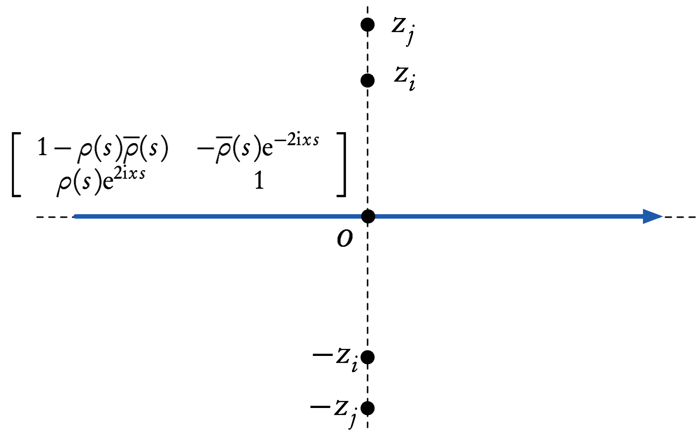

Figure 1.9. *The jump contour and jump functions for the Schrödinger inverse spectral RH problem.*

1. *$\Phi(z)$ is meromorphic in $\mathbb{C}\setminus\mathbb{R}$,*

$$\Phi^+(s)=\Phi^-(s)\begin{bmatrix} 1-\rho(s)\overline{\rho}(s) & -\overline{\rho}(s)e^{-2isx} \\ \rho(s)e^{2isx} & 1 \end{bmatrix}, \quad s\in\mathbb{R}, \quad \Phi(\infty)=[1,\ 1],$$

for a function ρ that satisfies $\rho(-s)=\overline{\rho}(s)$, $|\rho(s)|<1$ for $s\neq 0$,

2. *$\Phi(z)$ has a finite number of poles $\{z_j\}_{j=1}^n$, $\{-z_j\}_{j=1}^n$, for $z_j=i\sqrt{-\lambda_j}$, $\lambda_j<0$, on the imaginary axis where it satisfies*

$$\operatorname{Res}_{z=z_j}\Phi(z)=\lim_{z\to z_j}\Phi(z)\begin{bmatrix} 0 & 0 \\ c_j e^{2iz_j x} & 0 \end{bmatrix},$$
$$\operatorname{Res}_{z=-z_j}\Phi(z)=\lim_{z\to -z_j}\Phi(z)\begin{bmatrix} 0 & -c_j e^{2iz_j x} \\ 0 & 0 \end{bmatrix},$$
$$c_j\in i\mathbb{R}^+,$$

and

3. *$\Phi(z)$ satisfies the (essential) symmetry condition*

$$\Phi(-z)=\Phi(z)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

If we can solve this RH problem, we can recover the potential $V(x)$ by

$$V(x)=2i\lim_{z\to\infty} z\partial_x\Phi(z)_1. \tag{1.13}$$

This follows since for $z\in\mathbb{C}^+$

$$\Phi^-(x;z)_1=\phi_m(x;z)e^{izx}=1-\frac{1}{2iz}\int_{-\infty}^{x}\left[1-e^{2iz(x-\tau)}\right]V(\tau)\Phi^-(\tau;z)_1 d\tau,$$

where the integral representation can be verified by substitution $\phi_{\mathrm{m}}(x;z) = \Phi^{-}(x;z)_1 \mathrm{e}^{-\mathrm{i}zx}$ into the Schrödinger equation, $Lu = z^2 u$. Therefore,

$$\partial_x \Phi^{-}(z)_1 = -\mathrm{e}^{2\mathrm{i}zx} \int_{-\infty}^{x} \mathrm{e}^{-2\mathrm{i}z\tau} V(\tau) \Phi^{-}(\tau;z)_1 \mathrm{d}\tau \sim \frac{\Phi^{-}(x;z)_1 V(x)}{2\mathrm{i}z} \sim \frac{V(x)}{2\mathrm{i}z} \quad \text{as} \quad z \to \infty,$$

using integration by parts and the fact that $\Phi^{-}(x;z)_1 \to 1$ as $z \to \infty$. Similar relations follow for Φ^{+}.

It can be shown that $\Phi(z)$ is uniquely specified by the above RH problem. Most importantly, we have characterized the operator $-\mathrm{d}^2/\mathrm{d}^2 x - V(x)$ uniquely in terms of $\rho(z)$ defined on the essential spectrum and norming constants $\{c_j\}$ defined on the discrete spectrum $\{\lambda_j\}$. In this sense ρ, $\{c_j\}$ and $\{\lambda_j\}$ constitute the spectral data in the spectral analysis of Schrödinger operators. This procedure is critical in the solution of the Korteweg–de Vries equation with the inverse scattering transform as is discussed in great detail in Chapter 8.

