MATH 120A SAMPLE FINAL EXAM SOLUTIONS

FALL 2014

Problem 1 (16 points).

- (T) F "Every subgroup of a cyclic group is cyclic." True. We discussed this theorem in class.
- T (F) "There is a bijection from $\mathbb{Q}$ to $\mathbb{R}$." False: $\mathbb{Q}$ is countable, but $\mathbb{R}$ is uncountable (by Cantor).
- (T) F "Every finite group is isomorphic to a subgroup of S_n for some n." True: by Cayley's theorem every group G is isomorphic to a subgroup of S_G, which in turn is isomorphic to S_n where n is the order of G.
- (T) F "If $\sigma, \tau \in S_n$ are disjoint cycles, then $\sigma\tau = \tau\sigma$." True. We proved this in class.

- T (F) "If the two permutations $\sigma, \tau \in S_n$ are conjugate to one another, then σ and τ have the same orbits." False: consider (1 2) and (1 3) in S_3.
- T (F) "Let G be a group and let H be a normal subgroup of G. If G/H and H are cyclic, then G is cyclic." False: consider $G = \mathbb{Z} \times \mathbb{Z}$ and $H = \mathbb{Z} \times \{0\}$.
- (T) F "If G is a group and H is a normal subgroup of G, then H is the kernel of some homorphism $\phi : G \to G'$ for some group G'." True: consider $G' = G/H$ and $\phi(a) = aH$.
- T (F) "$\mathbb{Z}_3 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_9$." False: in $\mathbb{Z}_3 \times \mathbb{Z}_3$ every element has order 1 or 3, whereas in $\mathbb{Z}_9$ some elements have order 9.

- (T) F "Let G_1, G_2, G_1', and G_2' be groups. If G_1 is isomorphic to G_1' and G_2 is isomorphic to G_2', then $G_1 \times G_2$ is isomorphic to $G_1' \times G_2'$." True: it is routine to check that if $\phi_1 : G_1 \to G_1'$ and $\phi_2 : G_2 \to G_2'$ are isomorphisms, then we can define an isomorphism $\phi : G_1 \times G_2 \to G_1' \times G_2'$ by $\phi((a_1, a_2)) = (\phi_1(a_1), \phi_2(a_2))$.
- (T) F "Let G be a group and let H be a subgroup of G. If $ah = ha$ for all $h \in H$ and $a \in G$, then H is a normal subgroup of G." True. Note that this condition is strictly stronger than normality.
- (T) F "The direct product $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if m and n are relatively prime." True. We discussed this theorem in class.
- (T) F "There is a bijection from $\mathbb{Z}$ to $\mathbb{Q}$." True. See p. 5 of the textbook.

T (F) "For all $m, n \in \mathbb{Z}^+$ the direct product of symmetric groups $S_m \times S_n$ is isomorphic to S_{m+n}." False. They do not even have the same cardinality. (However, $S_m \times S_n$ is isomorphic to a *subgroup* of S_{m+n}.)

(T) F "Every element of the symmetric group S_n is a product of disjoint cycles." True: we discussed this theorem in class.

T (F) "If G is a group and n divides the order of G, then G has an element of order n." False: consider $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $n = 4$.

(T) F "Every group of prime order is cyclic." True: trivial groups are cyclic, and if G is a nontrivial group, take a nonidentity element $a \in G$ and note that its order must divide the order of G by Lagrange's theorem. So if G has prime order, the order of a is equal to the order of G, which implies that a generates G.

Problem 2 (4 points). Define three subgroups of $\mathbb{Z} \times \mathbb{Z}$ of index 2, and show that each of your three subgoups has index 2.

Solution.

(1) The subgroup $\mathbb{Z} \times 2\mathbb{Z}$ has cosets $\mathbb{Z} \times 2\mathbb{Z}$ and $(\mathbb{Z} \times 2\mathbb{Z}) + (0, 1)$, so its index is 2.
(2) The subgroup $2\mathbb{Z} \times \mathbb{Z}$ has cosets $2\mathbb{Z} \times \mathbb{Z}$ and $(2\mathbb{Z} \times \mathbb{Z}) + (1, 0)$, so its index is 2.
(3) The subgroup $H = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i + j \text{ is even}\}$ has cosets H and $H + (0, 1) = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i + j \text{ is odd}\}$, so its index is 2.

Problem 3 (4 points). Let N_1 and N_2 be normal subgroups of a group G. Prove that the set N_1N_2 defined by

$$N_1N_2 = \{x_1x_2 : x_1 \in N_1 \text{ and } x_2 \in N_2\}$$

is also a normal subgroup of G.

Solution.

Identity: Denote the identity element of G by e. Then $e = ee \in N_1N_2$.

Inverses: Let $z \in N_1N_2$, say $z = x_1x_2$ where $x_1 \in N_1$ and $x_2 \in N_2$. Then $z^{-1} = (x_1x_2)^{-1} = x_2^{-1}x_1^{-1} = x'x_2^{-1} \in N_1N_2$ for some $x' \in N_1$ because N_1 is normal.

