MATH 13 MIDTERM EXAM SOLUTIONS

WINTER 2014

Problem 1.

(a) For $n = 1, 2, 3, \ldots$ define the inverval $A_n = [1/n, n]$. Rewrite the union of intervals $\bigcup_{n \in \mathbb{N}} A_n$ as an interval. For this problem you do not need to prove that your answer is correct.

$(0, \infty)$.

(b) Give an example of sets A and B with the property that $\mathcal{P}(A) - \mathcal{P}(B) \not\subseteq \mathcal{P}(A - B)$, and show that your sets have this property. ($\mathcal{P}$ means "power set.")

Take $A = \{1, 2, 3, 4\}$ and $B = \{1, 2\}$. Then the set $\{1, 3\}$ is in $\mathcal{P}(A) - \mathcal{P}(B)$ but not in $\mathcal{P}(A - B)$.[1]

(c) Make a truth table for the compound statement $(P \vee Q) \implies R$.

P	Q	R	$P \vee Q$	$(P \vee Q) \implies R$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	T
F	F	F	F	T

[1]More generally, take any set A with at least two elements and let B be any nonempty proper subset of A. Say $x \in B$ and $y \in A - B$. Then $\{x, y\}$ is in $\mathcal{P}(A) - \mathcal{P}(B)$ but not in $\mathcal{P}(A - B)$.

Problem 2. For each statement below, circle T or F according to whether the statement is true or false. You do NOT need to justify your answers.

T (F) $(P \implies R) \vee (Q \implies R)$ logically implies $(P \vee Q) \implies R$.
(T) F For all a, b, c, d, if $\{a, b\} = \{c, d\}$ and $a = c$, then $b = d$.
(T) F For all sets A, B, C, D, and U, if $\{A, B, C, D\}$ is a partition of U, then so is $\{A \cup B, C \cup D\}$.

T (F) For all sets A and B, $(A \times B) \cup (B \times A) = (A \cup B) \times (A \cup B)$.
T (F) $\sim \forall x \in S, P(x)$ is logically equivalent to $\forall x \in S, \sim P(x)$.
T (F) $\emptyset \in \emptyset$.

(T) F $P \implies (\sim P \implies Q)$ is a tautology.
(T) F For all sets A and B, if $A \cap B = A$ then $A \subseteq B$.
(T) F $\sim(P \vee Q)$ is logically equivalent to $(\sim P) \wedge (\sim Q)$.

Problem 3. Let $x, y, z \in \mathbb{Z}$. Prove that the following sum of absolute values is even: $|x - y| + |y - z| + |z - x|$.

Proof. Without loss of generality we may assume $x \geq y \geq z$. Then $x - y \geq 0$, $y - z \geq 0$, and $z - x \leq 0$, so

$$|x - y| + |y - z| + |z - x| = (x - y) + (y - z) + (-(z - x)) = 2(x - z),$$

which is even. □

Alternative proof. We consider four cases.

- Case 1: x, y, and z are all even. Then $|x - y|$, $|y - z|$, and $|z - x|$ are all even, so their sum is even.
- Case 2: Exactly two of x, y, and z are even. Without loss of generality we may assume that x and y are even and that z is odd. Then $|x - y|$ is even, and $|y - z|$ and $|z - x|$ are odd. So their sum is even.
- Case 3: Exactly one of x, y, and z is even. Without loss of generality we may assume that x is even and that y and z are odd. Then $|x - y|$ and $|z - x|$ are odd and $|y - z|$ is even. So their sum is even.
- Case 4: x, y, and z are all odd. Then $|x - y|$, $|y - z|$, and $|z - x|$ are all even, so their sum is even.

In each case, $|x - y| + |y - z| + |z - x|$ is even.

□

Problem 4. Let $x \in \mathbb{Z}$. Prove that if $3 \nmid (x^2 + 2)$, then $3 \mid x$.

Proof. We prove the contrapositive. Assume that $3 \nmid x$. There are two cases to consider.

- Case 1: $x \equiv 1 \pmod 3$. Then $x^2 \equiv 1^2 \equiv 1 \pmod 3$, so $x^2 + 2 \equiv 1 + 2 \equiv 0 \pmod 3$.
- Case 2: $x \equiv 2 \pmod 3$. Then $x^2 \equiv 2^2 \equiv 1 \pmod 3$, so $x^2 + 2 \equiv 1 + 2 \equiv 0 \pmod 3$.

In each case we have $x^2 + 2 \equiv 0 \pmod 3$, meaning that $3 \mid (x^2 + 2)$ as desired.

□

Problem 5. PROVE or DISPROVE the following statement:

For all sets A, B, and C,

$$(A \subseteq C \vee B \subseteq C) \iff (A \cap B \subseteq C).$$

Solution. We disprove the statement.

Let $A = \{1\}$, $B = \{2\}$, and $C = \emptyset$. Then $A \cap B = \emptyset \subseteq C$, but $A \not\subseteq C$ and $B \not\subseteq C$.[2]

Problem 6. PROVE or DISPROVE the following statement:

There are $a, b \in \mathbb{Z}$ such that

$$a^2 - b^2 = 2.$$

Solution. We disprove the statement.

Let $a, b \in \mathbb{Z}$. Without loss of generality we may assume that $a, b \geq 0$ (because a^2 and b^2 do not depend on the signs of a and b.)

- Case 1: $a \leq b$. Then $a^2 - b^2 \leq 0$, so $a^2 - b^2 \neq 2$.
- Case 2: $a = b + 1$. Then $a^2 - b^2 = (b+1)^2 - b^2 = 2b + 1$, which cannot be equal to 2.
- Case 3: $a \geq b + 2$. Then $a^2 - b^2 = (b+2)^2 - b^2 = 4b + 4$, which is greater than 2.

In each case we have $a^2 - b^2 \neq 2$.

Solution (Alternative 1). We disprove the statement.

Let $a, b \in \mathbb{Z}$. We have $a^2 - b^2 = (a+b)(a-b)$. Assume to the contrary that this is equal to 2. The only ways to factor 2 are as $1 \cdot 2$, $2 \cdot 1$, $-1 \cdot -2$, and $-2 \cdot -1$. So 2 can only be factored as a product of consecutive integers. But $a + b$ and $a - b$ differ by $2b$, an even number, so they are not consecutive and we have a contradiction.

Solution (Alternative 2). We disprove the statement.

Let $a, b \in \mathbb{Z}$. We consider three cases.

- Case 1: a and b are both even, say $a = 2k$ and $b = 2l$ where $k, l \in \mathbb{Z}$. Then $a^2 - b^2 = 4(k^2 - l^2)$, which cannot be equal to 2.
- Case 2: a and b are both odd, say $a = 2k + 1$ and $b = 2l + 1$ where $k, l \in \mathbb{Z}$. Then $a^2 - b^2 = 4(k^2 + k - l^2 - l)$, which cannot be equal to 2.
- Case 3: a is odd and b is even. Then a^2 is odd and b^2 is even. So $a^2 - b^2$ is odd and therefore cannot be equal to 2.
- Case 4: a is even and b is odd. Then a^2 is even and b^2 is odd. So $a^2 - b^2$ is odd and therefore cannot be equal to 2.

In each case we have $a^2 - b^2 \neq 2$ as desired.

[2] More generally, take any sets A and B such that neither is a subset of the other, and let $C = A \cap B$. Then $A \cap B \subseteq C$, but $A \not\subseteq C$ and $B \not\subseteq C$.