Problem 1 (6 points). Find the general solution of the following system of equations. If it is inconsistent, say so.

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 3 \\ x_1 - x_2 - x_3 - x_4 &= -1 \\ 2x_1 - 2x_4 &= 2 \end{aligned}$$

Solution. Use Gaussian elimination on the augmented matrix of the system:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 1 & -1 & -1 & -1 & -1 \\ 2 & 0 & 0 & -2 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 0 & -2 & -2 & 0 & -4 \\ 0 & -2 & -2 & 0 & -4 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 0 & -2 & -2 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The free variables are x_3 and x_4. Let $x_3 = \alpha$ and $x_4 = \beta$. The first row says $x_1 - x_4 = 1$, so $x_1 = 1 + \beta$, and the second row says $x_2 + x_3 = 2$, so $x_2 = 2 - \alpha$. Therefore the general solution can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1+\beta \\ 2-\alpha \\ \alpha \\ \beta \end{pmatrix}.$$

where α and β are scalars. (There are other ways to write the general solution. If you got something that looks different, it is not necessarily wrong.)

Problem 2 (6 points). Compute the following product of matrices.

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$$

Solution. We are multiplying a 3×2 matrix by a 2×2 matrix, so the result will be a 3×2 matrix. Its $(1, 1)$ entry will be $1 \cdot 2 + (-1) \cdot 1 = 1$. Doing a similar calculation for all the other entries, we obtain

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 6 \\ 0 & -1 \end{pmatrix}.$$

Problem 3 (6 points). Is the following matrix invertible? If not, why not? If so, what is its inverse?

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 3 & 0 & 13 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution. We try to find the inverse by Gaussian elimination on $(A \mid I)$.[1] We have

$$\begin{aligned}
(A \mid I) &= \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 3 & 0 & 13 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & 0 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 13 & -4 & 0 \\ 0 & 1 & 0 & 3 & -1 & 1 \\ 0 & 0 & 1 & -3 & 1 & 0 \end{array}\right),
\end{aligned}$$

which means that

$$A^{-1} = \begin{pmatrix} 13 & -4 & 0 \\ 3 & -1 & 1 \\ -3 & 1 & 0 \end{pmatrix}.$$

To check our answer, we can multiply this matrix by A (on either side) and verify that we get the identity matrix.

[1] If you prefer, you can calculate the determinant first to see whether or not A is invertible.

Problem 4 (6 points). Calculate the following determinant.

$$\begin{vmatrix} 3 & 0 & -1 & 2 \\ -1 & 0 & 3 & 3 \\ 1 & 2 & 5 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix}$$

Solution. We do a cofactor expansion along the second column.[2] This gives

$$\begin{vmatrix} 3 & 0 & -1 & 2 \\ -1 & 0 & 3 & 3 \\ 1 & 2 & 5 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix} = (-1)^{3+2} \cdot 2 \cdot \begin{vmatrix} 3 & -1 & 2 \\ -1 & 3 & 3 \\ 2 & 0 & 1 \end{vmatrix}.$$

To compute this 3×3 determinant, we do a cofactor expansion along the third row and then use the formula for 2×2 determinants:

$$\begin{vmatrix} 3 & -1 & 2 \\ -1 & 3 & 3 \\ 2 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = -10.$$

Combining these two steps, we get

$$\begin{vmatrix} 3 & 0 & -1 & 2 \\ -1 & 0 & 3 & 3 \\ 1 & 2 & 5 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix} = 20.$$

[2]Any other row or column will ultimately give the same result, but we can save work by choosing rows or columns with many zeroes.

Problem 5 (6 points). Consider the matrix

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Is A diagonalizable? If not, why not? If so, find a diagonal matrix D and an invertible matrix X such that $XDX^{-1} = A$.

Solution. We begin by finding the eigenvalues of A, which are the roots of the characteristic polynomial $\det(A - \lambda I)$. We have

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = (2-\lambda)(-\lambda) + 1 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

Therefore $\lambda = 1$ is the only eigenvalue. Next we find the eigenvectors corresponding to $\lambda = 1$. These are the solutions $\vec{x}$ of the equation $(A - \lambda I)\vec{x} = \vec{0}$. The augmented matrix for this system is

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

so the eigenspace corresponding to $\lambda = 1$ is the set of all vectors of the form $(-\alpha, \alpha)^T$ where the free variable α is any scalar. Setting $\alpha = 1$ we obtain the eigenvector $\vec{x} = (-1, 1)^T$, but because there are no other free variables, all the eigenvectors corresponding to $\lambda = 1$ are multiples of this one. Moreover, there are no other eigenvalues. Therefore we cannot find two linearly independent eigenvectors as required to diagonalize a 2×2 matrix, so A is not diagonalizable.

Problem 6 (6 points). For each statement below, circle T or F according to whether the statement is true or false.

T / F For every invertible matrix A and every vector $\vec{x}$, if $A\vec{x} = \vec{0}$ then $\vec{x} = \vec{0}$.

T / F If A is a square matrix with real entries and λ is an eigenvalue of A, then the complex conjugate $\bar{\lambda}$ must also be an eigenvalue of A.

T / F If A and B are row-equivalent square matrices, then A and B must have the same eigenvalues.

T / F For all $n \times n$ matrices A and B, $\det(AB) = \det(BA)$.

T / F For every square matrix A, the transpose A^T has the same eigenvalues as A.

T / F For all $n \times n$ matrices A and B and all $i, j \in \{1, \ldots, n\}$, the (i, j) entry of AB is the (i, j) entry of A times the (i, j) entry of B.

Solution.

(1) True. To see this, multiply both sides on the left by A^{-1}.

(2) True. To see this, let $\vec{x}$ be an eigenvector of A corresponding to λ, and take the complex conjugate of both sides of the equation $A\vec{x} = \lambda\vec{x}$.

(3) False. Just pick an A, and apply a single row operation of type I or II to get B, and you will probably have a counterexample.

(4) True. To see this, use $\det(AB) = \det(A)\det(B)$.

(5) True. To see this, recall that transposing a matrix does not change its determinant, and then use the fact that the eigenvalues are the solutions of $\det(A - \lambda I) = 0$.

(6) False. Just pick any A and B, and you will probably have a counterexample.