Remark 1.6.2. *In the presentation we have ignored some technical details. One of which is the boundary behavior of $\psi_{\mathrm{m}}(x;z)/a(z)$ as z approaches the real axis. Specifically, one needs to show $\Phi(z)$ as defined in* (1.10) *is in an appropriate Hardy space (see Section 2.5). Additionally, we used implicitly that $a(z)$ and $A(z)$ limit to unity for large z. These details can be established rigorously, see [32].*

Bibliography

[1] M J Ablowitz. *Nonlinear dispersive waves*. Cambridge Texts in Applied Mathematics. Cambridge University Press, New York, 2011. pages 252

[2] M J Ablowitz and P A Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge University Press, 1991. pages 17, 19, 215, 236, 237, 255, 312

[3] M J Ablowitz and A S Fokas. *Complex Varibles: Introduction and Applications*. Cambridge University Press, second edition, 2003. pages 23, 33

[4] M J Ablowitz and H Segur. Asymptotic solutions of the Korteweg–de Vries equation. *Stud. in Appl. Math.*, 57:13–44, 1977. pages vi, 203, 212, 217, 257

[5] M J Ablowitz and H Segur. *Solitons and the inverse scattering transform*. SIAM, Philadelphia, PA, 1981. pages 198

[6] D V Anosov and A A Bolibrukh. *The Riemann-Hilbert problem*. Springer, New York, NY, 1994. pages 14

[7] K Atkinson and W Han. *Theoretical Numerical Analysis*. Springer, New York, NY, 2009. pages 340

[8] S Barnett. A companion matrix analogue for orthogonal polynomials. *Linear Algebra Appl.*, 12:197–208, 1975. pages 355

[9] Z Battles and L N Trefethen. An extension of MATLAB to continuous functions and operators. *SIAM J. Sci. Comput.*, 25:1743–1770, 2004. pages 196, 205, 213, 239

[10] R Beals and R R Coifman. Scattering and Inverse Scattering for First Order Systems. *Comm. Pure Appl. Math.*, 37(iv):39–90, 1984. pages 62, 95, 96, 97, 364

[11] E D Belokolos, A I Bobenko, V Z Enol'skii, A R Its, and V B Matveev. *Algebro-Geometric Approach to Nonlinear Integrable Equations*. Springer, 1994. pages 273, 274, 286

[12] R F Bikbaev and V O Tarasov. Initial-boundary value problem for the nonlinear Schrödinger equation. *J. Phys. A*, 24(11):2507–2516, 1991. pages 247, 248

[13] G Biondini and A Bui. On the nonlinear Schrödinger equation on the half line with homogeneous Robin boundary conditions. *Stud. Appl. Math.*, 129(3):249–271, 2012. pages 247, 248

[14] G Boffetta and A R Osborne. Computation of the direct scattering transform for the Nonlinear Schrödinger Equation. *J. Comp. Phys*, 102:252–264, 1995. pages 374

[15] A A Bolibrukh. The Riemann-Hilbert problem. *Russ. Math. Surv.*, 1990. pages 14

[16] F Bornemann. On the numerical evaluation of distributions in random matrix theory: a review. *Mar*, 16:803–866, September 2010. pages 258

[17] F Bornemann. On the numerical evaluation of Fredholm determinants. *Math. Comput.*, 70(4):871–915, 2010. pages 258, 375

[18] J P Boyd. *Chebyshev and Fourier spectral methods*. Dover Publications Inc., Mineola, NY, second edition, 2001. pages 299

[19] K F Clancey and I Gohberg. *Factorization of matrix functions and singular integral operators*, volume 3 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1981. pages 23, 61, 69

[20] P A Clarkson. Painlevé equations–nonlinear special functions. *J. Comput. Appl. Math.*, 153(1-2):127–140, 2003. pages 257

[21] C W Clenshaw and A R Curtis. A method for numerical integration on an automatic computer. *Numer. Math.*, 2(1):197–205, 1960. pages 358

[22] R R Coifman, P W Jones, and S Semmes. Two elementary proofs of the L 2 boundedness of Cauchy integrals on Lipschitz curves. *J. Am. Math. Soc.*, 2(3):553–564, 1989. pages 48, 49

[23] C W Curtis and B Deconinck. On the convergence of Hill's method. *Math. Comp.*, 79(269):169–187, 2010. pages 239

[24] B Deconinck, M Heil, A Bobenko, M van Hoeij, and M Schmies. Computing Riemann theta functions. *Math. Comp.*, 73:1417–1442, 2004. pages 273

[25] B Deconinck and J N Kutz. Computing spectra of linear operators using the Floquet–Fourier–Hill method. *J. Comp. Phys.*, 291:296–321, 2007. pages 196, 213, 215, 239

[26] B Deconinck and D O Lovit. Data analysis and reduction using stationary solutions of the NLS equation. *Appl. Anal.*, 89:611–626, 2010. pages 301

[27] B Deconinck, T Trogdon, and V Vasan. The Method of Fokas for Solving Linear Partial Differential Equations. *SIAM Rev.*, 56(1):159–186, January 2014. pages 51

[28] P Deift. *Orthogonal Polynomials and Random Matrices: a Riemann-Hilbert Approach*. Amer. Math. Soc., Providence, RI, 2008. pages 15, 16, 23, 97, 254

[29] P Deift. Some open problems in random matrix theory and the theory of integrable systems. In Baik J, Kriecherbauer T, Li L-C, K T-R McLaughlin, and C Tomei, editors, *Integr. Syst. Random Matrices Honor Percy Deift*. Providence, RI, 2008. pages 236, 256

[30] P Deift, A R Its, and X Zhou. Long-time asymptotics for integrable nonlinear wave equations. In A S Fokas and V E Zakharov, editors, *Important Dev. Solit. theory*, Springer Ser. Nonlinear Dynam., pages 181–204. Springer, Berlin, 1993. pages 103, 255

[31] P Deift and J Park. Long-time asymptotics for solutions of the NLS equation with a delta potential and even initial data. *Int. Math. Res. Not.*, 2011:5505–5624, 2011. pages 248

