Problem 1 (10 points). Evaluate the following limits. JUSTIFY your answers. If a limit does not exist, say so.

(1) $\lim_{n\to\infty} \frac{2+3n^2}{3+2n^2}$
(2) $\lim_{n\to\infty} \frac{(\ln n)^2}{n}$
(3) $\lim_{n\to\infty} \frac{\sin(n)+n}{2n}$
(4) $\lim_{n\to\infty}(-1)^n e^{-n}$
(5) $\lim_{n\to\infty} \sin\left(\frac{\pi n+3}{2n}\right)$

Solution.

(1) This converges to $3/2$. You can divide the numerator and denominator by n^2 and then take the limit of each, or you can use L'Hospital's rule.
(2) This converges to zero. Use L'Hopital's rule twice.
(3) This converges to $1/2$. You can divide the numerator and denominator by n and take the limit of each. We have $\lim_{n\to\infty} \sin(n)/n = 0$ by a straightforward application of the squeeze theorem. Note that L'Hospital's rule does not apply to this problem (see the condition in the last sentence of the book's statement of the rule.)
(4) This converges to zero because the sequence of absolute values converges to zero.
(5) This converges to 1 because $\frac{\pi n+3}{2n}$ converges to $\pi/2$, the sine function is continuous, and $\sin(\pi/2) = 1$.

Problem 2 (10 points). Determine whether the following series converge or diverge. JUSTIFY your answers. You do NOT need to find the sum.

(1) $\sum_{n=0}^{\infty} \frac{3n+2}{2n^3-n+1}$
(2) $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$
(3) $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$
(4) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$
(5) $\sum_{n=1}^{\infty} \frac{2^n n^2}{n!}$

Solution.

(1) This converges. Look at the highest powers of n to get $n/n^3 = 1/n^2$. The limit comparison test with $\sum_{n=0}^{\infty} 1/n^2$ gives a limit (of quotients) equal to $3/2$, and the series $\sum_{n=0}^{\infty} 1/n^2$ converges because it is a p-series with $p > 1$.
(2) This diverges. Apply the integral test and use the substitution $u = \ln n$ to integrate.
(3) This converges. Use the ratio test, and notice that the expression $e^{n^2}/e^{(n+1)^2}$ can be simplified after expanding $(n+1)^2$ to n^2+2n+1. The limit of quotients in the ratio test will be zero.
(4) This converges by the alternating series test.
(5) This converges by the ratio test. The limit of quotients in the ratio test will be zero.

Problem 3 (6 points). Find the Maclaurin series (Taylor series at zero) of the following functions. You do NOT need to find the radius of convergence.

(1) $f(x) = \begin{cases} \frac{1-\cos(x)}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(2) $f(x) = \sqrt{4 - x^2}$.

(3) $f(x) = \int_0^x z e^{z^2} \, dz$

Solution.

(1) First note that the clause defining $f(x) = 0$ when $x = 0$ is only there in order to make f defined and continuous at zero. Subtracting the Maclaurin series for the cosine function from 1, we obtain $\sum_{n=1}^{\infty} (-1)^{n+1} x^{2n}/(2n)!$. Note that there is no x^0 or x^1 term, so we can divide by x^2 to get the power series $\sum_{n=1}^{\infty} (-1)^{n+1} x^{2n-2}/(2n)!$. This can be reindexed to $\sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n+2)!$ if desired.

(2) Rewriting this as $2(1 + (-x^2/4))^{1/2}$, we see that this is a binomial series with exponent $k = 1/2$, and the answer is $2\sum_{n=0}^{\infty} \binom{1/2}{n} (-x^2/4)^n$, which can be simplified as $\sum_{n=0}^{\infty} \binom{1/2}{n} (-1)^n x^{2n}/2^{2n-1}$ if desired.

(3) By substitution of z^2 into the power series for e^z and multiplying by z we obtain $\sum_{n=0}^{\infty} z^{2n+1}/n!$ for the integrand. Integrating from zero to x term by term we obtain $\sum_{n=0}^{\infty} x^{2n+2}/(2n+2)n!$.

Problem 4 (6 points). Consider the power series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{5^n n^2}.$$

(1) Where is this power series centered?

(2) What is its radius of convergence?

(3) What is its interval of convergence?

Solution. Oops, the zeroeth term involves division by zero and so is undefined. Let's pretend the series starts at 1, which doesn't otherwise affect the argument.

(1) It is centered at 2 because it has the form of a power series in $x - 2$.

(2) The absolute value of the ratio of successive terms is $|(x-2)n^2/5(n+1)^2|$, which approaches $|x-2|/5$ as $n \to \infty$. Therefore the radius of convergence is 5 by the ratio test.

