MATH 13 SAMPLE FINAL EXAM SOLUTIONS
WINTER 2014

Problem 1 (15 points). For each statement below, circle T or F according to whether the statement is true or false. You do NOT need to justify your answers.

T F Every countably infinite subset of $\mathbb{R}$ is well-ordered.
T F For all integers $a$, $b$, and $c$, if $a \mid 2b$ and $b \mid 2c$ then $a \mid 2c$.
F T If $A$ is a set and $\mathcal{P}(A)$ is infinite, then $A$ must also be infinite.
F T There is a bijection from the closed interval $[0, \infty)$ to the open interval $(0, \infty)$.
F T For all sets $A$ and $B$, if there is a surjection from $A$ to $B$ and $A$ is countable, then $B$ is countable.
F T If $R$ is a symmetric relation on the set $A$, then $(A \times A) - R$ must also be a symmetric relation on $A$.
T F There are sets $A$, $B$, and $C$ and functions $f : A \to B$ and $g : B \to C$ such that $f$ is not injective and $g \circ f : A \to C$ is injective.
F T For every set $A$, there is a bijection from $\mathcal{P}(A)$ to $A^{[0,1]}$.
F T If $R$ is a reflexive relation on the set $A$, then $R^{-1}$ must also be a reflexive relation on $A$.
F T If $R_1$ and $R_2$ are equivalence relations on the set $A$, then $R_1 \cap R_2$ must also be an equivalence relation on $A$.
F T For all sets $A$, $B$, and $C$, there is a bijection from $A \times (B \times C)$ to $(A \times B) \times C$.
T F $\sim(P \land Q)$ is logically equivalent to $\sim P \land \sim Q$.
F T $\emptyset \subseteq \emptyset$.
F T For all integers $a$, $b$, and $n$ with $n \geq 2$, if $ab \equiv 0 \pmod{n}$ then $a = 0 \pmod{n}$ or $b = 0 \pmod{n}$.
T F Every well-ordered set of real numbers has a least element.

Remark. Andrés pointed out an error in my first draft of the solutions. The answer to the last T/F question above is ‘F’ because the empty set is a well-ordered set of real numbers without a least element—it has no nonempty subsets, so it satisfies the definition of “well-ordered” vacuously. (For the exam I will try to avoid asking questions that hinge on trivial cases.)
Problem 2 (5 points). Let $A$ and $B$ be sets. Prove that the following equality holds:

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A).$$

Solution.

$(\subseteq)$: Let $x \in (A \cup B) - (A \cap B)$, so $x \in A \cup B$ and $x \notin A \cap B$.

- Case 1: Assume $x \in A$. Then $x \notin B$ (or else $x$ would be in $A \cap B$) so $x \in A - B$.
- Case 2: Assume $x \in B$. Then $x \notin A$ (or else $x$ would be in $A \cap B$) so $x \in B - A$.

We have shown that, in either case, $x \in (A - B) \cup (B - A)$.

$(\supseteq)$: Let $x \in (A - B) \cup (B - A)$.

- Case 1: Assume $x \in A - B$; that is, $x \in A$ and $x \notin B$. Then $x \in A \cup B$ and $x \notin A \cap B$.
- Case 2: Assume $x \in B - A$; that is, $x \in B$ and $x \notin A$. Then $x \in A \cup B$ and $x \notin A \cap B$.

We have shown that, in either case, $x \in (A \cup B) - (A \cap B)$.

Solution (alternative).

$$(A \cup B) - (A \cap B) = (A \cup B) \cap (\overline{A} \cup \overline{B}) \quad \text{(De Morgan’s law)}$$

$$= ((A \cup B) \cap \overline{A}) \cup ((A \cup B) \cap \overline{B}) \quad \text{(distributive law)}$$

$$= ((A \cap \overline{A}) \cup (B \cap \overline{A})) \cup ((A \cap \overline{B}) \cup (B \cap \overline{B})) \quad \text{(distributive law)}$$

$$= (B \cap \overline{A}) \cup (A \cap \overline{B})$$

$$= (B - A) \cup (A - B)$$

$$= (A - B) \cup (B - A)$$

Problem 3 (5 points). Is this compound statement a tautology? Justify your answer.

$$(((P \implies Q) \land (\neg P \implies Q)) \implies Q)$$

Note: “$\neg P \implies Q$” means “$(\neg P) \implies Q$,” not “$\neg(P \implies Q)$.”