[32] P Deift and E Trubowitz. Inverse scattering on the line. *Commun. Pure Appl. Math.*, 32(2):121–251, March 1979. pages 17, 22

[33] P Deift, S Venakides, and X Zhou. An extension of the steepest descent method for Riemann-Hilbert problems: the small dispersion limit of the Korteweg-de Vries (KdV) equation. *Proc. Natl. Acad. Sci. USA*, 95:450–454, 1998. pages 217, 274

[34] P Deift and X Zhou. A steepest descent method for oscillatory Riemann–Hilbert problems. *Bull. Amer. Math. Soc.*, 26:119–124, 1992. pages 63, 175, 176, 205, 241

[35] P Deift and X Zhou. A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV Equation. *Ann. Math.*, 137(2):295–368, 1993. pages vi, 97, 199, 201, 205, 233

[36] P Deift and X Zhou. Long-time Asymptotics for Integrable Systems. Higher Order Theory. *Commun. Math. Phys.*, 191:175–191, 1994. pages 97, 256

[37] P Deift and X Zhou. *Long-time behavior of the non-focusing nonlinear Schrödinger equation — a case study*, volume 1 of *Lectures in Mathematical Sciences*. University of Tokyo, Tokyo, 1994. pages 99, 254, 255, 256

[38] P Deift and X Zhou. Asymptotics for the Painlevé II equation. *Commun. Pure Appl. Math.*, XLVIII(1995):277–337, 1995. pages 97, 259

[39] P Deift and X Zhou. Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space. *Comm. Pure Appl. Math.*, 56:1029–1077, August 2003. pages 97, 254, 255, 256

[40] P Deift, X Zhou, and S Venakides. The Collisionless Shock Region for the Long-Time Behavior of Solutions of the KdV Equation. *Comm. Pure Appl. Math.*, 47:199–206, 1994. pages vi, 203, 204, 205, 212, 217, 374

[41] R G Douglas. *Banach Algebra Techniques in Operator Theory*. Academic Press, 1972. pages 340

[42] P G Drazin and R S Johnson. *Solitons: An Introduction*. Cambridge University Press, New York, NY, 1996. pages 195, 227

[43] P Dubard, P Gaillard, C Klein, and V B Matveev. On multi-rogue wave solutions of the NLS equation and position solutions of the KdV equation. *Eur. Phys. J. - Spec. Top.*, 185(1):247–258, 2010. pages 252

[44] B A Dubrovin. Inverse problem for periodic finite zoned potentials in the theory of scattering. *Func. Anal. Its Appl.*, 9:61–62, 1975. pages 274, 280

[45] B A Dubrovin. Theta functions and non-linear equations. *Russ. Math. Surv.*, 36:11–92, 1981. pages 273, 275, 276

[46] B A Dubrovin. Integrable Systems and Riemann Surfaces Lecture Notes. `http://people.sissa.it/$\sim$dubrovin/rsnleq_web.pdf`, 2009. pages 274, 277

[47] P Duren. *Theory of H^p Spaces*. Academic Press, 1970. pages 45, 46, 363

[48] I Egorova, K Grunert, and G Teschl. On the Cauchy problem for the Korteweg-de Vries equation with steplike finite-gap initial data I. Schwartz-type perturbations. *Nonlinearity*, 22:1431–1457, 2009. pages 311

[49] L C Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010. pages 32, 58

[50] A S Fokas. *A Unified Approach to Boundary Value Problems*. SIAM, Philadelphia, PA, 2008. pages 51, 87, 88, 96, 248

[51] A S Fokas and I M Gel'fand. Integrability of Linear and Nonlinear Evolution Equations and the Associated Nonlinear Fourier Transforms. *Lett. Math. Phys.*, 32:189–210, 1994. pages 51

[52] A S Fokas, A R Its, A A Kapaev, and V Y Novokshenov. *Painlevé Transcendents: the Riemann–Hilbert Approach*. Amer. Math. Soc., 2006. pages vi, 14, 258, 259, 264, 265, 266, 270, 279

[53] G B Folland. *Real analysis*. John Wiley & Sons Inc., New York, 1999. pages 48, 91, 93, 365, 370, 371

[54] B Fornberg and J A C Weideman. A numerical methodology for the Painlevé equations. *J. Comp. Phys.*, 2011. pages 258

[55] J Frauendiener and C Klein. Hyperelliptic theta-functions and spectral methods: KdV and KP solutions. *Lett. Math. Phys.*, 76:249–267, 2006. pages 273, 282, 373

[56] C S Gardner, J M Greene, M D Kruskal, and R M Miura. Method for solving the Korteweg–de Vries equation. *Phys. Rev. Lett.*, 19:1095–1097, 1967. pages 195

[57] T Grava and C Klein. Numerical Solution of the Small Dispersion Limit of Korteweg âĂŞ de Vries and Whitham Equations. *Commun. Pure Appl. Math.*, LX:1623–1664, 2007. pages 373, 374

[58] E P Gross. Strucure of a quantized vortex in boson systems. *Neuvo Cim.*, 20:454–477, 1961. pages 235

[59] K Grunert and G Teschl. Long-time asymptotics for the KortewegâĂŞde Vries equation via nonlinear steepest descent. *Math. Physics, Anal. Geom.*, 12(3):287–324, 2009. pages 200, 203, 216, 229, 255

[60] O H Hald. Numerical solution of the Gel'fand-Levitan equation. *Linear Algebra Appl.*, 28(1979):99–111, December 1979. pages 375

