THE ENVELOPE OF A POINTCLASS UNDER A LOCAL DETERMINACY HYPOTHESIS

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ABSTRACT. Given an inductive-like pointclass $\Gamma$ and assuming the Axiom of Determinacy, Martin identified and analyzed the pointclass containing the norm relations of the next semiscale beyond $\Gamma$, if one exists. We show that much of Martin's analysis can be carried out assuming only $\text{ZF} + \text{DC}_R + \text{Det}(\Delta_\Gamma)$. This generalization requires arguments from Kechris–Woodin [10] and Martin [13]. The results of [10] and [13] can then be recovered as immediate corollaries of the general analysis. We also obtain a new proof of a theorem of Woodin on divergent models of $\text{AD}^+$, as well as a new result regarding the derived model at an indestructibly weakly compact limit of Woodin cardinals.

INTRODUCTION

Given an inductive-like pointclass $\Gamma$, Martin introduced a pointclass (which we call the envelope of $\Gamma$, following [22]) whose main feature is that it contains the prewellorderings of the next scale (or semiscale) beyond $\Gamma$, if such a (semi)scale exists. This work was distributed as “Notes on the next Suslin cardinal,” as cited in Jackson [3]. It is unpublished, so we use [3] as our reference instead for several of Martin’s arguments.

Martin's analysis used the assumption of the Axiom of Determinacy. We reformulate the notion of the envelope in such a way that many of its essential properties can by derived without assuming $\text{AD}$. Instead we will work in the base theory $\text{ZF} + \text{DC}_R$ for the duration of the paper, stating determinacy hypotheses only when necessary.

With the exception of Section 9 this paper is mainly expository. It places results of Martin, Kechris, Woodin, and others in a unified framework and in some cases substantially simplifies the original proofs.

Section 1 contains background information on descriptive set theory including the definition of “inductive-like pointclass.” In Section 2 we define the envelope of an inductive-like pointclass in a special case that will suffice for results in $L(\mathbb{R})$ and we prove some results about it, most notably its determinacy and its closure under real quantifiers. In Sections 3 and 4 we apply these results to the envelope of the pointclass of inductive sets and
then more generally to the envelopes of inductive-like pointclasses in $L(\mathbb{R})$. In particular we show that the Kechris–Woodin determinacy transfer theorem and a theorem of Martin on reflection of ordinal-definability for reals can be obtained as immediate consequences.

In Section 6 we give a more general definition of the envelope for boldface inductive-like pointclasses, which is phrased in terms of Moschovakis’s notion of “companion” for such pointclasses (see Section 5.) We then show that the results of Section 2 can readily be adapted to this more general setting.

In Section 7 we give an equivalent condition for the existence of semiscales that is phrased in terms of games. In the later sections these games are used to construct (under additional assumptions) semiscales on a universal $\Gamma$ set with prewellorderings in the envelope of $\Gamma$. In Section 8 we give a new proof of Woodin’s theorem on divergent models of $\text{AD}^+$. Finally, in Section 9 we show that the derived model at an indestructibly weakly compact limit of Woodin cardinals satisfies “every set of reals is Suslin.”

The results in this paper are mostly taken from the author’s PhD thesis [25, Ch. 3]. The author wishes to thank John Steel, who supervised this thesis work, for his guidance; Hugh Woodin, for explaining his argument for the result stated as Theorem 8.2 below; and Martin Zeman, for suggesting several corrections.

1. Pointclasses

We begin by recalling some standard notions from descriptive set theory. By convention we denote the Baire space $\omega^\omega$ by $\mathbb{R}$ and call its elements “reals.” A product space is a space of the form

$$\mathcal{X} = X_1 \times \cdots \times X_n, \quad X_i = \mathbb{R} \text{ or } X_i = \omega \text{ for all } i \leq n.$$ 

A pointset is a subset of a product space. A pointclass is a collection of pointsets, typically an initial segment of some complexity hierarchy for pointsets.

