MATH 120A MIDTERM EXAM SOLUTIONS (WHITE)

FALL 2014

Problem 1 (10 points). Mark each statement 'T' for true or 'F' for false. You do NOT need to justify your answers.

T F "The set of all 3×3 matrices with entries in the set $\{-1, 0, 1\}$ is closed under matrix multiplication." False. For example consider A^2 where the entries of A are all equal to 1.

T F "Addition is a binary operation on the set of all nonzero integers." False. Both 1 and -1 are nonzero, but their sum is not nonzero.

T F "A binary operation on a set S is a function from S to $S \times S$." False. It is a function from $S \times S$ to S.

T F "If two groups are isomorphic, then their orders (cardinalities) must be the same." True. This is because every isomorphism is a bijection.

T F "The groups $(2\mathbb{Z}, +)$ and $(3\mathbb{Z}, +)$ are isomorphic, where $n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$." True. The function $\{(2n, 3n) : n \in \mathbb{Z}\}$ is an isomorphism between them. (Also you can just use the fact that there is only one infinite cyclic group up to isomorphism.)

T F "Every group has only finitely many subgroups." False. For example, $\mathbb{Z}$ has infinitely many subgroups $2\mathbb{Z}$, $3\mathbb{Z}$, $4\mathbb{Z}$, *etc.*

T F "Every finite cyclic group is isomorphic to $(\mathbb{Z}_n, +_n)$ for some n." True. We discussed this theorem in class.

T F "The dihedral group D_n (the group of symmetries of a regular n-gon) has order $n!$." False. It has order $2n$ (there are n rotations and n reflections,) and $2n < n!$ when $n \geq 4$.

T F "The group of nonzero real numbers under multiplication has a subgroup that is isomorphic to $(\mathbb{Z}, +)$." True. The group of nonzero real numbers under multiplication has elements of infinite order, such as 2. Consider the cyclic subgroup generated by 2.

T F "Every cyclic group is abelian." True. Whenever a is a group element and $m, n \in \mathbb{Z}$ we have $a^m a^n = a^{m+n} = a^{n+m} = a^n a^m$.

Problem 2 (4 points).

(a) Let $(S, *)$ be a binary structure. Define what it means for the operation $*$ to be commutative.

(b) Define precisely what it means for two groups $(G, *)$ and $(G', *')$ to be isomorphic. (If you use the word "isomorphism" in your definition, define that word also.)

Solution.

(a) The operation $*$ is *commutative* if $a * b = b * a$ for all $a, b \in S$.

(b) The groups $(G, *)$ and $(G', *')$ are *isomorphic* if there is a bijection $\phi : G \to G'$ such that $\phi(a * b) = \phi(a) *' \phi(b)$ for all $a, b \in G$.

Problem 3 (4 points). Prove or disprove the following statement: There is a subgroup of the dihedral group D_5 that is isomorphic to $(\mathbb{Z}_2, +_2)$.

Solution. We prove the statement. Consider any reflection $\sigma \in D_5$. Then σ^2 is the identity (and σ is not) so σ has order 2. This means that the cyclic cubgroup of D_5 that it generates is isomorphic to $\mathbb{Z}_2$.

Problem 4 (4 points). Let $(G, +)$ be an abelian group (possibly infinite) and let S be the subset of G consisting of all elements of finite order:

$$S = \{a \in G : na = 0 \text{ for some positive integer } n\}.$$

(Note the use of additive notation.) Prove that S is a subgroup of $(G, +)$.

Solution. There are three conditions to verify:

Closure under $+$: Let $a, b \in S$. Take positive integers m and n such that $ma = 0$ and $nb = 0$. Then $(mn)(a + b) = (mn)a + (mn)b = (nm)a + (mn)b = n(ma) + m(nb) = n0 + m0 = 0$, so $a + b \in S$.[1]

Identity: The identity element 0 has finite order (namely order 1) so it is in S.

Closure under inverses: Let $a \in S$. Take a positive integer n such that $na = 0$. Then $n(-a) = -(na) = -0 = 0$, so $-a \in S$.[2]

[1] To see that $(pq)c = p(qc)$ for all positive integers p and q and all group elements $c \in G$, write out both sides in terms of $c + c + c + \cdots$ some number of times. It might help to consider an example such as $p = 2$ and $q = 3$. Formally, the general case is proved by a double induction on p and q.

[2] To see that $-(na) = n(-a)$, write na and $n(-a)$ as $a + \cdots + a$ and $(-a) + \cdots + (-a)$ respectively. When added together, each $-a$ cancels an a and we get 0.

Problem 5 (4 points). By drawing a table, define a binary operation $*$ on the two-element set $\{0,1\}$ such that $*$ is associative, but the structure $(\{0,1\},*)$ is not a group. Justify your answer.

Solution. In the following structure, for all $a,b,c \in \{0,1\}$ we have $a*(b*c) = (a*b)*c$ because both sides are equal to zero. But the structure is not a group because it has no identity element (alternatively, because elements are repeated in rows and columns of the table.)

$*$	0	1
0	0	0
1	0	0

Problem 6 (4 points). Draw the subgroup diagram for $\mathbb{Z}_8$. You do NOT need to prove that your diagram is complete, but it must be complete in order to get full credit.

Solution.

$$\begin{tikzcd} \mathbb{Z}_8 \arrow[d, dash] \\ \{0,2,4,6\} \arrow[d, dash] \\ \{0,4\} \arrow[d, dash] \\ \{0\} \end{tikzcd}$$