[61] N Hale and A Townsend. A Fast, Simple, and Stable Chebyshev–Legendre Transform Using an Asymptotic Formula. *SIAM J. Sci. Comput.*, 36(1):A148–A167, February 2014. pages 137

[62] G H Hardy. *A course of pure mathematics*. Cambridge University Press, Cambridge, centenary edition, 2008. pages vi

[63] S P Hastings and J B McLeod. *Classical methods in ordinary differential equations*, volume 129 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012. pages 203, 257

[64] A Its. The Riemann-Hilbert problem and integrable systems. *Not. AMS*, 50(11):1389, 2003. pages 4

[65] A R Its and D Shepelsky. Initial boundary value problem for the focusing NLS equation with Robin boundary conditions: half-line approach. *arXiv:1201.5948 [nlin.SI]*, 2012. pages 248

[66] M A Johnson and K Zumbrun. Convergence of Hill's Method for Non-self adjoint Operators. *SIAM J. Numer. Anal.*, 50(1):64–78, 2012. pages 239

[67] T Kato. *Perturbation theory for linear operators*. Springer, New York, NY, 1995. pages 17

[68] T W Körner. *Fourier analysis*. Cambridge University Press, Cambridge, UK, 1989. pages 346

[69] D J Korteweg and G de Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philos. Mag.*, 39:422–443, 1895. pages 195

[70] R Kress and I H Sloan. On the numerical solution of a logarithmic integral equation of the first kind for the Helmholtz equation. *Numer. Math.*, 66(1):199–214, December 1993. pages 114

[71] P D Lax. Periodic solutions of the KdV equation. *Comm. Pure Appl. Math.*, 28:141–188, 1975. pages 273

[72] J Lenells. Matrix Riemann-Hilbert problems with jumps across Carleson contours. *arXiv Prepr. arXiv1401.2506*, 2014. pages 23, 53

[73] R LeVeque. *Finite Difference Methods for Ordinary and Partial Differential Equations*. SIAM, Philadelphia, PA, 2007. pages 172, 248, 373

[74] S V Manakov. On the theory of two-dimensional stationary self-focusing of electromagnetic waves. *Sov. Phys. JETP*, 48:248–253, 1974. pages 235

[75] H P McKean. Algebraic curves of infinite genus arising in the theory of nonlinear waves. In *Proc. International Congress Mathematicians (Helsinki, 1978)*, pages 777–783, Helsinki, 1980. Acad. Sci. Fennica. pages 273

[76] H P McKean and E Trubowitz. Hill's surfaces and their theta functions. *Bull. Amer. Math. Soc.*, 84:1042–1085, 1978. pages 273

[77] Y Meyer and R R Coifman. *Wavelets: Calderón-Zygmund and Multilinear Operators*. Cambridge University Press, 1997. pages 49, 53, 62, 371

[78] S G Mikhlin and S Prössdorf. *Singular Integral Operators*. Springer, New York, 1980. pages 59, 335

[79] A Mikikits-Leitner and G Teschl. Long-time asymptotics of perturbed finite-gap Korteweg-de Vries solutions. *J. d'Analyse Mathématique*, 116(1):163–218, 2012. pages 323

[80] R M Miura. Korteweg–de Vries Equation and Generalizations. I. A Remarkable Nonlinear transformation. *J. Math. Phys.*, 9:1202–1204, 1968. pages 231

[81] N I Muskhelishvili. *Singular Integral Equations*. Groningen: Noordoff, 1953. pages 33, 34

[82] S P Novikov. A periodic problem for the Korteweg-de Vries equation. I. *Funkc. Anal. i Priložen.*, 8:54–66, 1974. pages 273

[83] S P Novikov, S V Manakov, L P Pitaevskii, and V E Zakharov. *Theory of Solitons*. Constants Bureau, New York, 1984. pages 273, 274, 279, 280, 281

[84] F W J Olver, D W Lozier, R F Boisvert, and C W Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010. pages 5, 8, 102, 103, 131, 141, 171, 227, 240, 264, 301, 355

[85] S Olver. Numerical solution of Riemann–Hilbert problems: Painlevé II. *Found. Comput. Math.*, 11(2):153–179, November 2010. pages 231, 259

[86] S Olver. RHPackage. `http://www.maths.usyd.edu.au/u/olver/projects/RHPackage.html`, 2010. pages 166, 169, 269

[87] S Olver. Computation of equilibrium measures. *J. Approx. Theory*, 163:1185–1207, 2011. pages 219, 377

[88] S Olver. Computing the Hilbert transform and its inverse. *Math. Comput.*, 80:1745–1767, 2011. pages 142

[89] S Olver. A general framework for solving Riemann-Hilbert problems numerically. *Numer. Math.*, 122(2):305–340, 2012. pages 166, 239

[90] S Olver and T Trogdon. Nonlinear Steepest Descent and Numerical Solution of Riemann–Hilbert Problems. *Comm. Pure Appl. Math.*, to appear:1–36, December 2013. pages 175, 257

[91] S Olver and T Trogdon. Numerical Solution of Riemann–Hilbert Problems: Random Matrix Theory and Orthogonal Polynomials. *Constr. Approx.*, 39(1):101–149, December 2013. pages 17, 257, 267

[92] A R Osborne. *Nonlinear ocean waves and the inverse scattering transform*, volume 97 of *International Geophysics Series*. Elsevier/Academic Press, Boston, MA, 2010. pages 274, 374