We say that a pointclass $\Gamma$ is $\omega$-parameterized (respectively, $\mathbb{R}$-parameterized) if for every product space $\mathcal{X}$ there is a $\Gamma$ subset of $\omega \times \mathcal{X}$ (respectively, of $\mathbb{R} \times \mathcal{X}$) that is universal for $\Gamma$ subsets of $\mathcal{X}$.

In this paper we consider the following types of pointclasses, which are named for their resemblance to the pointclasses IND and IND of inductive sets and boldface inductive sets respectively. The inductive pointsets are those that are definable without parameters by positive elementary induction on $\mathbb{R}$ (sometimes called “absolutely” or “lightface” inductive.) By “boldface” we mean that real parameters are allowed.

**Definition 1.1.** A pointclass $\Gamma$ is (lightface) inductive-like if it is $\omega$-parameterized, closed under $\exists^\mathbb{R}, \forall^\mathbb{R}$, and recursive substitution, and has the prewellordering property.
Definition 1.2. A pointclass $\Gamma$ is boldface inductive-like if it is $\mathbb{R}$-parameterized, closed under $\exists^\mathbb{R}$, $\forall^\mathbb{R}$, and continuous reducibility, and has the prewellordering property.

Note that some authors strengthen the prewellordering property to the scale property in these definitions.

In this paper we will typically use the following notational conventions. By $\Gamma$ and $\tilde{\Gamma}$ we will denote an inductive-like or boldface-inductive-like pointclass respectively. As usual we denote the dual pointclass of $\Gamma$ by $\hat{\Gamma} = \{\neg A : A \in \Gamma\}$ and the ambiguous part of $\Gamma$ by $\Delta_{\Gamma} = \Gamma \cap \hat{\Gamma}$. (Note that the existence of universal sets implies non-self-duality by a diagonal argument.) Similarly for a boldface pointclass we define the dual and ambiguous pointclasses $\tilde{\hat{\Gamma}}$ and $\tilde{\Delta}_{\Gamma}$.

Given a lightface pointclass $\Gamma$ we can define the corresponding relativizations $\Gamma(x)$ for $x \in \mathbb{R}$ and also the boldface pointclasses $\tilde{\Gamma} = \bigcup_{x \in \mathbb{R}} \Gamma(x)$, $\hat{\Gamma} = \bigcup_{x \in \mathbb{R}} \hat{\Gamma}(x)$, and $\Delta_{\Gamma} = \bigcup_{x \in \mathbb{R}} \Delta_{\Gamma}(x)$. Naturally, if $\Gamma$ is inductive-like then the corresponding boldface pointclass $\tilde{\Gamma}$ is boldface inductive-like.

Typically given a pointclass $\Gamma$ we will assume the local determinacy hypothesis $\text{Det}(\Delta_{\Gamma})$, meaning that every two-player game of perfect information on $\omega$ with a $\Delta_{\Gamma}$ payoff set is determined, although some of our results can proved under weaker hypotheses.

The prewellordering ordinal of an inductive-like pointclass $\Gamma$ (or more generally of a pointclass $\Gamma$ with the prewellordering property) is the supremum of the ordinal lengths of all $\Delta_{\Gamma}$ prewellorderings of $\mathbb{R}$, or equivalently the range of any regular $\Gamma$-norm on a complete $\Gamma$ pointset. We will often denote the prewellordering ordinal of a pointclass by $\kappa$.

The prototypical example of an inductive-like pointclass is, as noted above, the pointclass $\text{IND}$ of inductive sets. Other examples of inductive-like pointclasses include the pointclass $\Sigma^2_1 L(\mathbb{R})$ and more generally the pointclass $\Sigma^2_1$ under $\text{AD}^+$. Examples of boldface inductive-like pointclasses include, in addition to the boldface versions of the above lightface pointclasses, the pointclass $S(\kappa)$ of $\kappa$-Suslin sets when $\text{AD}$ holds, $\kappa$ is a Suslin cardinal, and $S(\kappa)$ is closed under $\forall^\mathbb{R}$.

