Problem 1.

(a) For \( n = 1, 2, 3, \ldots \) define the interval \( A_n = [1/n, n] \). Rewrite the union of intervals \( \bigcup_{n \in \mathbb{N}} A_n \) as an interval. For this problem you do not need to prove that your answer is correct.

\( (0, \infty) \).

(b) Give an example of sets \( A \) and \( B \) with the property that \( \mathcal{P}(A) - \mathcal{P}(B) \not\subseteq \mathcal{P}(A - B) \), and show that your sets have this property. (\( \mathcal{P} \) means “power set.”)

Take \( A = \{1, 2, 3, 4\} \) and \( B = \{1, 2\} \). Then the set \( \{1, 3\} \) is in \( \mathcal{P}(A) - \mathcal{P}(B) \) but not in \( \mathcal{P}(A - B) \).\(^1\)

(c) Make a truth table for the compound statement \( (P \lor Q) \implies R \).

\begin{array}{cccc}
P & Q & R & (P \lor Q) \implies R \\
T & T & T & T \\
T & T & F & F \\
T & F & T & T \\
T & F & F & F \\
F & T & T & T \\
F & T & F & F \\
F & F & T & T \\
F & F & F & T \\
\end{array}

\(^1\)More generally, take any set \( A \) with at least two elements and let \( B \) be any nonempty proper subset of \( A \). Say \( x \in B \) and \( y \in A - B \). Then \( \{x, y\} \) is in \( \mathcal{P}(A) - \mathcal{P}(B) \) but not in \( \mathcal{P}(A - B) \).
Problem 2. For each statement below, circle T or F according to whether the statement is true or false. You do NOT need to justify your answers.

T ① (P \implies R) \lor (Q \implies R) logically implies (P \lor Q) \implies R.

① F For all a, b, c, d, if \{a, b\} = \{c, d\} and a = c, then b = d.

① F For all sets A, B, C, D, and U, if \{A, B, C, D\} is a partition of U, then so is \{A \cup B, C \cup D\}.

T ⑤ For all sets A and B, \((A \times B) \cup (B \times A) = (A \cup B) \times (A \cup B)\).

T ⑤ \forall x \in S, P(x) is logically equivalent to \forall x \in S, \sim P(x).

T ⑤ \emptyset \in \emptyset.

① F P \implies (\sim P \implies Q) is a tautology.

① F For all sets A and B, if \(A \cap B = A\) then \(A \subseteq B\).

① F \sim(P \lor Q) is logically equivalent to \(\sim P \land \sim Q\).
Problem 3. Let \( x, y, z \in \mathbb{Z} \). Prove that the following sum of absolute values is even: \(|x - y| + |y - z| + |z - x|\).

Proof. Without loss of generality we may assume \( x \geq y \geq z \). Then \( x - y \geq 0, \ y - z \geq 0, \) and \( z - x \leq 0, \) so
\[
|x - y| + |y - z| + |z - x| = (x - y) + (y - z) + (- (z - x)) = 2(x - z),
\]
which is even. \(\square\)

Alternative proof. We consider four cases.

- Case 1: \( x, y, \) and \( z \) are all even. Then \(|x - y|, |y - z|, \) and \(|z - x| \) are all even, so their sum is even.
- Case 2: Exactly two of \( x, y, \) and \( z \) are even. Without loss of generality we may assume that \( x \) and \( y \) are even and that \( z \) is odd. Then \(|x - y| \) is even, and \(|y - z| \) and \(|z - x| \) are odd. So their sum is even.
- Case 3: Exactly one of \( x, y, \) and \( z \) is even. Without loss of generality we may assume that \( x \) is even and that \( y \) and \( z \) are odd. Then \(|x - y| \) and \(|z - x| \) are odd and \(|y - z| \) is even. So their sum is even.
- Case 4: \( x, y, \) and \( z \) are all odd. Then \(|x - y|, |y - z|, \) and \(|z - x| \) are all even, so their sum is even.