[93] L P Pitaevskii. Vortex lines in a imperfect Bose gas. *Sov. Phys., JETP*, 13:451–454, 1961. pages 235

[94] J Plemelj. Riemannsche Funktionenscharen mit gegebener Monodromiegruppe. *Monatshefte für Math.*, 1908. pages 14

[95] M Prähofer and Spoh. Exact scaling functions for one-dimensional stationary KPZ growth, 2003. pages 258

[96] M Prähofer and H Spohn. Exact scaling functions for one-dimensional stationary KPZ growth. *J. Stat. Phys.*, 115:255–279, 2004. pages 258

[97] L G Reyna. L 2 estimates for Chebyshev collocation. *J. Sci. Comput.*, 3(1):1–23, March 1988. pages 190

[98] T J Rivlin. *Chebyshev polynomials*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, second edition, 1990. pages 357

[99] W Rudin. *Principles of mathematical analysis*. McGraw-Hill, Inc., New York, NY, 3rd edition, 1964. pages 337

[100] E M Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970. pages 364

[101] A Tovbis, S Venakides, and X Zhou. On semiclassical (zero dispersion limit) solutions of the focusing nonlinear Schrödinger equation. *Commun. Pure Appl. Math.*, LVII:877–985, 2004. pages 240

[102] C A Tracy and H Widom. Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.*, 159:151–174, 1994. pages 257

[103] T Trogdon. ISTPackage. `http://students.washington.edu/trogdon/software.html`, 2013. pages 209, 226, 246

[104] T Trogdon. Rational approximation, oscillatory Cauchy integrals and Fourier transforms. *arXiv Prepr. arXiv1403.2378*, 2014. pages 120

[105] T Trogdon and B Deconinck. A Riemann–Hilbert problem for the finite-genus solutions of the KdV equation and its numerical solution. *Phys. D Nonlinear Phenom.*, 251:1–18, 2013. pages 273

[106] T Trogdon and B Deconinck. A numerical dressing method for the nonlinear superposition of solutions of the KdV equation. *Nonlinearity*, May 2014. pages 311

[107] T Trogdon and B Deconinck. Dispersive and soliton perturbations of finite-genus solutions of the KdV equation: Computational results. *Phys. Lett. A*, 378(7-8):617–622, May 2014. pages 311

[108] T Trogdon and S Olver. A Riemann–Hilbert approach to Jacobi operators and Gaussian quadrature. *arXiv Prepr. arXiv1311.5838*, November 2013. pages 17

[109] T Trogdon and S Olver. Numerical inverse scattering for the focusing and defocusing nonlinear Schrödinger equations. *Proc. R. Soc. A*, 469(2149), November 2013. pages 235

[110] T Trogdon, S Olver, and B Deconinck. Numerical inverse scattering for the Korteweg–de Vries and modified Korteweg–de Vries equations. *Phys. D Nonlinear Phenom.*, 241(11):1003–1025, 2012. pages 195, 229, 231

[111] T T Wu, B M McCoy, C A Tracy, and E Barouch. Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region. *Phys. Rev. B*, 13(1):316–374, January 1976. pages vi

[112] V E Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J Appl. Mech. Tech. Phys.*, 9:190–194, 1968. pages 235

[113] V E Zakharov and S V Manakov. Asymptotic behavior of non-linear wave systems integrated by the inverse scattering method. *Sov. Phys. JETP*, 44:106–112, 1976. pages 256

[114] X Zhou. The Riemann-Hilbert problem and inverse scattering. *SIAM J. Math. Anal.*, 20(4):966–986, 1989. pages 23, 61, 62, 68, 69, 70, 76, 324

[115] X Zhou. Riemann–Hilbert problems and integrable systems. *Lect. MSRI*, 1999. pages 47, 61, 64, 318, 324

Notation and Abreviations

(f)	The divisor of a meromorphic function f	276
$[\diamond,\diamond]$	The standard matrix commutator: $[M,N] = MN - NM$	88
$[G;\Gamma]$	An inhomogeneous L^2 RH problem	62
$\mathrm{Z}(\Gamma)$	Functions that satisfy the absolutely converging zero-sum condition	154
$\mathrm{Z}_n(\Gamma)$	A finite-dimensional subspace of $\mathrm{Z}(\Gamma)$	162
$\mathrm{Ai}(z)$	The Airy function	9
$\arg z$	The argument of complex number, $\arg z \in (-\pi, \pi]$	
$\bar{\mathrm{d}}s$	$\bar{\mathrm{d}}s = 1/(2\pi \mathrm{i})\mathrm{d}s$	25
$\mathscr{C}$	Equivalent to $\mathscr{C}_{\mathbb{U}}$	128
$\check{f}_k$	The kth Chebyshev coefficient of f	116
$\operatorname{codim} X$	The dimension of the quotient space Y/X where Y is clear from context	340
$\deg D$	The degree of a divisor D	276
$\dim X$	The dimension of a vector space X	
$\mathrm{e}^{\alpha\hat{\sigma}_3}A$	$\mathrm{e}^{\alpha\sigma_3}A\mathrm{e}^{-\alpha\sigma_3}$	88
ℓ^p	The Banach space of p-summable complex sequences. Context determines if sequences run over $\mathbb{N}$ or $\mathbb{Z}$.	338
$\ell^{(\lambda,R),p}$	p-summable complex sequences with algebraic and exponential decay	339
$\ell^{\lambda,p}$	p-summable complex sequences with algebraic decay	339
$\ell^{(\lambda,R),p}_{\pm}$	$\ell^{(\lambda,R),p}$ with a zero-sum condition	339
$\operatorname{erfc} z$	The complimentary error function	5
$\mathscr{F}^{-\mathrm{z}}$	The map to vanishing basis coefficients on the line	122
$\mathtt{F}_{\alpha,\beta}$	The DFT matrix operator	111
$\mathscr{T}$	The operator that maps a function to its Chebyshev coefficients	116
$\mathscr{T}^{\pm\mathrm{z}}$	The map to vanishing Chebyshev basis coefficients	123

