MATH 13 SAMPLE FINAL EXAM

WINTER 2014

Student name:

Student ID number:

INSTRUCTIONS

• Books, notes, and electronic devices may NOT be used. These items must be kept in a closed backpack or otherwise hidden from view during the exam.
• Cheating in any form may result in an F grade for the course as well as administrative sanctions.
• The time remaining will be written on the board periodically.
• You may hand in your exam and leave early, but please do not do this during the last 5 minutes of the exam period because it may disturb other students.

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Problem 1 (15 points). For each statement below, circle T or F according to whether the statement is true or false. You do NOT need to justify your answers.

T  F  Every countably infinite subset of $\mathbb{R}$ is well-ordered.
T  F  For all integers $a$, $b$, and $c$, if $a \mid 2b$ and $b \mid 2c$ then $a \mid 2c$.
T  F  If $A$ is a set and $\mathcal{P}(A)$ is infinite, then $A$ must also be infinite.
T  F  There is a bijection from the closed interval $[0, \infty)$ to the open interval $(0, \infty)$.
T  F  For all sets $A$ and $B$, if there is a surjection from $A$ to $B$ and $A$ is countable, then $B$ is countable.

T  F  If $R$ is a symmetric relation on the set $A$, then $(A \times A) - R$ must also be a symmetric relation on $A$.
T  F  There are sets $A$, $B$, and $C$ and functions $f : A \rightarrow B$ and $g : B \rightarrow C$ such that $f$ is not injective and $g \circ f : A \rightarrow C$ is injective.
T  F  For every set $A$, there is a bijection from $\mathcal{P}(A)$ to $A^{\{0,1\}}$.
T  F  If $R$ is a reflexive relation on the set $A$, then $R^{-1}$ must also be a reflexive relation on $A$.
T  F  If $R_1$ and $R_2$ are equivalence relations on the set $A$, then $R_1 \cap R_2$ must also be an equivalence relation on $A$.

T  F  For all sets $A$, $B$, and $C$, there is a bijection from $A \times (B \times C)$ to $(A \times B) \times C$.
T  F  $\sim (P \land Q)$ is logically equivalent to $\sim P \land \sim Q$
T  F  $\emptyset \subseteq \emptyset$
T  F  For all integers $a, b, n$ with $n \geq 2$, if $ab \equiv 0 \pmod{n}$ then $a = 0 \pmod{n}$ or $b = 0 \pmod{n}$.
T  F  Every well-ordered set of real numbers has a least element.
Problem 2 (5 points). Let $A$ and $B$ be sets. Prove that the following equality holds:

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A).$$

Problem 3 (5 points). Is this compound statement a tautology? Justify your answer.

$$((P \implies Q) \wedge (\sim P \implies Q)) \implies Q$$
Problem 4 (5 points). Let $a, a', b, b', n \in \mathbb{Z}$ with $n \geq 2$. Prove that if $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$, then $a + b \equiv a' + b' \pmod{n}$.

Problem 5 (5 points). Define a sequence $(a_1, a_2, a_3, \ldots)$ by $a_1 = 3$, and for all $n \in \mathbb{N}$,

$$a_{n+1} = \begin{cases} 
2a_n & \text{if } n \text{ is odd} \\
4a_n & \text{if } n \text{ is even.}
\end{cases}$$

Prove that $a_n \leq 3^n$ for all $n \in \mathbb{N}$. 
Problem 6 (5 points). Let $A$ be a set and let $f : A \rightarrow A$ be a function such that $f(f(x)) = x$ for all $x \in A$. Define the relation $R$ on $A$ by

$$R = \{(x, y) \in A : y = x \text{ or } y = f(x)\}.$$ 

Prove that $R$ is an equivalence relation.

Problem 7 (5 points). Prove that there is a bijection from $A \times A$ to $A^{\{0,1\}}$. 
Problem 8 (5 points). Prove that there is a partition $P$ of $\mathbb{N}$ such that

(1) $P$ is infinite, and
(2) every element of $P$ is infinite.

(Hint: You may use the existence of a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.)