Solution. Assume that “$(P \implies Q) \land (\neg P \implies Q)$” is true. So “$P \implies Q$” and “$\neg P \implies Q$” are both true.

- Case 1: Assume $P$. Then because “$P \implies Q$” is true, $Q$ is true.
- Case 2: Assume $\neg P$. Then because “$\neg P \implies Q$” is true, $Q$ is true.

We have shown that in all the cases where the hypothesis is true, the conclusion is also true. So the implication is a tautology.

Remark. We could also solve this problem by making a truth table with four rows for the possible combinations of truth values for $P$ and $Q$. The type of solution above has the advantage that it is more intuitive (in my opinion.) It also might be faster, especially on T/F questions where you are not asked to write out your argument.
**Problem 4** (5 points). Let $a, a', b, b', n \in \mathbb{Z}$ with $n \geq 2$. Prove that if $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$, then $a + b \equiv a' + b' \pmod{n}$.

**Solution.** Assume that $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$. So $n \mid (a' - a)$ and $n \mid (b' - b)$. Therefore $n \mid ((a' - a) + (b' - b)) = ((a' + b') - (a + b))$, so $a + b \equiv a' + b' \pmod{n}$.

**Problem 5** (5 points). Define a sequence $(a_1, a_2, a_3, \ldots)$ by $a_1 = 3$, and for all $n \in \mathbb{N}$,

$$a_{n+1} = \begin{cases} 2a_n & \text{if } n \text{ is odd} \\ 4a_n & \text{if } n \text{ is even} \end{cases}$$

Prove that $a_n \leq 3^n$ for all $n \in \mathbb{N}$.

**Proof.** We prove this using strong induction on $n$.

- Base case: $a_1 = 3 \leq 3^1$.
- Induction step: Let $n \in \mathbb{N}$ and assume that $a_i \leq 3^i$ for all $i \leq n$. We want to show that $a_{n+1} \leq 3^{n+1}$. If $n$ is odd, then we have

$$a_{n+1} = 2a_n \leq 2 \cdot 3^n \leq 3 \cdot 3^n = 3^{n+1},$$

as desired. If $n$ is even, then (using the fact that $n - 1 \in \mathbb{N}$ and $n - 1$ is odd) we have

$$a_{n+1} = 4a_n = 4 \cdot 2a_{n-1} = 8a_{n-1} \leq 8 \cdot 3^{n-1} \leq 9 \cdot 3^{n-1} = 3^{n+1},$$

as desired.

Therefore $a_n \leq 3^n$ for all $n \in \mathbb{N}$. \qed
Problem 6 (5 points). Let $A$ be a set and let $f : A \rightarrow A$ be a function such that $f(f(x)) = x$ for all $x \in A$. Define the relation $R$ on $A$ by

$$R = \{(x,y) \in A : y = x \text{ or } y = f(x)\}.$$  

Prove that $R$ is an equivalence relation.

Solution.

- We show that $R$ is reflexive. Let $x \in A$. Then $(x,x) \in R$ because $x = x$.
- We show that $R$ is symmetric. Let $x, y \in A$ and assume that $(x, y) \in R$. That is, $y = x$ or $y = f(x)$. If $y = x$, then $x = y$, so $(y, x) \in R$. If $y = f(x)$, then $f(y) = f(f(x)) = x$, so $(y, x) \in R$ in this case also.
- We show that $R$ is transitive. Let $x, y, z \in A$ and assume that $(x, y), (y, z) \in R$. If $x = y$ or $y = z$ then clearly $(x, z) \in R$. If not, then $y = f(x)$ and $z = f(y)$, so $z = f(f(x)) = x$ and again $(x, z) \in R$.

Problem 7 (5 points). Prove that there is a bijection from $A \times A$ to $A^{(0,1)}$.

