MATH 120A MIDTERM EXAM SOLUTIONS (YELLOW)

FALL 2014

Problem 1 (10 points). Mark each statement 'T' for true or 'F' for false. You do NOT need to justify your answers.

T (F) "If G is an infinite group and $a \in G$ is not the identity, then the order of a must be infinite." False. For two examples, the infinite dihedral group D_∞ has elements of order 2, and the infinite group $\mathrm{GL}(2, \mathbb{R})$ has elements of every finite order.

(T) F "There is only one binary operation that can be defined on the set $\{0\}$." True. The only function from $\{0\} \times \{0\}$ to $\{0\}$ is $\{(0, 0), 0\}$, *i.e.* the function that sends the pair $(0, 0)$ to 0.

(T) F "Any two groups of order 3 are isomorphic." True. For a group with three elements e, a, and b where e is the identity, you can check that there is only one way to fill in the table for the group, and it looks like the table for $\mathbb{Z}_3$.

T (F) "Every finite group G has an element a such that the order of a is equal to the order of G." False. For example, the Klein group has order 4 but its elements have orders 1, 2, 2, and 2.

T (F) "The only automorphism of $(\mathbb{Z}_3, +_3)$ is the identity. (An *automorphism* of a group is an isomorphism of that group with itself.)" False. The function that switches 1 and 2 (and fixes 0) is also an automorphism of $\mathbb{Z}_3$. You can think of this automorphism as negation modulo 3.

(T) F "Every finite group has only finitely many subgroups." True. Every subgroup of a group G is a subset of G, and if G has finite order n then there are only 2^n many subsets of G. (In fact the number of subgroups is strictly less than 2^n unless G is trivial.)

T (F) "There is a surjection from $\mathbb{Q}$ onto $\mathbb{R}$." False. $\mathbb{Q}$ is countable but $\mathbb{R}$ is uncountable. (Ask me if you want to see proofs of these facts.)

T (F) "Any two abelian groups of order 4 are isomorphic." False. The Klein group is not isomorphic to $\mathbb{Z}_4$.

T (F) "The dihedral group D_5 has an element of order 3." False. The reflections have order 2, the identity has order 1, and the other elements (nonidentity rotations) have order 5.

(T) F "There is a binary operation $*$ on the set $\{3, 4, 5\}$ such that $(\{3, 4, 5\}, *)$ is a group." True. Take any bijection $\phi : \{3, 4, 5\} \to \mathbb{Z}_3$ and define an operation $*$ on $\{3, 4, 5\}$ by $a * b = \phi^{-1}(\phi(a) +_3 \phi(b))$. Then ϕ is an isomorphism of $(\{3, 4, 5\}, *)$ with $(\mathbb{Z}_3, +_3)$, so the binary structure $(\{3, 4, 5\}, *)$ is a group.

Problem 2 (4 points).

(a) Let $(S, *)$ be a binary structure. Define what it means for the operation $*$ to be associative.
(b) Let A be a set. How is the symmetric group S_A defined? Be sure to say what its elements are *and* what the operation is.

Solution.

(a) The operation $*$ is *associative* if $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$.
(b) The symmetric group S_A consists of permutations of the set A, with the operation given by composition.

Problem 3 (4 points). Prove or disprove the following statement: There is a subgroup of $(\mathbb{Z}, +)$ that is isomorphic to $(\mathbb{Z}_6, +_6)$.

Solution. We disprove the statement. Let H be a subgroup of $\mathbb{Z}$. If $H = \{0\}$ then clearly it is not isomorphic to $\mathbb{Z}_6$. Otherwise H has a nonzero element a. Because every nonzero element of $\mathbb{Z}$ has infinite order, H is infinite, so again it cannot be isomorphic to $\mathbb{Z}_6$.

Problem 4 (4 points). Let $(G, *)$ and $(G', *')$ be groups and let ϕ be a homomorphism from $(G, *)$ to $(G', *')$, meaning that $\phi(x * y) = \phi(x) *' \phi(y)$ for all $x, y \in G$. Let S be the range of ϕ:

$$S = \{x' \in G' : x' = \phi(x) \text{ for some } x \in G\}.$$

Prove that S is a subgroup of G'.

Solution. Let e and e' denote the identity elements of G and G' respectively. There are three conditions to verify:

Closure under $*'$: Let $x', y' \in S$. Take $x, y \in G$ such that $x' = \phi(x)$ and $y' = \phi(y)$. Then $x' *' y' = \phi(x) *' \phi(y) = \phi(x * y) \in S$.

Identity: $\phi(e)$ is the identity element of G', and it is in S. (I mentioned in class that homomorphisms preserve the identity, so you can use this fact. For completeness, here is the rest of the proof: $\phi(e) *' \phi(e) = \phi(e * e) = \phi(e) = \phi(e) *' e'$ where e' is the identity of G'. Then cancel $\phi(e)$ on the left.)

Closure under inverses: Let $x' \in S$. Take $x \in G$ such that $x' = \phi(x)$. Then $\phi(x^{-1})$, which is in S, is the inverse of x'. (I mentioned in class that homomorphisms preserve inverses, so you can use this fact. For completeness, here is the rest of the proof: $\phi(x^{-1}) *' x' = \phi(x^{-1}) *' \phi(x) = \phi(x^{-1} * x) = \phi(e) = e'$ and similarly $x' *' \phi(x^{-1}) = e'$.)

Problem 5 (4 points). By drawing a table, define a binary operation $*$ on the two-element set $\{0, 1\}$ that is not associative. Explain why your operation is not associative.

Solution. In the following structure we have $0 * (0 * 1) = 0 * 1 = 1$ but $(0 * 0) * 1 = 1 * 1 = 0$. (Clearly the table for $\mathbb{Z}_2$ won't work, and it can't be all zeroes or all ones. I found this example by trial and error on the remaining possibilities.)

$*$	0	1
0	1	1
1	0	0

Problem 6 (4 points). Draw the subgroup diagram for $\mathbb{Z}_{14}$. You do NOT need to prove that your diagram is complete, but it must be complete in order to get full credit.

Solution.