MATH 13 SAMPLE FINAL EXAM SOLUTIONS

WINTER 2014

Problem 1 (15 points). For each statement below, circle T or F according to whether the statement is true or false. You do NOT need to justify your answers.

T	(F)	Every countably infinite subset of $\mathbb{R}$ is well-ordered.
T	(F)	For all integers a, b, and c, if $a \mid 2b$ and $b \mid 2c$ then $a \mid 2c$.
(T)	F	If A is a set and $\mathcal{P}(A)$ is infinite, then A must also be infinite.
(T)	F	There is a bijection from the closed interval $[0, \infty)$ to the open interval $(0, \infty)$.
(T)	F	For all sets A and B, if there is a surjection from A to B and A is countable, then B is countable.
(T)	F	If R is a symmetric relation on the set A, then $(A \times A) - R$ must also be a symmetric relation on A.
T	(F)	There are sets A, B, and C and functions $f : A \to B$ and $g : B \to C$ such that f is not injective and $g \circ f : A \to C$ is injective.
T	(F)	For every set A, there is a bijection from $\mathcal{P}(A)$ to $A^{\{0,1\}}$.
(T)	F	If R is a reflexive relation on the set A, then R^{-1} must also be a reflexive relation on A.
(T)	F	If R_1 and R_2 are equivalence relations on the set A, then $R_1 \cap R_2$ must also be an equivalence relation on A.
(T)	F	For all sets A, B, and C, there is a bijection from $A \times (B \times C)$ to $(A \times B) \times C$.
T	(F)	$\sim(P \wedge Q)$ is logically equivalent to $\sim P \wedge \sim Q$
(T)	F	$\emptyset \subseteq \emptyset$
T	(F)	For all integers a, b, n with $n \geq 2$, if $ab \equiv 0 \pmod{n}$ then $a = 0 \pmod{n}$ or $b = 0 \pmod{n}$.
T	(F)	Every well-ordered set of real numbers has a least element.

Remark. Andrés pointed out an error in my first draft of the solutions. The answer to the last T/F question above is 'F' because the empty set is a well-ordered set of real numbers without a least element—it has no nonempty subsets, so it satisfies the definition of "well-ordered" vacuously. (For the exam I will try to avoid asking questions that hinge on trivial cases.)

Problem 2 (5 points). Let A and B be sets. Prove that the following equality holds:

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A).$$

Solution.

($\subseteq$): Let $x \in (A \cup B) - (A \cap B)$, so $x \in A \cup B$ and $x \notin A \cap B$.

- Case 1: Assume $x \in A$. Then $x \notin B$ (or else x would be in $A \cap B$) so $x \in A - B$.
- Case 2: Assume $x \in B$. Then $x \notin A$ (or else x would be in $A \cap B$) so $x \in B - A$.

We have shown that, in either case, $x \in (A - B) \cup (B - A)$.

($\supseteq$): Let $x \in (A - B) \cup (B - A)$.

- Case 1: Assume $x \in A - B$; that is, $x \in A$ and $x \notin B$. Then $x \in A \cup B$ and $x \notin A \cap B$.
- Case 2: Assume $x \in B - A$; that is, $x \in B$ and $x \notin A$. Then $x \in A \cup B$ and $x \notin A \cap B$.

We have shown that, in either case, $x \in (A \cup B) - (A \cap B)$.

Solution (alternative).

$$\begin{aligned}
(A \cup B) - (A \cap B) &= (A \cup B) \cap \overline{(A \cap B)} \\
&= (A \cup B) \cap (\overline{A} \cup \overline{B}) \quad \text{(De Morgan's law)} \\
&= ((A \cup B) \cap \overline{A}) \cup ((A \cup B) \cap \overline{B}) \quad \text{(distributive law)} \\
&= ((A \cap \overline{A}) \cup (B \cap \overline{A})) \cup ((A \cap \overline{B}) \cup (B \cap \overline{B})) \quad \text{(distributive law)} \\
&= (B \cap \overline{A}) \cup (A \cap \overline{B}) \\
&= (B - A) \cup (A - B) \\
&= (A - B) \cup (B - A)
\end{aligned}$$

Problem 3 (5 points). Is this compound statement a tautology? Justify your answer.

$$((P \implies Q) \wedge (\sim P \implies Q)) \implies Q$$

Note: "$\sim P \implies Q$" means "$(\sim P) \implies Q$," not "$\sim(P \implies Q)$."