In each case, \(|x - y| + |y - z| + |z - x| \) is even. \(\square\)

Problem 4. Let \( x \in \mathbb{Z} \). Prove that if \( 3 \nmid (x^2 + 2) \), then \( 3 \mid x \).

Proof. We prove the contrapositive. Assume that \( 3 \nmid x \). There are two cases to consider.

- Case 1: \( x \equiv 1 \pmod{3} \). Then \( x^2 \equiv 1^2 \equiv 1 \pmod{3} \), so \( x^2 + 2 \equiv 1 + 2 \equiv 0 \pmod{3} \).
- Case 2: \( x \equiv 2 \pmod{3} \). Then \( x^2 \equiv 2^2 \equiv 1 \pmod{3} \), so \( x^2 + 2 \equiv 1 + 2 \equiv 0 \pmod{3} \).

In each case we have \( x^2 + 2 \equiv 0 \pmod{3} \), meaning that \( 3 \mid (x^2 + 2) \) as desired. \(\square\)
Problem 5. PROVE or DISPROVE the following statement:

For all sets $A$, $B$, and $C$,

$$(A \subseteq C \lor B \subseteq C) \iff (A \cap B \subseteq C).$$

**Solution.** We disprove the statement.

Let $A = \{1\}$, $B = \{2\}$, and $C = \emptyset$. Then $A \cap B = \emptyset \subseteq C$, but $A \not\subseteq C$ and $B \not\subseteq C$.

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Problem 6. PROVE or DISPROVE the following statement:

There are $a, b \in \mathbb{Z}$ such that

$$a^2 - b^2 = 2.$$

**Solution.** We disprove the statement.

Let $a, b \in \mathbb{Z}$. Without loss of generality we may assume that $a, b \geq 0$ (because $a^2$ and $b^2$ do not depend on the signs of $a$ and $b$.)

- Case 1: $a \leq b$. Then $a^2 - b^2 \leq 0$, so $a^2 - b^2 \neq 2$.
- Case 2: $a = b + 1$. Then $a^2 - b^2 = (b + 1)^2 - b^2 = 2b + 1$, which cannot be equal to 2.
- Case 3: $a \geq b + 2$. Then $a^2 - b^2 = (b + 2)^2 - b^2 = 4b + 4$, which is greater than 2.

In each case we have $a^2 - b^2 \neq 2$.

**Solution (Alternative 1).** We disprove the statement.

Let $a, b \in \mathbb{Z}$. We have $a^2 - b^2 = (a + b)(a - b)$. Assume to the contrary that this is equal to 2. The only ways to factor 2 are as $1 \cdot 2$, $2 \cdot 1$, $-1 \cdot -2$, and $-2 \cdot -1$. So 2 can only be factored as a product of consecutive integers. But $a + b$ and $a - b$ differ by $2b$, an even number, so they are not consecutive and we have a contradiction.

**Solution (Alternative 2).** We disprove the statement.

Let $a, b \in \mathbb{Z}$. We consider three cases.

- Case 1: $a$ and $b$ are both even, say $a = 2k$ and $b = 2l$ where $k, l \in \mathbb{Z}$. Then $a^2 - b^2 = 4(k^2 - l^2)$, which cannot be equal to 2.
- Case 2: $a$ and $b$ are both odd, say $a = 2k + 1$ and $b = 2l + 1$ where $k, l \in \mathbb{Z}$. Then $a^2 - b^2 = 4(k^2 + k - l^2 - l)$, which cannot be equal to 2.
- Case 3: $a$ is odd and $b$ is even. Then $a^2$ is odd and $b^2$ is even. So $a^2 - b^2$ is odd and therefore cannot be equal to 2.
- Case 4: $a$ is even and $b$ is odd. Then $a^2$ is even and $b^2$ is odd. So $a^2 - b^2$ is odd and therefore cannot be equal to 2.

In each case we have $a^2 - b^2 \neq 2$ as desired.

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2More generally, take any sets $A$ and $B$ such that neither is a subset of the other, and let $C = A \cap B$. Then $A \cap B \subseteq C$, but $A \not\subseteq C$ and $B \not\subseteq C$. 