Math 120A HW 8

Due Wednesday, March 4.

1. Consider the group $G = \mathbb{Z}_9$ (under addition modulo 9) and the cyclic subgroup $H = \langle 3 \rangle$. The group is abelian, so we can just talk about cosets rather than left or right cosets.
   a. Compute all of the cosets of H in G (for example, write $2 + H = \{2, 5, 8\}$.)
   b. Which cosets in part (a) are equal to which other cosets in part (a)?
2. Let G be the group of all nonzero real numbers under multiplication and let H be the cyclic subgroup of G generated by -2. Let $a = 1$, $b = 1/2$, and $c = 1/3$. (Again, the group is abelian, so we can just talk about cosets rather than left or right cosets.)
   a. "Compute" the cosets aH, bH, and cH (for example, write $aH = \{\ldots \text{some elements} \ldots\}$ where you list enough elements that the pattern is clear.)
   b. Let $d = -4$ and prove that dH is equal to one of the cosets from part (a). (Give an actual proof, don't just say that they look the same so far.)
3. Let G be a group and let H be a subgroup of G. Recall that the left coset relation $\sim_{\mathrm{L}}$ on G is defined by $a \sim_{\mathrm{L}} b \iff b \in aH$. Prove that the left coset relation is transitive. (In class we proved that it is reflective and symmetric, so you can then conclude that it is an equivalence relation.) *Hint:* $b \in aH$ *means* $b = ah$ *for some element* $h \in H$.
4. Let G be a group and let H be a subgroup of G.
   a. Assuming that G is finite, prove that the number of left cosets of H in G is equal to the number of right cosets of H in G by using a counting argument.
   b. Explain why your counting argument for part (a) does not work when G and H are infinite.
   c. Prove in general (not assuming that G is finite) that the number of left cosets of H in G is equal to the number of right cosets of H in G. In particular, prove that there is a bijection f from the set of all left cosets of H in G to the set of all right cosets of H in G, given by $f(aH) = Ha^{-1}$. (Don't forget to prove that f is well-defined! If we did the more obvious thing and wrote $f(aH) = Ha$, it would not define a function, because $aH = bH$ does not generally imply $Ha = Hb$.)
5. Consider the group $\mathbb{Z}_3 \times \mathbb{Z}_3$ (with the operation of addition mod 3 in each coordinate.)
   a. Compute the cyclic subgroup generated by each element.
   b. Is $\mathbb{Z}_3 \times \mathbb{Z}_3$ cyclic? How could you answer this part without doing part (a) first?
   c. Prove that if p is a prime number, then every element of $\mathbb{Z}_p \times \mathbb{Z}_p$ has order 1 or p.

6. Prove that the group $\mathbb{Z} \times \mathbb{Z}_2$ (with the operation of ordinary addition in the first coordinate and addition mod 2 in the second coordinate) is not cyclic. (Note that it doesn't make any sense to talk about a number being relatively prime to infinity, so you will a different argument.)