The following notion of countable approximation for pointsets is central to the paper.

Definition 1.3 (Martin, see [3, Defn. 3.9]). Let $\mathcal{X}$ be a product space, let $\kappa$ be an ordinal, and let $(A_\alpha : \alpha < \kappa)$ be a sequence of subsets of $\mathcal{X}$. For a pointset $A \subset \mathcal{X}$, we say $A \in (A_\alpha : \alpha < \kappa)$ if for every countable set $\sigma \subset \mathcal{X}$ there is an ordinal $\alpha < \kappa$ such that $A \cap \sigma = A_\alpha \cap \sigma$.

Martin then made the following definition. It is similar to the definition of "envelope" that we will use in this paper, but we will need to make a significant modification. We include the original definition below in order to provide historical context.
Definition 1.4 (Martin, see [3, Defn. 3.9]). Let $\Delta$ be a pointclass and let $\kappa$ be an ordinal. For a product space $\mathcal{X}$ and a pointset $A \subset \mathcal{X}$, we say $A \in \Lambda(\Delta, \kappa)$ if $A \in \left( A_\alpha : \alpha < \kappa \right)$ for some sequence $(A_\alpha : \alpha < \kappa)$ of $\Delta$ subsets of $\mathcal{X}$.

If $\text{AC}$ holds then the pointclass $\Lambda(\Delta, \kappa)$ is not likely to be very useful because there will be too many well-ordered sequences of pointsets. For example, if $\Delta$ is a nontrivial boldface pointclass (so it contains every countable pointset) and $\kappa \geq \mathfrak{c}$ then any pointset whatsoever is in $\Lambda(\Delta, \kappa)$. To get a useful definition without assuming $\text{AD}$ we need to require some kind of uniformity for the sequence of pointsets $(A_\alpha : \alpha < \kappa)$. First we deal with a special case.

2. The envelope of $\Sigma^J_1(\mathbb{R})$

We will need some background on the fine structure of $L(\mathbb{R})$ that can be found, for example, in [23, §1]. We use the Jensen hierarchy $(J_\alpha : \alpha \in \text{Ord})$ for $L(\mathbb{R})$ but the reader would not miss much by reading “$L_\kappa(\mathbb{R})$” in place of “$J_\kappa(\mathbb{R})$.” However we should note that the ordinal height of $J_\alpha(\mathbb{R})$ is $\omega \alpha$ rather than $\alpha$.

By convention we allow $\mathbb{R}$ as an unstated parameter in our formulas. For example, by $\Sigma_1$ we mean $\Sigma_1(\{ \mathbb{R} \})$. Therefore quantification over the reals always counts as bounded quantification.

Given ordinals $\kappa$ and $\beta$ with $\kappa \leq \beta$, we say $J_\kappa(\mathbb{R}) \prec^R_1 J_\beta(\mathbb{R})$ if $J_\kappa(\mathbb{R})$ is a $\Sigma^0_1(\mathbb{R} \cup \{ \mathbb{R} \})$-elementary substructure of $J_\beta(\mathbb{R})$. As usual, $\Theta$ denotes the least ordinal that is not a surjective image of the reals.

Definition 2.1. A gap

A gap in $L(\mathbb{R})$ is a maximal interval of ordinals $[\kappa, \beta]$ such that $J_\kappa(\mathbb{R}) \prec^R_1 J_\beta(\mathbb{R})$ and $\beta \leq \Theta^{L(\mathbb{R})}$.

