MATH 120A MIDTERM EXAM
(YELLOW PAPER)

WINTER 2015

Student name:

Student ID number:

INSTRUCTIONS

• Books, notes, and electronic devices may NOT be used. These items must be kept in a closed backpack or otherwise hidden from view during the exam.
• Cheating in any form may result in an F grade for the course as well as administrative sanctions.
• When time is called, you must stop working immediately, close your exam, and remain seated until your exam is collected.
• If you want to leave your seat for any reason before time is called, raise your hand and remain seated until acknowledged.

<table>
<thead>
<tr>
<th></th>
<th>/ 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>/ 4</td>
</tr>
<tr>
<td>3</td>
<td>/ 4</td>
</tr>
<tr>
<td>4</td>
<td>/ 4</td>
</tr>
<tr>
<td>5</td>
<td>/ 4</td>
</tr>
<tr>
<td>6</td>
<td>/ 4</td>
</tr>
<tr>
<td>Total</td>
<td>/ 30</td>
</tr>
</tbody>
</table>
Problem 1 (10 points). Mark each statement ‘T’ for true (meaning always true) or ‘F’ for false (meaning sometimes false). You do NOT need to justify your answers to this problem.

T  F  If $G$ is a group whose operation is associative, then $G$ is called an abelian group.
T  F  Let $(S, *)$ be a binary structure and let $A, B \subseteq S$. If $A$ and $B$ are closed under $*$, then their union $A \cup B$ is closed under $*$.
T  F  The set of all nonzero $3 \times 3$ matrices with real entries forms a group under the operation of matrix multiplication.
T  F  The group $(\mathbb{C}^*, \cdot)$ of nonzero complex numbers under multiplication has a subgroup that is isomorphic to $\mathbb{Z}_4$.
T  F  If a group $G$ has an element of order 2, then it also has a subgroup of order 2.
T  F  Any two groups of order 2 are isomorphic to each other.
T  F  Being abelian is a structural property of groups.
T  F  Let $G$ be a group and $a, b, c \in G$. If $aba = aca$, then $b = c$.
T  F  Every cyclic group is abelian.
T  F  There is a bijection from $\mathbb{R}$ to $\mathbb{Q}$.

Problem 2 (4 points). Define the italicized terms:
(a) What is an identity element for a binary structure $(S, *)$?

(b) Assume that $(S, *)$ has an identity element and let $a \in S$.
What is an inverse of $a$ in $(S, *)$?
Problem 3 (4 points). Let \( n \in \mathbb{N} \) and recall that \( GL_n(\mathbb{R}) \) denotes the group of all invertible \( n \times n \) matrices with real entries under the operation of matrix multiplication. Let \( \vec{x} \in \mathbb{R}^n \) and define the set
\[
H = \{ A \in GL_n(\mathbb{R}) : A\vec{x} = \vec{x} \},
\]
where \( A\vec{x} \) denotes the product of a matrix and a vector in the usual sense. Prove that \( H \) is a subgroup of \( GL_n(\mathbb{R}) \).

Problem 4 (4 points). Let \((S, \ast)\) denote an arbitrary binary structure. Prove that the property \( P \) saying
\[
a \ast a = a \text{ for every } a \in S
\]
is a structural property.
Problem 5 (4 points). For each part, make sure to justify your answer:
(a) Let $\mathbb{R}^*$ denote the set of all nonzero real numbers.
   Give an example of a nontrivial subgroup of $(\mathbb{R}^*, \cdot)$ that is finite.

(b) Give an example of a proper subgroup of $(\mathbb{Q}, +)$ that is NOT cyclic.

Problem 6 (4 points). Consider the group $\mathbb{Z}_{12}$ with the operation of addition modulo 12.
(a) List all of the subgroups of $\mathbb{Z}_{12}$.
   How do you know that your list is complete?

(b) Draw the subgroup diagram of $\mathbb{Z}_{12}$. 