MATH 13
SETS AND BIJECTIONS

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This is an outline of the material that I have covered (or will cover) from Chapter 10. It is almost all in the book, but I am skipping around a bit so I thought it would help to collect the relevant material here. I added a few things that I didn't have time to discuss in the lecture but which are relevant and might be helpful to you. The proofs here are mostly just sketches and so you can get some practice by filling them in. For some of the results stated below, the full proofs would be rather long, but ideally you should be able to complete them all given enough time.

1. The relation "≈"

Definition 1.1. Let A and B be sets. We take "$A \approx B$" to mean "there is a bijection from A to B."

Proposition 1.2. *The relation $A \approx B$ is an equivalence relation.*[1]

Proof idea. For reflexivity, use identity functions. For symmetry, use inverse functions. For transitivity, use composition of functions. □

Proposition 1.3. *If $m \in \mathbb{N}$ then $\mathbb{N} \approx \mathbb{N} - \{m\}$.*

Proof idea. Define a function $f : \mathbb{N} \to \mathbb{N} - \{m\}$ by

$$f(x) = \begin{cases} x & \text{if } x < m, \text{ and} \\ x+1 & \text{if } x \geq m. \end{cases}$$

Show that f is a bijection. □

Proposition 1.4. $\mathbb{N} \approx \mathbb{Z}$.

Proof idea. Define a function $f : \mathbb{N} \to \mathbb{Z}$ by $f(1) = 0$, $f(2) = 1$, $f(3) = -1$, $f(4) = 2$, $f(5) = -2$, *etc.* □

Proposition 1.5. *Let A be a set. Then $\{0,1\}^A \approx \mathcal{P}(A)$.*[2]

Proof idea. Define a function $G : \{0,1\}^A \to \mathcal{P}(A)$ as follows. For an element $f \in \{0,1\}^A$ (which is itself a function) we define $G(f) = f^{-1}(\{1\})$. In other words, $G(f)$ is the subset of A consisting of all of the elements of A that are mapped to 1 (rather than 0) by the function f. You can show that G is a bijection. □

[1] Technically we should say that for every set S of sets, the relation R on S defined by $R = \{(A,B) \in S : A \approx B\}$ is an equivalence relation. The proof would look exactly the same, so we don't bother.

[2] Recall that B^A denotes the set of functions from A to B, and $\mathcal{P}(A)$ denotes the set of all subsets of A.

2. Finite sets

Definition 2.1. We say a set A is *finite* if A is empty or $\{1, \ldots, m\} \approx A$ for some $m \in \mathbb{N}$. Otherwise (if A is nonempty and there is no such bijection) we say that A is *infinite*.

In other words, a nonempty set A is finite if $A = \{a_1, \ldots, a_m\}$ for some natural number m, where $a_i \neq a_j$ whenever $i \neq j$. The notations a_i and $f(i)$, where f is a bijection from $\{1, \ldots, m\}$ to A as in the definition of "finite," are two ways of writing the same thing.

Example 2.2. The set $A = \{5, 7, 8\}$ is finite because there is a bijection from $\{1, 2, 3\}$ to A: for example $\{(1, 5), (2, 7), (3, 8)\}$.

Proposition 2.3. *The set* $\mathbb{N}$ *is infinite.*

Proof idea. Show by induction that for every $m \in \mathbb{N}$, every function $f : \{1, \ldots, m\} \to \mathbb{N}$ has the property that its range has a largest element. Because $\mathbb{N}$ has no largest element, it cannot be the range of such a function. □

Proposition 2.4. *Let* $m \in \mathbb{N}$*. If* $A \subseteq \{1, \ldots, m\}$*, then* A *is finite.*

Proof idea. Use induction on m. Let $A \subseteq \{1, \ldots, m + 1\}$. The nontrivial case is when $m + 1 \in A$. If there is a bijection $f : \{1, \ldots, n\} \to A \cap \{1, \ldots, m\}$ for some $n \in \mathbb{N}$, then consider $f \cup \{(n + 1, m + 1)\}$. □

Proposition 2.5. *Let* A *and* B *be sets.*

(1) *If* B *is finite and* $A \subseteq B$*, then* A *is finite.*
(2) *If* B *is finite and there is an injection* $f : A \to B$*, then* A *is finite.*
(3) *If* A *is finite and there is a surjection* $f : A \to B$*, then* B *is finite.*
(4) *If* A *and* B *are finite, then* $A \cup B$ *is finite.*
(5) *If* A *and* B *are finite, then* $A \times B$ *is finite.*
(6) *If* A *and* B *are finite, then* B^A *is finite.*

Proof. Exercise. □

Remark 2.6. A few weeks ago, when we were being less rigorous, we defined the *cardinality* of a nonempty finite set A to be the unique natural number m such that $\{1, \ldots, m\} \approx A$. It is a good exercise to show that m is indeed unique; that is, if $m, n \in \mathbb{N}$ and $\{1, \ldots, m\} \approx \{1, \ldots, n\}$ then $m = n$.

Although there are ways to define cardinality for infinite sets, we won't do this. The existence or nonexistence of bijections between various infinite sets is a very subtle issue and talking about cardinalities and "number of elements" too early invites confusion.

3. Countable sets

Definition 3.1. Let A be a set. We say that A is *countably infinite* if $\mathbb{N} \approx A$.