$\mathtt{T}_n$ The DCT matrix operator 117

$\Gamma(\zeta)$ The Gamma function 103

Γ^+ The region above a Lipschitz graph Γ 48

$\Gamma^\dagger$ The Schwarz conjugate of contour Γ: $\Gamma^\dagger = \{\bar{z} : z \in \Gamma\}$ 76

γ_0 The set of non-smooth points of a contour Γ 57

$\hat{\mathbb{C}}$ A cut version of the complex plane 279

i The imaginary unit $\sqrt{-1}$

$\operatorname{ind}\mathscr{T}$ The Fredholm index of an operator $\mathscr{T}$: $\dim\ker\mathscr{T} - \dim\operatorname{codim}\operatorname{ran}\mathscr{T}$ 340

$\operatorname{ind}_\Gamma g(s)$ The normalized increment of the argument of g as Γ is traversed 36

$J_\downarrow^{-1}(x)$ An inverse of the Joukowsky map from the unit circle 351

$J_+^{-1}(z)$ A right inverse of the Joukowsky map 350

$J_-^{-1}(z)$ A right inverse of the Joukowsky map 350

$J_\uparrow^{-1}(x)$ An inverse of the Joukowsky map from the unit circle 351

$\ker\mathscr{T}$ The kernel of an operator $\mathscr{T}$ 340

$\|\diamond\|_p$ $\|\diamond\|_{L^p(\Gamma)}$ when Γ is clear from context 334

$\|\diamond\|_u$ The uniform norm 334

$\|\diamond\|_X$ The norm on a Banach space X 333

$\mathbb{C}$ The field of complex numbers

$\mathbb{C}^\pm$ The open upper- (+) and lower-half (−) planes

$\mathbb{C}^{n\times m}$ The vector space of $n \times m$ complex matrices 4

$\mathbb{D}$ The unit disk $\{z \in \mathbb{C} : |z| < 1\}$ 45

$\mathbb{I}$ The closed unit interval $[-1,1]$ oriented from left to right

$\mathbb{N}$ The natural numbers $\{1,2,3,\ldots\}$

$\mathbb{R}^\pm$ The open left- (−) and right-half (+) lines

$\mathbb{U}$ The unit circle $\{z \in \mathbb{C} : |z| = 1\}$ with counter-clockwise orientation

$\mathbb{Z}$ The integers $\{\ldots,-2,-1,0,1,2,\ldots\}$

$\mathscr{B}^+(\Gamma)$ The weighted Bergman space above the Lipschitz graph Γ 48

$\mathscr{C}'[G;\Gamma]$ The operator $u \mapsto u - \mathscr{C}_\Gamma^-[u(G-I)]$ 63

$\mathscr{C}'[X_+,X_-;\Gamma]$ The operator $u \mapsto \mathscr{C}_\Gamma^+[uX_-] - \mathscr{C}_\Gamma^-[uX_+]$ 66

$\mathscr{C}[G;\Gamma]$ The operator $u \mapsto u - \mathscr{C}_\Gamma^- u(G-I)$ 63

$\mathscr{C}[X_+,X_-;\Gamma]$ The operator $u \mapsto \mathscr{C}_\Gamma^+ u X_+^{-1} - \mathscr{C}_\Gamma^- u X_-^{-1}$ 66

$\mathscr{C}_\Gamma^\pm f(z)$ The boundary values $(\mathscr{C}_\Gamma f(z))^\pm$ 30

$\mathscr{C}_\Gamma f(z)$ The Cauchy integral (transform) of f 25

$\mathscr{C}_n[G;\Gamma]$ A finite-dimensional approximation of $\mathscr{C}[G;\Gamma]$ 177

$\mathscr{E}^\pm(\Gamma)$ The Hardy space to the left/right of an admissible contour Γ 56

$\mathscr{E}^p(D)$ The Hardy space on a general domain D 46

$\mathscr{F}$ The operator that maps to either Fourier or Laurent coefficients 345

$\mathscr{H}^p$ The L^p-based Hardy space on the disk 45

$\mathscr{K}(X,Y)$ The subspace of compact operators from X to Y 340

$\mathscr{L}(X,Y)$ The Banach algebra of bounded linear operators from X to Y 340

$\mathscr{S}(\mathbb{R})$ The Schwartz class on $\mathbb{R}$:
$\mathscr{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) : \sup_x |x^j f^{(k)}(x)| < \infty \text{ for all } j,k \geq 0 \right\}$

$\mathscr{S}_\delta(\mathbb{R})$ The Schwartz class with exponential decay:
$\mathscr{S}_\delta(\mathbb{R}) = \left\{ f \in \mathscr{S}(\mathbb{R}) : \sup_x e^{\delta|x|} |f(x)| < \infty \right\}$