Solution. Define a function $G : A \times A \rightarrow A^{(0,1)}$ by $G((x,y)) = \{(0,x),(1,y)\}$.\(^1\)

- We show that $G$ is injective. Let $(x,y), (x',y') \in A \times A$ and assume that $G((x,y)) = G((x',y'))$. That is, $\{(0,x),(1,y)\} = \{(0,x'),(1,y')\}$. Clearly this means that $(0,x) = (0,x')$ and $(1,y) = (1,y')$ and not the other way around. So $x = x'$ and $y = y'$, which means that $(x,y) = (x',y')$.
- We show that $G$ is surjective. Let $f \in A^{(0,1)}$. Then we have $G((f(0),f(1))) = \{(0,f(0)),(1,f(1))\} = f$, so $f$ is in the range of $G$.

Solution (alternative). Because every bijection has an inverse that is also a bijection, it suffices to prove that there is a bijection from $A^{(0,1)}$ to $A \times A$.

Define a function $G : A^{(0,1)} \rightarrow A \times A$ by $G(f) = (f(0), f(1))$.

- We show that $G$ is injective. Let $f, f' \in A^{(0,1)}$ and assume that $G(f) = G(f')$. That is, $(f(0),f(1)) = (f'(0), f'(1))$. So $f(0) = f'(0)$ and $f(1) = f'(1)$. Therefore $f = f'$.
- We show that $G$ is surjective. Let $(x,y) \in A \times A$. Define the function $f \in A^{(0,1)}$ by $f(0) = x$ and $f(1) = y$.\(^2\) Then $G(f) = (x,y)$.

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\(^1\)Equivalently, we could define the function $G((x,y))$ in terms of its values: $G((x,y))(0) = x$ and $G((x,y))(1) = y$.

\(^2\)Equivalently, we could write $f = \{(0,x),(1,y)\}$. 


Problem 8 (5 points). Prove that there is a partition $P$ of $\mathbb{N}$ such that
(1) $P$ is infinite, and
(2) every element of $P$ is infinite.
(Hint: You may use the existence of a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.)

Proof. Take a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Define a set $P$ of subsets of $\mathbb{N}$ by

$$P = \{A_i : i \in \mathbb{N}\}$$

where

$$A_i = f^{-1}(\{i\} \times \mathbb{N}) = \{x \in \mathbb{N} : \text{the ordered pair } f(x) \text{ has first coordinate } i\}.$$ 

We show that $P$ is a partition.

- Every element of $P$ is nonempty, because $\{i\} \times \mathbb{N}$ is nonempty and $f$ is surjective.
- The elements of $P$ are pairwise disjoint because $(\{i\} \times \mathbb{N}) \cap (\{j\} \times \mathbb{N}) = \emptyset$ whenever $i \neq j$, and the inverse images of two disjoint sets (under any function) are disjoint.
- $\mathbb{N} = \bigcup_{i \in \mathbb{N}} A_i$, because for all $x \in \mathbb{N}$ we have $x \in A_i$ where $i$ is the first coordinate of the pair $f(x)$.

We check the remaining two conditions.

- $P$ is infinite because the function $\mathbb{N} \to P$ that sends $i$ to $A_i$ is a bijection. (The fact that it is an injection follows from the argument above, which shows that if $i \neq j$ then $A_i$ and $A_j$ are disjoint nonempty sets, hence $A_i \neq A_j$.)
- Every element $A_i$ of $P$ is infinite because the function $A_i \to \mathbb{N}$ that sends $x$ to the second coordinate of $f(x)$ is easily shown to be a bijection. 

□

Remark. The details of this solution take a bit longer to write out than I intended. The main point is that the set $\mathbb{N} \times \mathbb{N}$ has an infinite partition into the infinite sets $\{1\} \times \mathbb{N}$, $\{2\} \times \mathbb{N}$, etc. and we can then use a bijection of $\mathbb{N}$ with $\mathbb{N} \times \mathbb{N}$ to “transfer” this partition over to a partition of $\mathbb{N}$. Once you come up with this idea (or some other idea) you should just write as much detail as you have time for and not worry about the rest.

Remark. A more number-theoretic proof idea (without using the hint) would be to define, for all $i \in \mathbb{N}$, the set

$$A_i = \{x \in \mathbb{N} : 2^{i-1} \mid x \text{ and } 2^i \nmid x\},$$

and to show that the sets $A_1, A_2, \ldots$ are all distinct infinite sets and that the set $P = \{A_1, A_2, \ldots\}$ is a partition of $\mathbb{N}$.