MATH 120A SAMPLE MIDTERM EXAM SOLUTIONS

FALL 2014

Problem 1 (10 points). Mark each statement 'T' for true or 'F' for false. You do NOT need to justify your answers.

T (F) "Any two groups of order 4 are isomorphic." False: $\mathbb{Z}_4$ is not isomorphic to the Klein group V.

T (F) "For every $n \in \mathbb{Z}^+$ there is an element of order n in the group $(\mathbb{Z}, +)$." False: every nonzero element of $\mathbb{Z}$ has infinite order.

(T) F "Multiplication is an associative operation on the set of all 2×2 real matrices." True: this is a tedious calculation, but the result is important to know.

(T) F "The group $(\mathbb{Z}, +)$ is isomorphic to one of its proper subgroups." True: for example $\mathbb{Z}$ is isomorphic to $2\mathbb{Z}$ (the subgroup consisting of all even integers) via the isomorphism that sends n to $2n$.

T (F) "The relation $\{(a, x), (a, y)\}$ is a function from the set $\{a, b\}$ to the set $\{x, y, z\}$." False: it assigns two values to a and no value to b.

T (F) "Multiplication (meaning composition) is a commutative operation on the set of all permutations of $\{1, 2, 3\}$." False: the permutation that switches 1 and 2 does not commute with the permutation that switches 2 and 3, for example.

T (F) "Up to isomorphism, there is only one infinite abelian group." False: $\mathbb{Q}$ and $\mathbb{Z}$ are both infinite abelian groups, but they are not isomorphic.

(T) F "If G and H are isomorphic groups and every element of G has order 2, then every element of H must have order 2 also." True: if $\varphi : G \to H$ is an isomorphism and $b \in H$, take $a \in G$ with $\varphi(a) = b$. Then $b^2 = \varphi(a)^2 = \varphi(a^2) = \varphi(e) = e$. (Actually the question as stated is vacuously true because the identity element of G cannot have order 2; I meant to say "every *nonidentity* element.")

T (F) "Every abelian group is cyclic." False: $\mathbb{R}$ and $\mathbb{Q}$ (under addition) and the Klein group V are all examples of abelian groups that are not cyclic.

T (F) "The group $(\mathbb{Z}_7, +_7)$ has an element of order 6." False: if $6a = 0$ in $\mathbb{Z}_7$, then because $7a = 0$ also, we can subtract to get $a = 0$. (More generally, if p is prime then one can show that every nonidentity element of $\mathbb{Z}_p$ has order p.)

Problem 2 (4 points). Let G be a group and let $a \in G$.

(a) Define the *order* of a. (Some examples of correct definitions follow.)
- The order of a is the cardinality of $\langle a \rangle$.
- The order of a is the order of the cyclic subgroup of G generated by a.
- The order of a is $|\langle a \rangle|$.
- The order of a is the least positive integer n such that $a^n = e$, or infinity if there is no such positive integer n.

(b) Define what it means for a to be a *generator* of G. (Some examples of correct definitions follow.)
- a is a generator of G if $\langle a \rangle = G$.
- a is a generator of G if the cyclic subgroup of G it generates is equal to G itself.
- a is a generator of G if every element of G is equal to a^n for some $n \in \mathbb{Z}$.
- a is a generator of G if $G = \{a^n : n \in \mathbb{Z}\}$.

Problem 3 (4 points). Prove or disprove the following statement: The group $(\mathbb{Z}_6, +_6)$ has a subgroup H that is isomorphic to $(\mathbb{Z}_4, +_4)$.

Solution. The statement is false. If H is isomorphic to $\mathbb{Z}_4$ then it has some element a of order 4 (namely the image of a generator of $\mathbb{Z}_4$ under such an isomorphism.) But if H is a subgroup of $\mathbb{Z}_6$ then the only possible orders of its elements are 1, 2, 3, or 6, because these are the only orders of elements in $\mathbb{Z}_6$.

Alternatively:

Solution. The statement is false. The only subgroups of $\mathbb{Z}_6$ are $\{0\}$, $\mathbb{Z}_6$, $\{0, 2, 4\}$, and $\{0, 3\}$. None of these subgroups has order 4, so none of them can be isomorphic to $\mathbb{Z}_4$.

Problem 4 (4 points). Let G be a group and let S be a subset of G. Define another subset C of G by

$$C = \{g \in G : sg = gs \text{ for all } s \in S\}.$$

Prove that C is a subgroup of G.

Solution. We show that C contains the identity element, is closed under inverses, and is closed under the operation.

identity: The identity element e of G satisfies $es = s = se$ for all $s \in S$, so $e \in C$.

inverses: If $g \in C$, then for all $s \in S$ we have $sg = gs$. Multiplying both sides on the left and right by g^{-1} we get $g^{-1}s = sg^{-1}$ for all $s \in S$, so $g^{-1} \in C$.

closure: If $g_1, g_2 \in C$, then for all $s \in S$ we have $sg_1 = g_1 s$ and $sg_2 = g_2 s$. Therefore $sg_1 g_2 = g_1 s g_2 = g_1 g_2 s$ for all $s \in S$, so $g_1 g_2 \in C$.

Problem 5 (4 points). By drawing a table, define a binary operation $*$ on the two-element set $\{a, b\}$ such that the structure $(\{a, b\}, *)$ has an identity element, but is not a group. Explain why your structure has the properties you claim.

Solution. The following structure has a as an identity element, but it is not a group because cancellation fails: $ba = bb$. (Also, b has no inverse.)

$*$	a	b
a	a	b
b	b	b

Problem 6 (4 points). Up to isomorphism, there are exactly two groups of order 4, namely $\mathbb{Z}_4$ and another group V whose elements we will call e, a, b, and c where e is the identity. Draw a subgroup diagram for $\mathbb{Z}_4$ and another subgroup diagram for V.

Solution.