(3) We must determine convergence at the endpoints $2 - 5 = -3$ and $2 + 5 = 7$. Plugging in $x = 7$ we get the series $\sum 1/n^2$, which converges because it is a p-series with $p = 2 > 1$. At the other endpoint we get $\sum(-1)^n/n^2$, which converges because it converges absolutely (or one can use the alternating series test.) Therefore the interval of convergence is $[-3, 7]$.

Problem 5 (10 points). For each statement below, circle T or F according to whether the statement is true or false. You do NOT need to justify your answers.

- T "If the series $\sum_{n=0}^{\infty} |a_n|$ converges, then the series $\sum_{n=0}^{\infty} a_n$ must also converge" is true—it is the statement that absolute convergence implies convergence.
- T "If the series $\sum_{n=0}^{\infty} b_n$ converges and $0 \le a_n \le b_n$, then the series $\sum_{n=0}^{\infty} a_n$ must also converge" is true—it is an instance of the comparison theorem.
- F "The series $\sum_{n=1}^{\infty}(-4/3)^n$ converges" is false because it is a geometric series with common ratio $-4/3$.
- T "The series $\sum_{n=1}^{\infty}(\sqrt{n+1} - \sqrt{n})$ diverges" is true. Write out the first few partial sums and you should see why. Note that the "telescoping" method establishes convergence in some cases and divergence in others.
- T "The series $\sum_{n=1}^{\infty}(1 - 1/n^2)$ diverges" is true because the terms approach 1 (not zero) as $n \to \infty$.

- T "If the series $\sum_{n=0}^{\infty} a_n$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ converges to $S - a_0$" is true—we have just pulled the first term out of the sum.
- F "If the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge, then $\lim_{n\to\infty} a_n/b_n = L$ for some nonzero real number L" is false. Consider $a_n = 1/n^2$ and $b_n = 1/n^3$ or *vice versa*.
- F "If the function f is continuous, positive, and decreasing on $[1, \infty)$ and $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} f(n)$ converges to the same value" is false—it converges, but not to the same value.
- F "If the power series $\sum_{n=1}^{\infty} c_n x^n$ converges for $x = 1$ then it must also converge for $x = -1$" is false, as the example $c_n = (-1)^n/n$ shows.
- T "If the series $\sum_{n=0}^{\infty} a_n$ converges, then the sequence of terms $\{a_n\}_{n=0}^{\infty}$ must converge to zero" is true—this is the contrapositive of the "test for divergence".

Problem 6 (10 points). For each statement below, circle T or F according to whether the statement is true or false. You do NOT need to justify your answers.

F "The series $\sum_{n=1}^{\infty}(-1)^n/n^2$ converges conditionally" is false—it converges absolutely.

T "If the sequence $\{b_n\}_{n=0}^{\infty}$ is bounded and increasing, then it must converge" is true—it follows from the monotonic sequence theorem.

T "If the series $\sum_{n=0}^{\infty} a_n$ diverges and $0 \leq a_n \leq b_n$ for all n, then the series $\sum_{n=0}^{\infty} b_n$ must also diverge" is true—it is an instance of the comparison test.

F "If the series $\sum_{n=0}^{\infty} a_n$ converges to L, then the series $\sum_{n=0}^{\infty}(a_n+3)$ converges to $L+3$" is false. If the first series converges, then the limit of its terms is zero, so the limit of the terms of the second series is 3, and the second series diverges.

T "The series $\sum_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n}\right)$ converges" is true, by the "telescoping" method.

F "Given a sequence $\{a_n\}_{n=0}^{\infty}$, if the sequences $\{a_{2n}\}_{n=0}^{\infty}$ and $\{a_{2n+1}\}_{n=0}^{\infty}$ both converge then $\{a_n\}_{n=0}^{\infty}$ itself must converge" is false—consider $a_n = (-1)^n$.

F "If the series $\sum_{n=0}^{\infty} b_n$ converges and $a_n/b_n \to 0$ as $n \to \infty$, then the series $\sum_{n=0}^{\infty} a_n$ must also converge" is false: consider $b_n = (-1)^n/\sqrt{n+1}$ and $a_n = 1/(n+1)$.

T "The series $\sum_{n=1}^{\infty} n^{-1.1}$ converges" is true—it is a p-series with $p = 1.1 > 1$.

T "If the power series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely at $x=-2$ then it must also converge absolutely at $x=2$" is true—the only difference between the series at the two endpoints is in the signs of terms, and the signs of the terms do not affect absolute convergence.

F "The series $\sum_{n=1}^{\infty} 1/3^{n-1}$ converges to $4/3$" is false by the formula for sums of geometric series (and also the fact that the third partial sum is already larger than $4/3$.)