MATH 161 SAMPLE FINAL EXAM SOLUTIONS

SPRING 2014

1. Euclidean geometry

In this section, you may use the parallel postulate and its consequences.

Problem 1 (4 points).

(1) State Playfair's postulate.
(2) Define the reflection across a line ℓ.

Solution.

(1) For every line ℓ and every point P not on ℓ, there is one and only one line through P that is parallel to ℓ.
(2) The reflection across ℓ is the unique non-identity isometry that fixes all points of ℓ.
(Alternative statement: The reflection of a point P across ℓ is the unique point P' such that ℓ is the perpendicular bisector of the segment $\overline{PP'}$.)

Problem 2 (4 points). Circle 'T' or 'F' according to whether the statement is true or false. You do NOT need to justify your answers.

(1) Given triangles $\triangle ABC$ and $\triangle A'B'C'$, if $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $\overline{AC} \cong \overline{A'C'}$, then $\triangle ABC$ must be congruent to $\triangle A'B'C'$.

True. (This is the AAS congruence theorem.)

(2) For any non-collinear points A, B, and C, there is a unique circle through A, B, and C.

True. (This is proved in Chapter 2.)

(3) If r_ℓ and r_m are reflections across parallel lines ℓ and m, we must have $r_\ell \circ r_m = r_m \circ r_\ell$.

False. (These two compositions are translations by vectors of the same length, but in opposite directions.)

(4) For every isometry f and all points A and B we have $\overline{Af(A)} \cong \overline{Bf(B)}$.

False. (The only isometries with this property are translations.)

Problem 3 (4 points). Suppose that the triangles $\triangle ABC$ and $\triangle A'B'C'$ are isosceles triangles with $AC = BC$ and $A'C' = B'C'$. Suppose that the bases $\overline{AB}$ and $\overline{A'B'}$ are congruent and the triangles $\triangle ABC$ and $\triangle A'B'C'$ have the same height. Prove that $\triangle ABC \cong \triangle A'B'C'$.

Solution. Let D and D' be the midpoints of the two bases. Then we have $\triangle ADC \cong \triangle BDC$ by the SSS congruence theorem. In particular, the angles at D are equal so they are both right angles. This means CD is the height of $\triangle ABC$. A similar argument shows that the angles at D' are right angles and $C'D'$ is the height of $\triangle A'B'C'$. Then our hypothesis on the heights gives $CD = C'D'$. Now by the SAS congruence theorem, the four smaller triangles $\triangle ADC$, $\triangle BDC$, $\triangle A'D'C'$, and $\triangle B'D'C'$ are all congruent. In particular the legs $\overline{AC}$ and $\overline{BC}$ of $\triangle ABC$ are congruent to the legs $\overline{A'C'}$ and $\overline{B'C'}$ of $\triangle A'B'C'$, so we have $\triangle ABC \cong \triangle A'B'C'$ by the SSS congruence theorem.

(Note: there are a few different ways to prove this, but in any case to get started you should think of a way to construct a segment satisfying the definition of "height" so that you can use the hypothesis on the heights.)

Problem 4 (4 points). Prove that if f is an isometry and $f \circ f$ is the identity function, then f has a fixed point.

Solution. Assume that f is an isometry and $f \circ f$ is the identity function. We consider cases according to the classification of isometries.

If f is the identity, its set of fixed points is the entire plane and we're done.

If f is a reflection, then its set of fixed points is a line and we're done.

If f is a non-identity translation then $f \circ f$ is not the identity (it is a translation with displacement vector twice that of f,) so this case can't happen.

If f is a rotation then its center of rotation is a fixed point.

Finally, if f is a glide reflection that is not a reflection, say $f = r_\ell \circ T_{\vec{v}}$ where ℓ is a line and $\vec{v}$ is a vector parallel to ℓ, then $f \circ f = r_\ell \circ T_{\vec{v}} \circ r_\ell \circ T_{\vec{v}} = r_\ell \circ r_\ell \circ T_{\vec{v}} \circ T_{\vec{v}} = T_{\vec{v}} \circ T_{\vec{v}} = T_{2\vec{v}}$, which is not the identity, so this case can't happen.

Problem 5 (4 points). Let $\overline{AB}$ and $\overline{A'B'}$ be congruent line segments. Prove that there is one and only one even isometry f such that $f(A) = A'$ and $f(B) = B'$. (An isometry is called *even* if it is the composition of an even number of reflections.)

Solution. (We will prove this from the theorem which says that for any two congruent triangles, there is a unique isometry taking one to the other. You can also prove it from scratch, similarly to the way the theorem is proved.)