The gaps partition the interval $[0, \Theta^{L(\mathbb{R})}]$. We say an ordinal $\kappa$ begins a gap in $L(\mathbb{R})$ if $[\kappa, \beta]$ is a gap for some ordinal $\beta$. If $\kappa$ is a limit ordinal, this means that new $\Sigma^0_1$ facts about reals are witnessed cofinally often below $\kappa$ in the Jensen hierarchy. When stating results about the pointclass $\Sigma^J_1(\mathbb{R})$ we will assume without loss of generality that $\kappa$ begins a gap because otherwise $\Sigma^J_1(\mathbb{R}) = \Sigma^J_1(\mathbb{R})$ for some $\alpha < \kappa$.

If $\kappa$ begins a gap then there is a $\Sigma^J_1(\mathbb{R})$ partial surjection $\mathbb{R} \rightarrow J_\kappa(\mathbb{R})$ (see [23, Lem. 1.11]) and therefore every $\Sigma^J_1(\mathbb{R})$ pointset is in $\Sigma^J_1(\mathbb{R})(x)$ for some real parameter $x$. That is, we can replace arbitrary parameters in $J_\kappa(\mathbb{R})$ with real parameters.

If the level $J_\kappa(\mathbb{R})$ is admissible (equivalently, if it satisfies the $\Sigma^0_1$-collection axiom) then the pointclass $\Sigma^J_1(\mathbb{R})$ is closed under $\forall^\mathbb{R}$. This implies that $\Sigma^J_1(\mathbb{R})$ is inductive-like, as the other clauses of the definition are easy to verify. (To verify the prewellordering property, for example, observe that

\footnote{More precisely called a $\Sigma_1$-gap}
whenever \( \kappa \) is a limit ordinal, a \( \Sigma_1^{J_\kappa(\mathbb{R})} \)-norm on a \( \Sigma_1^{J_\kappa(\mathbb{R})} \) set is given by mapping every element of the set to the least ordinal \( \alpha < \kappa \) such that the level \( J_\alpha(\mathbb{R}) \) contains a witness to the relevant \( \Sigma_1 \) property of that element.)

We define a special case of the envelope as follows. (This definition is not standard; see Section 6 for remarks on other definitions.)

\textbf{Definition 2.2.} Let \( J_\kappa(\mathbb{R}) \) be an admissible level of \( L(\mathbb{R}) \) that begins a gap. For a pointset \( A \), we say:

- \( A \in \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})}) \) if \( A \in (A_\alpha : \alpha < \kappa) \) for some sequence of pointsets \( (A_\alpha : \alpha < \kappa) \) that is \( \Delta_1 \)-definable over \( J_\kappa(\mathbb{R}) \).
- \( A \in \text{Env}(\Sigma_1^{\tilde{J}_\kappa(\mathbb{R})}) \) if \( A \in (A_\alpha : \alpha < \kappa) \) for some sequence of pointsets \( (A_\alpha : \alpha < \kappa) \) that is \( \Delta_{\tilde{\kappa}} \)-definable over \( J_\kappa(\mathbb{R}) \).

Note that \( \text{Env}(\Sigma_1^{\tilde{J}_\kappa(\mathbb{R})}) = \bigcup_{x \in \mathbb{R}} \text{Env}(\Sigma_1^{J_x(\mathbb{R})}(x)) \) where the relativization \( \text{Env}(\Sigma_1^{J_x(\mathbb{R})}(x)) \) is defined in the obvious way. The definition of \( \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})}) \) can be rephrased in terms of a notion of local ordinal-definability.

\textbf{Definition 2.3.} Let \( \beta \) be an ordinal and let \( A \) be a pointset. Let \( p \in J_\beta(\mathbb{R}) \) be a parameter. \(^2\) Then we say:

- \( A \in \text{OD}^\beta(p) \) if \( A \) is first-order definable from \( p \) and ordinal parameters over the structure \( (J_\beta(\mathbb{R}); \in) \).
- \( A \in \text{OD}^{<\beta}(p) \) if \( A \in \text{OD}^\alpha(p) \) for some ordinal \( \alpha < \beta \).