$\Omega_\pm$ Components to the left/right of a complete contour 24

∂D The boundary of a set D 45

$\phi(z,b,a)$ The Lerch transcendent 140

$\Phi^\pm(s) = \Phi_\pm(s)$ The boundary values of Φ from the left/right 24

$\operatorname{Res}_{z=a} f(z)$ The residue of a function $f(z)$ at a point $z=a$ 3

$\operatorname{sgn} x$ The sign of real number x, $\operatorname{sgn} 0 = 0$

$\sigma(q)$ The Bloch spectrum of q 278

σ_1 $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ 205

σ_3 $\operatorname{diag}(1,-1)$ 88

Σ_∞ The class of Jordan curves tending to straight lines at infinity 47

$\Subset$ $D \Subset A$ when D is a connected component of A 56

θ^m Evenly-spaced points on the periodic interval 110

$\tilde{H}_\pm^k(\Gamma)$ $H_\pm^k(\Gamma) \oplus \mathbb{C}^{n\times n}$ when Γ is unbounded 66

$\triangleq$ Used for inline definitions 47

$\boldsymbol{x}^n$ The Chebyshev points 117

$\{s_1,s_2,s_3\}$ Stokes' constants 257

$^\top$ Transpose of a matrix or vector

$A(D)$ The Abel map of a divisor D 276

$B(x,\delta)$ The ball centered at x of radius δ

$B_{\theta,\phi}(x,\delta)$ $e^{i\phi}(\{y \in B(x,\delta) : |\mathrm{Im}(x-y)|/|x-y| > \sin\theta\} - x) + x$ 27

C A generic positive (unless otherwise specified) constant

$C^k(A)$ The Banach space of k-times continuously differentiable functions 334

$C^{0,\alpha}(\Gamma)$ The Banach space of α-Hölder continuous functions 32

$C_c^k(A)$ The space of k-times continuously differentiable functions with compact support 334

$C_n[G;\Gamma]$ The collocation matrix for $\mathscr{C}[G;\Gamma]$ 163

D^{-1} $\{z^{-1} : z \in D\}$ 47

$D_\nu(\zeta)$ The parabolic cylinder function 102

Df The weak differentiation operator applied to f 58

$f|_A$ The restriction of a function $f : D \to R$ to a set $A \subset D$

$f^\dagger$ The Schwarz conjugate of a function f: $f^\dagger(z) = \overline{f(\overline{z})}^\top$ 76

$H^k(\Gamma)$ The kth-order Sobolev space on a self-intersecting admissible contour Γ 58

$H_\pm^k(\Gamma)$ The Sobolev spaces of Zhou 61

$H_z^k(\Gamma)$ $H^k(\Gamma)$ with the $(k-1)$th-order zero-sum condition 59

I The identity matrix/operator with dimensionality implied

$L^p(\Gamma)$ The Lebesgue space on a self-intersecting curve Γ 334

n_i The number of collocation points on Γ_i, $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_L$ 162

$P_{\mathrm{II}}(s_1, s_2, s_3; x)$ The solution of the Painlevé II ODE with Stokes constants s_1, s_2, s_3 .. 257

$q(n,x,t)$ An approximation to the solution of the PDE with n collocation points per contour 209

$R_\pm(\Gamma)$ Functions that are a.e. rational on each component of $\Omega_\pm$ 66

$T_k(x)$ The Chebyshev polynomials 353

$T_k^{\pm z}(x)$ The vanishing Chebyshev basis 123

U_k The Chebyshev U polynomials 355

$\mathscr{H}_\Gamma f(z)$ The Hilbert transform of a function f 126

$\mathscr{T}$ The Chebyshev transform 353

$\mathscr{V}[f]$ The total variation of f 333

$FP\mathscr{C}_\Gamma f(z)$	The finite-part Cauchy transform of f	147
${}_2F_1$	The Gauss hypergeometric function	141
RH problem	Riemann–Hilbert problem	3
BA	Baker–Akhiezer	280
DCT	Discrete cosine transform	117
DFT	Discrete Fourier transform	111
IST	Inverse scattering transform	87
KdV	Korteweg-de Vries	195
NLS	Nonlinear Schrödinger	235
ODE	Ordinary differential equation	257
PDE	Partial differential equation	195
SIE	Singular integral equation	62

Index

(α,γ)-Hölder condition, 32
Γ-differentiable, 334
α-Hölder continuous, 25
    Banach space, 32
    uniformly, 25
k-regular, 63

Abel's formula, 18, 94
Abelian differential, 277
Ablowitz–Segur solutions, 257
admissible numerical method, 178
Airy
    equation, 9
    function, 10
aliasing, 113
analyticity, 349
arc, 359

Baker–Akhiezer function, 274, 280
Banach algebra, 343
bands, 278
Bergman space, 49
Bernstein ellipse, 354
binomial theorem, 353
Bloch spectrum, 278

Cauchy integral, 25
    formula, 45, 47
    operator, 50
Cauchy transform, *see* Cauchy integral
    finite-part, 145, 147, 149
Chebyshev
    coefficients, 116, 355
    expansion, 115
    points, 116
    polynomials, 353
    polynomials of the first kind, *see* Chebyshev polynomials
    polynomials of the second kind, *see* Chebyshev U polynomials
    series, 353
    series, discrete, 117
    transform, 353
    U polynomials, 355
circular contours, 197
closure of rational functions, 67
collocation matrix, 163
collocation method, 157, 159
combination solutions, 311
complementary error function, 5
complete Elliptic integral, 8
completely integrable, 198, 236
comrade matrix, 355
conjugation, 83
contour, 359
    admissible, 56
    augmentation, 80

complete, 23
decoupling, 84
incomplete, 79
scaling, 84
self-intersecting, 24
truncation, 82
conversion relationship, 356
cosine series, 352
curve, 359
cyclic junction condition, 166