Existence: Take any point C not on $\overleftrightarrow{AB}$. Take a point C' such that $AC = A'C'$ and $\angle BAC = \angle B'A'C'$. Then $\triangle ABC \cong \triangle A'B'C'$ by the SAS congruence theorem, so there is an isometry f such that $f(A) = A'$, $f(B) = B'$, and $f(C) = C'$. If f is even then we're done. If f is odd, then we define $g = r_\ell \circ f$ where $\ell = \overleftrightarrow{A'B'}$. Then g is an even isometry, $g(A) = A'$, and $g(B) = B'$, and we're done.

Uniqueness: Suppose toward a contradiction that f and g are two distinct even isometries such that $f(A) = g(A) = A'$ and $f(B) = g(B) = B'$. Then $g \circ f^{-1}$ is a non-identity isometry fixing A' and B', so $g \circ f^{-1} = r_\ell$ where $\ell = \overleftrightarrow{A'B'}$. Then r_ℓ is the composition of two even isometries, so it is even, contradicting the fact that reflections are not even.

2. Hyperbolic geometry

In this section, assume the hyperbolic postulate instead of the parallel postulate.

Problem 6 (4 points).

(1) State the hyperbolic postulate.
(2) Define "Saccheri quadrilateral".

Solution.

(1) For every line ℓ and every point P not on ℓ, there is more than one line through P and parallel to ℓ.

(Note: some books say "for *some* line ℓ and *some* point P not on ℓ...," so this version would also be acceptable, although it is not very obvious that the two versions are equivalent.)

(2) A Saccheri quadrilateral is a quadrilateral $ABCD$ such that $AB = CD$ and the angles $\angle B$ and $\angle C$ are right angles.

(Note: it is possible to write the vertices in a different order and get an equivalent definition.)

Problem 7 (4 points). Circle 'T' or 'F' according to whether the statement is true or false. You do NOT need to justify your answers. In hyperbolic geometry:

(1) The sum of angles of every triangle is more than $180°$.

False. (In fact, it is always less than $180°$.)

(2) If ℓ_1, ℓ_2, and ℓ_3 are lines, ℓ_1 is parallel to ℓ_2, and ℓ_2 is parallel to ℓ_3, then ℓ_1 must be parallel to ℓ_3.

False. (This statement is easily seen to be equivalent to Playfair's postulate, which contradicts the hyperbolic postulate.)

(3) There is an SAS similarity theorem.

False. (Here is a counterexample that is easy to see: In the Poincaré model, consider isosceles right triangles with the right angle at the origin. As the legs get longer and approach the boundary of the unit disk, the base angles approach zero. So these triangles are not similar to one another, although any two of them satisfy the hypothesis of SAS similarity. SAS *congruence*, however, does hold in hyperbolic geometry.)

(4) Let f be a rotation (defined in terms of reflections as usual.) If f fixes an omega point, then f must be the identity function.

True. (Let f be a non-identity rotation around a point O. Every omega point Ω is represented by a ray of the form $\overrightarrow{OA}$. Because $\overrightarrow{OA}$ and $f[\overrightarrow{OA}]$ are not limiting parallel to one another, they represent different omega points. In other words, Ω is not fixed by f. As an illustration, note that in the special case of the Poincaré disk model where the center of rotation O is the center of the unit disk, the rotation extends in a natural way to the boundary of the unit disk.)

Problem 8 (4 points)*.* Work in hyperbolic geometry. Let ℓ and m be parallel lines. Suppose that there are two distinct points P and P' on ℓ such that the perpendicular distance from P to m equals the perpendicular distance from P' to m. Prove that ℓ and m have a common perpendicular.

Solution. Take points Q and Q' on m such that the segments $\overline{PQ}$ and $\overline{P'Q'}$ are perpendicular to m. By hypothesis we have $PQ = P'Q'$. Let Q'' be the midpoint of $\overline{QQ'}$ and let n be the line though Q'' that is perpendicular to m.

The line n does not intersect $\overline{PQ}$ or $\overline{P'Q'}$ because they share a common perpendicular, so n must intersect ℓ at some point P''. We have $PQQ''P'' \cong P'Q'Q''P''$ by SASAS congruence, so in particular the two angles at P'' are congruent. This means they are both right angles, so n is perpendicular to ℓ. So we showed that ℓ and m have a common perpendicular n.

(Note: this proof also works in Euclidean geometry as well as in hyperbolic geometry. However, it is more interesting in hyperbolic geometry because the angles at P and at P' will not be right angles and the distance $P''Q''$ will be less than the distances PQ and $P'Q'$.)