In the case where \( J_\kappa(\mathbb{R}) \) is admissible we get another characterization of \( \text{OD}^{<\kappa} \). This is surely well-known, but we isolate it here in order to make an analogy later in Section 6.

\textbf{Lemma 2.4.} Let \( J_\kappa(\mathbb{R}) \) be an admissible level of \( L(\mathbb{R}) \) and let \( A \) be a pointset. Then the following statements are equivalent:

- \( A \in \text{OD}^{<\kappa} \).
- \( A \in \Delta_1^{J_\kappa(\mathbb{R})}(s) \) for some finite sequence of ordinal parameters \( s \in \kappa^{<\omega} \).

\textit{Proof.} If \( A \in \text{OD}^\alpha \) where \( \alpha < \kappa \), say \( A \) is first-order definable over \( J_\alpha(\mathbb{R}) \) from a finite sequence of ordinals \( s_0 \), then because the sequence of levels \( (J_\alpha(\mathbb{R}) : \alpha < \kappa) \) is \( \Delta_1^{J_\kappa(\mathbb{R})} \) we have \( A \in \Delta_1^{J_\kappa(\mathbb{R})}(s_0) \).

Conversely, if \( A \in \Delta_1^{J_\kappa(\mathbb{R})}(s) \) where \( s \in \kappa^{<\omega} \) then by admissibility, for any sufficiently large \( \alpha < \kappa \) all the relevant witnesses to \( \Sigma_1 \) facts about reals and \( s \) have already appeared, so \( A \) is \( \Delta_1^{J_\kappa(\mathbb{R})}(s) \). In particular, \( A \in \text{OD}^\alpha \). \( \square \)

The following result is easily seen to be a consequence of the existence of a \( \Sigma_0 \) well-ordering of finite sequences of ordinals together with the fact that the sequence of levels \( (J_\alpha : \alpha < \kappa) \) is \( \Delta_1^{J_\kappa(\mathbb{R})} \).

\(^2\)In our applications, \( p \) will be a real or a Turing degree.
Lemma 2.5. Let $J_\kappa(\mathbb{R})$ be an admissible level of $L(\mathbb{R})$. Then there is a $\Delta^1_1(J_\kappa(\mathbb{R}))$ sequence enumerating all OD$^{<\kappa}$ pointsets.

From Lemmas 2.4 and 2.5 we immediately get the following characterization of the envelope, which we will use many times without comment in the sequel.

Lemma 2.6. Let $J_\kappa(\mathbb{R})$ be an admissible level of $L(\mathbb{R})$ that begins a gap. Let $\mathcal{X}$ be a product space. For every pointset $A \subseteq \mathcal{X}$ the following statements are equivalent.

- $A \in \text{Env}(\Sigma^1_1 J_\kappa(\mathbb{R}))$.
- For every countable set $\sigma \subseteq \mathcal{X}$ there is a pointset $A' \in \text{OD}^{<\kappa}$ with $A \cap \sigma = A' \cap \sigma$.

Next we establish some basic closure properties of the envelope. Some of the corresponding arguments for Martin’s pointclass $\Lambda(\Delta_1, \kappa)$ can be found in [3, Lem 3.10 and Cor. 3.11].

Proposition 2.7. Let $J_\kappa(\mathbb{R})$ be an admissible level of $L(\mathbb{R})$ that begins a gap. Then $\text{Env}(\Sigma^1_1 J_\kappa(\mathbb{R}))$ contains $\Sigma^1_1 J_\kappa(\mathbb{R})$ and is closed under Boolean combinations, number quantification, and recursive substitution.