Darboux transformation, 247
decomposing algebras, 77
deformation technique, 82–84
differentiation, 355
direct scattering, 196
Dirichlet boundary conditions, 247
discete
Laurent series, 115
discrete
cosine transform, 117
Fourier series, 110
Fourier transform, 110
semi-inner product, 110
divisor, 276
nonspecial, 276
dressing method, 96
KdV, 312
NLS, 324
dual pairing, 334

elliptic integral, 136
of the first kind, 7
endpoint, 359

fast cosine transform, 118
finite degree, 24
finite section method, 158
finite-genus solutions, 273
finite-part Cauchy transform, 147
forward scattering, 237
forward spectral problem, 199, 200, 237
Fourier
coefficients, 109
series, 109
transform, inverse discrete, 113
transform, periodic, 345
Fourier coefficients, 345
Fourier series, 345
Fredholm
determinant, 375
index, 340
Fuchsian ODE, 11
fundamental solution, 37, 38

Gamma function, 103, 227
gaps, 278
Gauss hypergeometric function, 141
GK factorization theory, 69

Hardy space
general domain, 46
on the disk, 45
Hastings–McLeod solution, 257
Hilbert space, 342
Hilbert transform, 126
Hilbert–Schmidt operator, 85
Hill's method, 213, 239
holomorphic differentials, 275
homogeneous Painlevé II equation, *see* Painlevé II
homology, 275

indefinite integrals, 136, 357
index
Fredholm, 340
of a function, 36
total, 69
integration, 355
interpolation, 112
inverse scattering, 20, 196, 238
inverse scattering transform, 87, 196, 236
inverse spectral map, 14, 17
inverse spectral problem, 14, 199, 200

Jacobi
elliptic functions, 8
operator, 14
variety, 276
Jordan curve, 46
jump
condition, 4
matrix, 4

KdV equation, 195
defocusing, modified, 195
kernel, 340

Laurent series, 3, 115
Lax pair, 52, 87, 198, 236
Lebesque space, 334
Legendre polynomials, 137, 190
lensing, 82
Lerch transcendent, 140
Lipschitz graph, 48
locally integrable, 359

Möbius transformation, 119, 120
matrix
-valued function, 342
norm, 335
meromorphic, 20
meromorphic differential, 276
method of nonlinear steepest descent, 87, 97
method of stationary phase, v
method of steepest descent, v
Miura transformation, 231
model problem, 264
moment conditions, 37, 284, 325
monodromy matrix, 12

Neumann boundary conditions, 247
NLS equation
defocusing, 87, 235
focusing, 235
nonlinear superposition, 318
nonsingular junction condition, 167
norming constants, 19, 215
numerical stability, 143

open endpoint, 359
open mapping theorem, 76
operator
(semi-)Fredholm, 340
compact, 340
orthogonal polynomials, 15

Painlevé II, 257
Painlevé transcendents, vi
parabolic cylinder function, 102, 264
parametrix, 84, 262
BA function, 283
global, 100
local, 173, 221, 289
partial indices, 69
Plemelj's lemma, 30, 125
Poisson kernel, 105
positons, 252
product condition, 63
projection
method, 157
operator, 158

range, 340

rapid oscillations, 175
rate of convergence, 348
reflection coefficient, 18, 95, 199, 236
region
  collisionless shock, 203
  dispersive, 201, 204
  Painlevé, 201, 203
  soliton, 201, 203
  transition, 203
regulator, 340
residue condition, 20, 200, 238
Riemann
  matrix, 275
  surface
    hyperelliptic, 274
Riemann theta functions, 273
Riemann–Hilbert
  problem, $\mathscr{E}^{\pm}(\Gamma)$ solution, 56
  correspondence, 13
  problem, 3
  problem, L^p, 24
  problem, continuous, 24
  problem, diagonal, 40
  problem, matrix, 40
  problem, scalar, 34
  problem, triangular, 43
right-standard factorization
    relative, 69
Robin boundary conditions, 246

scattering data, 17, 199, 200
  numerical computation, 205, 213, 238
scattering matrix, 18
scattering problem, 198
  Zakharov–Shabat, 198, 236
Schrödinger operator, 17
Schur decomposition, 44
Schwartz conjugate, 76
Schwarz invariant, 76
sectionally analytic function, 3, 23
singular integral equations, 62
smooth, 347
Sobolev space, 334
Sobolev spaces, 57
  of Zhou, 61
soliton, 195, 214, 235, 237
special function, vi
  nonlinear, vi
spectral data, 17, 19
spectral map, 14
spectral measure, 14
step-like finite-gap function, 311
Stokes constants, 257
Stokes’ phenomenon, 9
strongly uniform, 184

three-term recurrence relation, 15
transcendental, vi
trapezoidal rule, 110
truncation operator, 158

Uniform approximation, 231, 308
uniform approximation, 252, 264
uniformly accurate, 196

Vandermonde system, 112, 281
vanishing
  basis, 122
  Chebyshev basis, 123
  coefficient space, 121
vanishing lemma, 76, 96
vanishing solution, 76

weakly uniform, 184
Wiener algebra, 109
Wronskian, 18

zero-sum condition, 59