Closure: Let $z, t \in N_1N_2$, say $z = x_1x_2$ and $t = y_1y_2$ where $x_1, y_1 \in N_1$ and $x_2, y_2 \in N_2$. Then $zt = x_1x_2y_1y_2 = x_1y'x_2y_2 \in N_1N_2$ for some $y' \in N_1$ because N_1 is normal.

Normality: Let $z \in N_1N_2$, say $z = x_1x_2$ where $x_1 \in N_1$ and $x_2 \in N_2$. Let $g \in G$. Then $gzg^{-1} = gx_1x_2g^{-1} = x'gx_2g^{-1} = x'x''gg^{-1} = x'x'' \in N_1N_2$ for some $x' \in N_1$ and $x'' \in N_2$ because N_1 and N_2 are normal.

Problem 4 (4 points). Let ρ be the element of D_4 corresponding to a rotation by a half-turn ($180°$). Note that $\rho\sigma = \sigma\rho$ for every element $\sigma \in D_4$ (you can use this fact without proof). Calculate the factor group $D_4/\langle\rho\rangle$ (*i.e.* show that it is isomorphic to some familiar group).

Solution. Note that ρ^2 is the identity element but ρ is not, so ρ has order 2. Therefore $D_4/\langle\rho\rangle$ has order $8/2 = 4$, and it must be isomorphic either to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Look at the orders of elements $\sigma\langle\rho\rangle \in D_4/\langle\rho\rangle$ (recall that the order of $\sigma\langle\rho\rangle$ is equal to the least $n \in \mathbb{Z}^+$ such that $\sigma^n \in \langle\rho\rangle$.) There are three cases:

- If σ is a rotation by $0°$ or $180°$, then $\sigma \in \langle\rho\rangle$, so $\sigma\langle\rho\rangle$ is the identity.
- If σ is a rotation by $90°$ or $270°$, then $\sigma \notin \langle\rho\rangle$ but $\sigma^2 = \rho \in \langle\rho\rangle$, so $\sigma\langle\rho\rangle$ has order 2.
- If σ is a reflection, then $\sigma \notin \langle\rho\rangle$, but $\sigma^2 = \text{identity} \in \langle\rho\rangle$, so $\sigma\langle\rho\rangle$ has order 2.

Therefore $D_4/\langle\rho\rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (rather than $\mathbb{Z}_4$.)

Problem 5 (6 points). Give definitions of the following terms:

(a) Let G be a group. What is a *subgroup* of G?

(b) Let G and G' be groups. What is a *homomorphism* from G to G'?

(c) Let X and Y be sets, let $f : X \to Y$ be a function, and let B be a subset of Y. What is the *inverse image* of B under f (denoted by $f^{-1}[B]$)?

Solution.

(a) A *subgroup* of G is a subset of G that is a group under the (restriction of the) operation from G. (Alternatively: a *subgroup* of G is a subset of G that contains the identity element of G, is closed under inverses in the sense of G, and is closed under the operation of G.)

(b) A *homomorphism* from G to G' is a function $f : G \to G'$ such that $f(ab) = f(a)f(b)$ for all $a, b \in G$.

(c) $f^{-1}[B] = \{x \in X : f(x) \in B\}$.

Problem 6 (4 points). In the symmetric group S_4:

- How many elements have order 1?
- How many elements have order 2?
- How many elements have order 3?
- How many elements have order 4?
- How many elements have order 5?

Solution.

- One element has order 1 (the identity.)
- Nine elements have order 2 (six transpositions and three products of disjoint transpositions.)
- Eight elements have order 3 (they are all cycles of length 3.)
- Six elements have order 4 (they are all cycles of length 4.)
- No elements have order 5 (beware that this is not just a consequence of the fact that $5 > 4$; there are elements of order 6 in S_5, for example.)

Problem 7 (4 points). List, up to isomorphism, all abelian groups of order 225. Which group in your list is isomorphic to $\mathbb{Z}_{15} \times \mathbb{Z}_{15}$? Why?

Solution. $225 = 15^2 = 3^2 5^2$, so our list is:

(1) $\mathbb{Z}_{3^2} \times \mathbb{Z}_{5^2}$
(2) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{5^2}$
(3) $\mathbb{Z}_{3^2} \times \mathbb{Z}_5 \times \mathbb{Z}_5$
(4) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$.

We have

$$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \cong (\mathbb{Z}_3 \times \mathbb{Z}_5) \times (\mathbb{Z}_3 \times \mathbb{Z}_5) \cong \mathbb{Z}_{15} \times \mathbb{Z}_{15}.$$

($\mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{15}$ because 3 and 5 are relatively prime.)

Problem 8 (4 points). Let G be a group and suppose that $a \in G$ has order 2, but no other element of G has order 2. Prove that $ax = xa$ for all $x \in G$.

Solution. Denote the identity element of G by e. Note that $xax^{-1} \neq e$: otherwise, multiplying on the left by x^{-1} and on the right by x, we would get $a = e$. However, $(xax^{-1})^2 = xax^{-1}xax^{-1} = xa^2x^{-1} = xex^{-1} = e$. Therefore xax^{-1} has order 2, so by our hypothesis we have $xax^{-1} = a$. Multiplying by x on the right gives $xa = ax$.