MATH 161 SAMPLE MIDTERM EXAM SOLUTIONS

2014 MAY 5

Problem 1 (3 points). State, but do *not* prove, the SAS (side-angle-side) similarity theorem.

Solution. For any two triangles $\triangle ABC$ and $\triangle A'B'C'$, if $\angle A \cong \angle A'$ and $AB/A'B' = AC/A'C'$, then $\triangle ABC \sim \triangle A'B'C'$.

Problem 2 (5 points). Let $ABCD$ be a quadrilateral and let ℓ be a line that does not pass through any of the vertices A, B, C, or D. Prove that if ℓ intersects three sides of $ABCD$, then it also intersects the fourth side of $ABCD$.

Solution. Without loss of generality, we may assume that ℓ intersects the three sides $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$. Then A and B are on opposite sides of ℓ and B and C are on opposite sides of ℓ, so (by the plane separation property) A and C are on the same side of ℓ. Because C and D are on opposite sides of ℓ and A and C are on the same side of ℓ, we have that A and D are on opposite sides of ℓ (again by the plane separation property.) In other words, the line ℓ intersects $\overline{AD}$, which is the fourth side of the quadrilateral $ABCD$.

Solution (alternative). Without loss of generality, we may assume that ℓ intersects the three sides $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$. Then ℓ intersects two sides of $\triangle ABC$, so by Pasch's axiom it does not intersect the third side $\overline{AC}$. Now consider $\triangle ACD$: the line ℓ intersects the side $\overline{CD}$ and does not intersect the side $\overline{AC}$, so again by Pasch's axiom it must intersect the side $\overline{AD}$. We have shown that the line ℓ intersects the fourth side $\overline{AD}$ of the quadrilateral $ABCD$, as desired.

Problem 3 (5 points). Given a triangle $\triangle ABC$, let D, E, and F be the midpoints of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ respectively. Show that the four smaller triangles $\triangle ADE$, $\triangle DBF$, $\triangle FED$, and $\triangle EFC$ are all congruent to one another.

Solution. By SAS similarity, $\triangle ABC \sim \triangle ADE$ with similarity constant 2, so $DE = BC/2$. Therefore $DE = BF = FC$. An analogous argument shows that $DF = AE = EC$ and $EF = AD = DB$. Therefore the four smaller triangles are congruent to one another by SSS congruence. (You may want to draw a diagram to see this.)

Problem 4 (5 points). Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles such that $\triangle ABC \sim \triangle A'B'C'$ with similarity constant k (so $A'B' = kAB$, *etc.*) Prove that $\text{Area}(\triangle A'B'C') = k^2\,\text{Area}(\triangle ABC)$. You may use the usual formula for the area of a triangle.

Solution. Consider the sides $\overline{AB}$ and $\overline{A'B'}$ as the bases of $\triangle ABC$ and $\triangle A'B'C'$ respectively. Let D and D' denote the unique points on the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ respectively such that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ and $\overleftrightarrow{A'B'} \perp \overleftrightarrow{C'D'}$. By definition, CD is the height of $\triangle ABC$ and $C'D'$ is the height of $\triangle A'B'C'$. Now $\triangle ADC \sim \triangle A'D'C'$ by AA similarity (using the congruent angles at A and A' and the right angles at D and D) and we have $C'D'/CD = A'C'/AC = k$. In other words, the height of $\triangle A'B'C'$ is k times the height of $\triangle ABC$. Because the base of $\triangle A'B'C'$ is k times the base of $\triangle ABC$, the claim follows by the "one-half base times height" formula for the area of a triangle.

Problem 5 (5 points). Prove that $\cos(30°) = \sqrt{3}/2$.

Solution. We begin by constructing a right triangle containing a 30° angle. First, take an equilateral triangle $\triangle ABC$ with sides of length 1. Its angles are equal to one another (because $\triangle ABC \cong \triangle BCA$ by SSS congruence) and they sum to 180°, so they are all 60°. Let D be the midpoint of $\overline{AB}$. Then $\triangle ADC \cong \triangle BDC$ by SSS congruence, so in particular we have $\angle ADC \cong \angle BDC$ and $\angle ACD \cong \angle BCD$. Therefore $\angle ADC$ and $\angle BDC$ are both right angles and $\angle ACD$ and $\angle BCD$ are both 30°. By the definition of cosine we have $\cos(30°) = DC/AC$. Finally, we have $AC = 1$ and $DC = \sqrt{(AC)^2 - (AD)^2} = \sqrt{1 - (1/2)^2} = \sqrt{3}/2$ by Pythagoras's theorem.

Problem 6 (5 points). Let O_1, A and O_2 be collinear points with A between O_1 and O_2. Let c_1 be the circle with center O_1 and radius O_1A, and let c_2 be the circle with center O_2 and radius O_2A. Prove that the circles c_1 and c_2 do not intersect at any point other than A.

Solution. Suppose toward a contradiction that c_1 and c_2 intersect at some other point B. Because A and B lie on a circle with center O_1, we have $\overline{O_1A} \cong \overline{O_1B}$, so the triangle O_1AB is isosceles. Therefore the base angles of this triangle at A and B are congruent. By an analogous argument the triangle O_2AB is isosceles, so its base angles at A and B are also congruent. Because A lies on the segment $\overline{O_1O_2}$, the angles O_1AB and O_2AB are supplementary. Therefore the angles O_1BA and O_2BA are also supplementary, so B lies on the segment $\overline{O_1O_2}$ also. Because $\overline{O_1A} = \overline{O_1B}$ this means that $A = B$, a contradiction.