Solution. Assume that "$(P \implies Q) \wedge (\sim P \implies Q)$" is true. So "$P \implies Q$" and "$\sim P \implies Q$" are both true.

- Case 1: Assume P. Then because "$P \implies Q$" is true, Q is true.
- Case 2: Assume $\sim P$. Then because "$\sim P \implies Q$" is true, Q is true.

We have shown that in all the cases where the hypothesis is true, the conclusion is also true. So the implication is a tautology.

Remark. We could also solve this problem by making a truth table with four rows for the possible combinations of truth values for P and Q. The type of solution above has the advantage that it is more intuitive (in my opinion.) It also might be faster, especially on T/F questions where you are not asked to write out your argument.

Problem 4 (5 points). Let $a, a', b, b', n \in \mathbb{Z}$ with $n \geq 2$. Prove that if $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$, then $a + b \equiv a' + b' \pmod{n}$.

Solution. Assume that $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$. So $n \mid (a' - a)$ and $n \mid (b' - b)$. Therefore $n \mid ((a' - a) + (b' - b)) = ((a' + b') - (a + b))$, so $a + b \equiv a' + b' \pmod{n}$.

Problem 5 (5 points). Define a sequence $(a_1, a_2, a_3, \ldots)$ by $a_1 = 3$, and for all $n \in \mathbb{N}$,

$$a_{n+1} = \begin{cases} 2a_n & \text{if } n \text{ is odd} \\ 4a_n & \text{if } n \text{ is even.} \end{cases}$$

Prove that $a_n \leq 3^n$ for all $n \in \mathbb{N}$.

Proof. We prove this using strong induction on n.

- Base case: $a_1 = 3 \leq 3^1$.
- Induction step: Let $n \in \mathbb{N}$ and assume that $a_i \leq 3^i$ for all $i \leq n$. We want to show that $a_{n+1} \leq 3^{n+1}$. If n is odd, then we have

  $a_{n+1} = 2a_n \leq 2 \cdot 3^n \leq 3 \cdot 3^n = 3^{n+1}$,

  as desired. If n is even, then (using the fact that $n - 1 \in \mathbb{N}$ and $n - 1$ is odd) we have

  $a_{n+1} = 4a_n = 4 \cdot 2a_{n-1} = 8a_{n-1} \leq 8 \cdot 3^{n-1} \leq 9 \cdot 3^{n-1} = 3^{n+1}$,

  as desired.

Therefore $a_n \leq 3^n$ for all $n \in \mathbb{N}$. □

Problem 6 (5 points). Let A be a set and let $f : A \to A$ be a function such that $f(f(x)) = x$ for all $x \in A$. Define the relation R on A by

$$R = \{(x, y) \in A : y = x \text{ or } y = f(x)\}.$$

Prove that R is an equivalence relation.

Solution.

- We show that R is reflexive. Let $x \in A$. Then $(x, x) \in R$ because $x = x$.
- We show that R is symmetric. Let $x, y \in A$ and assume that $(x, y) \in R$. That is, $y = x$ or $y = f(x)$. If $y = x$, then $x = y$, so $(y, x) \in R$. If $y = f(x)$, then $f(y) = f(f(x)) = x$, so $(y, x) \in R$ in this case also.
- We show that R is transitive. Let $x, y, z \in A$ and assume that $(x, y), (y, z) \in R$. If $x = y$ or $y = z$ then clearly $(x, z) \in R$. If not, then $y = f(x)$ and $z = f(y)$, so $z = f(f(x)) = x$ and again $(x, z) \in R$.

Problem 7 (5 points). Prove that there is a bijection from $A \times A$ to $A^{\{0,1\}}$.

