MATH 120A SAMPLE FINAL EXAM

FALL 2014

Student name:

Student ID number:

Instructions

- Books, notes, and electronic devices may NOT be used. These items must be kept in a closed backpack or otherwise hidden from view during the exam.
- Cheating in any form may result in an F grade for the course as well as administrative sanctions.
- The time remaining will be written on the board periodically.
- When time is called, you must stop working immediately.
- If you want to leave your seat for any reason before time is called, raise your hand and remain seated until acknowledged.

1	/ 16
2	/ 4
3	/ 4
4	/ 4
5	/ 6
6	/ 4
7	/ 4
8	/ 4
Total	/ 46

Problem 1 (16 points). Mark each statement 'T' for true (meaning always true) or 'F' for false (meaning sometimes false). You do NOT need to justify your answers.

T F Every subgroup of a cyclic group is cyclic.
T F There is a bijection from $\mathbb{Q}$ to $\mathbb{R}$.
T F Every finite group is isomorphic to a subgroup of S_n for some n.
T F If $\sigma, \tau \in S_n$ are disjoint cycles, then $\sigma\tau = \tau\sigma$.

T F If the two permutations $\sigma, \tau \in S_n$ are conjugate to one another, then σ and τ have the same orbits.
T F Let G be a group and let H be a normal subgroup of G. If G/H and H are cyclic, then G is cyclic.
T F If G is a group and H is a normal subgroup of G, then H is the kernel of some homorphism $\phi : G \to G'$ for some group G'.
T F $\mathbb{Z}_3 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_9$.

T F Let G_1, G_2, G_1', and G_2' be groups. If G_1 is isomorphic to G_1' and G_2 is isomorphic to G_2', then $G_1 \times G_2$ is isomorphic to $G_1' \times G_2'$.
T F Let G be a group and let H be a subgroup of G. If $ah = ha$ for all $h \in H$ and $a \in G$, then H is a normal subgroup of G.
T F The direct product $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if m and n are relatively prime.
T F There is a bijection from $\mathbb{Z}$ to $\mathbb{Q}$.

T F For all $m, n \in \mathbb{Z}^+$ the direct product of symmetric groups $S_m \times S_n$ is isomorphic to S_{m+n}.
T F Every element of the symmetric group S_n is a product of disjoint cycles.
T F If G is a group and n divides the order of G, then G has an element of order n.
T F Every group of prime order is cyclic.

Problem 2 (4 points). Define three subgroups of $\mathbb{Z} \times \mathbb{Z}$ of index 2, and show that each of your three subgoups has index 2.

Problem 3 (4 points). Let N_1 and N_2 be normal subgroups of a group G. Prove that the set $N_1 N_2$ defined by

$$N_1 N_2 = \{x_1 x_2 : x_1 \in N_1 \text{ and } x_2 \in N_2\}$$

is also a normal subgroup of G.

Problem 4 (4 points). Let ρ be the element of D_4 corresponding to a rotation by a half-turn (180°). Note that $\rho\sigma = \sigma\rho$ for every element $\sigma \in D_4$ (you can use this fact without proof). Calculate the factor group $D_4/\langle\rho\rangle$ (*i.e.* show that it is isomorphic to some familiar group).

Problem 5 (6 points). Give definitions of the following terms:

(a) Let G be a group. What is a *subgroup* of G?

(b) Let G and G' be groups. What is a *homomorphism* from G to G'?

(c) Let X and Y be sets, let $f : X \to Y$ be a function, and let B be a subset of Y. What is the *inverse image* of B under f (denoted by $f^{-1}[B]$)?

Problem 6 (4 points). In the symmetric group S_4:

- How many elements have order 1?
- How many elements have order 2?
- How many elements have order 3?
- How many elements have order 4?
- How many elements have order 5?

Problem 7 (4 points). List, up to isomorphism, all abelian groups of order 225. Which group in your list is isomorphic to $\mathbb{Z}_{15} \times \mathbb{Z}_{15}$? Why?

Problem 8 (4 points). Let G be a group and suppose that $a \in G$ has order 2, but no other element of G has order 2. Prove that $ax = xa$ for all $x \in G$.