In other words, A is countably infinite if its elements can be "listed" as $A = \{a_1, a_2, a_3, \ldots\}$ with one element a_i for each natural number i, and $a_i \neq a_j$ whenever $i \neq j$. The notations a_i and $f(i)$, where f is a bijection from $\mathbb{N}$ to A as in the definition of "countably infinite," are two ways of writing the same thing.

Example 3.2. It follows from the existence of bijections constructed above that $\mathbb{Z}$ is countably infinite, and if $m \in \mathbb{N}$ then $\mathbb{N} - \{m\}$ is countably infinite.

Here is another important example:

Proposition 3.3. $\mathbb{N} \times \mathbb{N}$ *is countably infinite.*

Proof idea. Look at Figure 10.2 on p. 247 of the book. (A picture really helps to understand this.) □

Definition 3.4. Let A be a set. We say that A is *countable* if A is finite or countably infinite. In other words, either:

(1) A is empty,
(2) $\{1, \ldots, m\} \approx A$ for some $m \in \mathbb{N}$, or
(3) $\mathbb{N} \approx A$.

Otherwise, we say that A is *uncountable.*

Proposition 3.5. *Let* $A \subseteq \mathbb{N}$. *Then* A *is countable.*

Proof idea. If A is empty then it is finite by definition. If A has a largest element, say m, then $A \subseteq \{1, \ldots, m\}$ so again it is finite. If A is nonempty and has no largest element then it is countably infinite: you can get a bijection from $\mathbb{N}$ to A by listing the elements of A in increasing order. □

Proposition 3.6. *Let* A *and* B *be sets.*

(1) If B *is countable and* $A \subseteq B$*, then* A *is countable.*
(2) If B *is countable and there is an injection* $f : A \to B$*, then* A *is countable.*
(3) If A *is countable and there is a surjection* $f : A \to B$*, then* B *is countable.*
(4) If A *and* B *are countable then* $A \cup B$ *is countable.*
(5) If A *and* B *are countable then* $A \times B$ *is countable.*

Proof. Exercise. □

Proposition 3.7. *The set* $\mathbb{Q}$ *is countably infinite.*

Proof idea. There is a surjection $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ given by $f((a, b)) = a/b$. Show that $\mathbb{Z} \times \mathbb{N}$ is countable. Conclude that $\mathbb{Q}$ is countable. (Don't forget to also show that $\mathbb{Q}$ is infinite.) Alternatively, you can construct a bijection from $\mathbb{N}$ to $\mathbb{Q}$ directly as in the textbook (pp. 247–250.) □

4. Uncountable sets

Every set is either (1) finite, (2) countably infinite, or (3) neither finite nor countably infinite. Sets that are neither finite nor countably infinite are called "uncountable." Nobody even suspected the existence of uncountable sets until Georg Cantor proved the following theorem, which can be used to show that many sets are uncountable. In particular we can use it to show that the set $\mathbb{R}$ of real numbers is uncountable.

Theorem 4.1 (Cantor)**.** *Let A be a set. Then there is no surjection from A to $\mathcal{P}(A)$.*

Proof. Let $f : A \to \mathcal{P}(A)$. We will show that f is not a surjection by defining an element B of $\mathcal{P}(A)$ that is not in the range of f. Define the set

$$B = \{x \in A : x \notin f(x)\}.$$

Then $B \in \mathcal{P}(A)$. For every $x \in A$ we have $x \in B \iff x \notin f(x)$ by the definition of B. Therefore the sets B and $f(x)$ cannot be equal, because one of them contains x and the other does not. This shows that B is not in the range of f. □

The proof of Cantor's theorem is simple but confusing. You may need to read it a few times if you haven't seen it before. You can try applying it to a small set such as $A = \{1, 2, 3\}$.

Corollary 4.2. *The set $\mathcal{P}(\mathbb{N})$ is uncountable.*

Proof. First note that the set $\mathcal{P}(\mathbb{N})$ is infinite because $\mathbb{N}$ is infinite and there is an injection $g : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ given by $g(x) = \{x\}$ (for example.) However, it is not countably infinite because there is no bijection from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$ by Cantor's theorem. □

The set $\mathcal{P}(\mathbb{N})$ is a bit abstract; it contains some familiar sets like the set of all prime numbers, the set of all even numbers, and the set of all odd numbers, but it also contains many other sets that we can't describe in a nice way. So let's look at a more familiar set.

Corollary 4.3. *The open interval $(0, 1)$ in $\mathbb{R}$ is uncountable.*

Proof. Define a function $f : \mathcal{P}(\mathbb{N}) \to (0, 1)$ by

$$f(A) = 0.d_1d_2d_3\ldots$$

where the decimal digit d_i is 4 if $i \in A$ and 5 if $i \notin A$.[3] This function f is an injection. (This is not entirely trivial, because of things like $0.999\ldots = 1.000\ldots$, but if we just use the digits 4 and 5 then different sequences of digits correspond to different real numbers.) So if the interval $(0, 1)$ were countable, then the set $\mathcal{P}(\mathbb{N})$ would also be countable, contradicting Cantor's theorem. □

Corollary 4.4. *The set $\mathbb{R}$ is uncountable.*

Proof. If it were countable, then the subset $(0, 1)$ would be countable, contradicting the above result. □

Note that $(0, 1) \approx \mathbb{R}$. For example, you can show that the function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = (x - 1/2)/(x(1 - x))$ is a bijection. Does *every* uncountable subset A of $\mathbb{R}$ satisfy $A \approx \mathbb{R}$? No one knows!

[3] There is nothing special about the choice of digits 4 and 5 here.