Solution. Define a function $G : A \times A \to A^{\{0,1\}}$ by $G((x, y)) = \{(0, x), (1, y)\}$.[1]

- We show that G is injective. Let $(x, y), (x', y') \in A \times A$ and assume that $G((x, y)) = G((x', y'))$. That is, $\{(0, x), (1, y)\} = \{(0, x'), (1, y')\}$. Clearly this means that $(0, x) = (0, x')$ and $(1, y) = (1, y')$ and not the other way around. So $x = x'$ and $y = y'$, which means that $(x, y) = (x', y')$.
- We show that G is surjective. Let $f \in A^{\{0,1\}}$. Then we have $G((f(0), f(1))) = \{(0, f(0)), (1, f(1))\} = f$, so f is in the range of G.

Solution (alternative). Because every bijection has an inverse that is also a bijection, it suffices to prove that there is a bijection from $A^{\{0,1\}}$ to $A \times A$.

Define a function $G : A^{\{0,1\}} \to A \times A$ by $G(f) = (f(0), f(1))$.

- We show that G is injective. Let $f, f' \in A^{\{0,1\}}$ and assume that $G(f) = G(f')$. That is, $(f(0), f(1)) = (f'(0), f'(1))$. So $f(0) = f'(0)$ and $f(1) = f'(1)$. Therefore $f = f'$.
- We show that G is surjective. Let $(x, y) \in A \times A$. Define the function $f \in A^{\{0,1\}}$ by $f(0) = x$ and $f(1) = y$.[2] Then $G(f) = (x, y)$.

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[1] Equivalently, we could define the function $G((x, y))$ in terms of its values: $G((x, y))(0) = x$ and $G((x, y))(1) = y$.

[2] Equivalently, we could write $f = \{(0, x), (1, y)\}$.

Problem 8 (5 points). Prove that there is a partition P of $\mathbb{N}$ such that

(1) P is infinite, and
(2) every element of P is infinite.

(Hint: You may use the existence of a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.)

Proof. Take a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Define a set P of subsets of $\mathbb{N}$ by

$$P = \{A_i : i \in \mathbb{N}\}$$

where

$$A_i = f^{-1}[\{i\} \times \mathbb{N}] = \{x \in \mathbb{N} : \text{the ordered pair } f(x) \text{ has first coordinate } i\}.$$

We show that P is a partition.

- Every element of P is nonempty, because $\{i\} \times \mathbb{N}$ is nonempty and f is surjective.
- The elements of P are pairwise disjoint because $(\{i\} \times \mathbb{N}) \cap (\{j\} \times \mathbb{N}) = \emptyset$ whenever $i \neq j$, and the inverse images of two disjoint sets (under any function) are disjoint.
- $\mathbb{N} = \bigcup_{i \in \mathbb{N}} A_i$ because for all $x \in \mathbb{N}$ we have $x \in A_i$ where i is the first coordinate of the pair $f(x)$.

We check the remaining two conditions.

- P is infinite because the function $\mathbb{N} \to P$ that sends i to A_i is a bijection. (The fact that it is an injection follows from the argument above, which shows that if $i \neq j$ then A_i and A_j are disjoint nonempty sets, hence $A_i \neq A_j$.)
- Every element A_i of P is infinite because the function $A_i \to \mathbb{N}$ that sends x to the second coordinate of $f(x)$ is easily shown to be a bijection.... □

Remark. The details of this solution take a bit longer to write out than I intended. The main point is that the set $\mathbb{N} \times \mathbb{N}$ has an infinite partition into the infinite sets $\{1\} \times \mathbb{N}$, $\{2\} \times \mathbb{N}$, *etc.* and we can then use a bijection of $\mathbb{N}$ with $\mathbb{N} \times \mathbb{N}$ to "transfer" this partition over to a partition of $\mathbb{N}$. Once you come up with this idea (or some other idea) you should just write as much detail as you have time for and not worry about the rest.

Remark. A more number-theoretic proof idea (without using the hint) would be to define, for all $i \in \mathbb{N}$, the set

$$A_i = \{x \in \mathbb{N} : 2^{i-1} \mid x \text{ and } 2^i \nmid x\},$$

and to show that the sets $A_1, A_2, \ldots$ are all distinct infinite sets and that the set $P = \{A_1, A_2, \ldots\}$ is a partition of $\mathbb{N}$.