MATH 120B SAMPLE FINAL EXAM

SPRING 2015

Student name:

Student ID number:

Instructions

- Books, notes, and electronic devices may NOT be used. These items must be kept in a closed backpack or otherwise hidden from view during the exam.
- Cheating in any form may result in an F grade for the course as well as administrative sanctions.
- When time is called, you must stop working immediately, close your exam, and remain seated until your exam is collected.
- If you want to leave your seat for any reason before time is called, raise your hand and remain seated until acknowledged.

1	/ 16
2	/ 4
3	/ 4
4	/ 4
5	/ 6
6	/ 4
7	/ 4
8	/ 4
Total	/ 46

Problem 1 (16 points). Mark each statement 'T' for true (meaning always true) or 'F' for false (meaning sometimes false). You do NOT need to justify your answers to this problem.

T F $\mathbb{Z}_2 \times \mathbb{Z}_{10} \cong \mathbb{Z}_{20}$.

T F For any two positive integers a and b, if $\gcd(a, b) = 1$ then $\varphi(ab) = \varphi(a)\varphi(b)$, where φ denotes Euler's phi function.

T F If F_1 and F_2 are integral domains, then their direct product $F_1 \times F_2$ is an integral domain.

T F If the fields F_1 and F_2 are isomorphic to each other, then their multiplicative groups F_1^* and F_2^* are isomorphic to each other.

T F There is a finite field of order 6.

T F If R is a ring and $f(x), g(x) \in R[x]$ are nonzero then $\deg f(x)g(x) = \deg f(x) + \deg g(x)$.

T F If p is a prime number and $f(x) \in \mathbb{Z}_p[x]$ is nonzero, then $f(x)$ has a zero in $\mathbb{Z}_p$.

T F Let K be a field and let $f(x) \in K[x]$. If $f(x)$ has no zeroes in K, then $f(x)$ is irreducible in $K[x]$.

T F If D is an integral domain then the polynomial ring $D[x]$ is an integral domain.

T F Every proper ideal of an integral domain is prime.

T F Every ideal is the kernel of some homomorphism.

T F Let R and R' be rings and let $\phi : R \to R'$ be a homomorphism. Then $\ker \phi$ is an ideal of R.

T F Let R and R' be commutative rings, let $\phi : R \to R'$ be a homomorphism, and let N' be a prime ideal of R'. Then $\varphi^{-1}[N']$ is either a prime ideal of R or the improper ideal of R.

T F $\mathbb{Q}$ is an extension field of $\mathbb{Z}_2$.

T F $\mathbb{Z}_2$ has an extension field E of order 4 (meaning $|E| = 4$.)

T F $\mathbb{Z}_2$ has an extension field E of order 5 (meaning $|E| = 5$.)

Problem 2 (4 points). For each part, answer yes or no, and *justify your answer.*
(a) Does $\mathbb{Z}_8$ have a subring that is isomorphic to $\mathbb{Z}_2$?

(b) Does $\mathbb{Z}_6$ have a subring that is isomorphic to $\mathbb{Z}_2$?

Problem 3 (4 points). Compute 3^{70} mod 23. Show your work, along with appropriate explanation and justification.

Problem 4 (4 points). Find the quotient and remainder when the polynomial x^4+3x^2+2x+1 in $\mathbb{Z}_5[x]$ is divided by the polynomial x^2+2x+3 in $\mathbb{Z}_5[x]$.

Problem 5 (6 points). Give the requested definitions.

(a) Let R be a ring. What is a *subring of* R?

(b) Let F be a field, let E be an extension field of F, and let $\alpha \in E$.
What does it mean for α to be *algebraic* over F?

(c) Let R and R' be rings and let $\phi : R \to R'$ be a homomorphism.
What is the *kernel* of ϕ?

Problem 6 (4 points). Let R be a commutative ring and define

$$N = \{a \in R : a^n = 0 \text{ for some } n \in \mathbb{Z}^+\}.$$

Prove that N is an ideal of R.

Problem 7 (4 points). For each part, make sure to justify your answer.

(a) Give an example of a commutative ring R with unity such that the trivial ideal $\{0\}$ of R is prime but not maximal.

(b) If R is a commutative ring with unity and N is a nontrivial prime ideal of R, must N be maximal? Prove it or give a counterexample.

Problem 8 (4 points). Define the real number $\alpha = 2^{2/3} - 1$.

(a) Prove that α is algebraic (over $\mathbb{Q}$).

(b) Find $\deg(\alpha, \mathbb{Q})$, meaning the degree of α over $\mathbb{Q}$, and $\mathrm{irr}(\alpha, \mathbb{Q})$, meaning the irreducible polynomial of α over $\mathbb{Q}$. Justify your answer.