Proof. Fix a product space $\mathcal{X}$. To see that $\Sigma^1_1 J_\kappa(\mathbb{R}) \subseteq \text{Env}(\Sigma^1_1 J_\kappa(\mathbb{R}))$, let $A \subseteq \mathcal{X}$ be a pointset that is definable over $J_\kappa(\mathbb{R})$ by a $\Sigma^1_1$ formula $\theta$. Let $\sigma \subseteq \mathcal{X}$ be countable. Because $J_\kappa(\mathbb{R})$ is admissible the ordinal $\kappa$ has uncountable cofinality, so if we choose a large enough $\alpha < \kappa$ then the OD$^{\alpha}$ set $A' \subseteq \mathcal{X}$ defined by $x \in A' \iff (J_\kappa(\mathbb{R}); \in) \models \theta(x)$ satisfies $A \cap \sigma = A' \cap \sigma$. Therefore $A \in \text{Env}(\Sigma^1_1 J_\kappa(\mathbb{R}))$.

The closure of $\text{Env}(\Sigma^1_1 J_\kappa(\mathbb{R}))$ under Boolean combinations is an immediate consequence of the closure of OD$^{<\kappa}$ under Boolean combinations. The closure of $\text{Env}(\Sigma^1_1 J_\kappa(\mathbb{R}))$ under number quantification is a consequence of the closure of OD$^{<\kappa}$ under number quantification, using the fact that if the pointset $\sigma \subseteq \mathcal{X}$ is countable then so is the pointset $\omega \times \sigma \subseteq \omega \times \mathcal{X}$. The closure of $\text{Env}(\Sigma^1_1 J_\kappa(\mathbb{R}))$ under recursive substitution follows from the closure of OD$^{<\kappa}$ under recursive substitution, using the fact that the pointwise image of a countable set $\sigma$ (under a recursive function) is countable. □

Note that all results so far can be relativized from $\Sigma^1_1 J_\kappa(\mathbb{R})$ and OD$^{<\kappa}$ to $\Sigma^1_1 J_\kappa(\mathbb{R})(x)$ and OD$^{<\kappa}(x)$ respectively where $x$ is a real. In particular, the pointclass $\text{Env}(\Sigma^1_1 J_\kappa(\mathbb{R}))$ is closed under continuous reducibility.

The following “anti-uniformization” lemma limits the extent of the envelope. It is essentially a localization from OD to OD$^{<\kappa}$ of an observation due to Solovay (as attributed in [23, p. 149].)

Lemma 2.8. Let $J_\kappa(\mathbb{R})$ be an admissible level of $L(\mathbb{R})$ that begins a gap. If $A \subseteq \mathbb{R} \times \omega$ is a pointset whose sections $A_x$ (defined by $A_x(n) \iff A(x, n)$) satisfy $A_x \notin \text{OD}^{<\kappa}(x)$ for all reals $x$, then $A \notin \text{Env}(\Sigma^1_1 J_\kappa(\mathbb{R}))$. 
Proof. Let \( A \subset \mathbb{R} \times \omega \) be a pointset in \( \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})}) \), say \( A \in \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})}(x)) \) where \( x \in \mathbb{R} \). Take a pointset \( A' \in \text{OD}^{<\kappa}(x) \) such that \( A' \cap \sigma = A \cap \sigma \) where \( \sigma \) is the countable pointset \( \{x\} \times \omega \). Then from \( A' \) and \( x \) we can compute the section \( A_x \), so \( A_x \in \text{OD}^{<\kappa}(x) \). \( \square \)

Remark 2.9. We remark that an anti-uniformization result more along the lines of Solovay's original observation would state that, if the \( \Pi^J_1 \kappa(\mathbb{R}) \) relation \( A \) on \( \mathbb{R} \times \mathbb{R} \) given by \( (x, y) \in A \iff y \notin \text{OD}^{<\kappa}(x) \) is total, then it has no uniformization in \( \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})}) \). Later we will see that if \( \text{AD} \) holds in the admissible set \( J_\kappa(\mathbb{R}) \) then this relation is indeed total and moreover the pointclass \( \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})}) \) is sufficiently closed (in particular it is closed under real quantifiers) that these two anti-uniformization statements are equivalent.

Next we proceed to establish some properties of \( \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})}) \) that follow from the assumption of \( \text{AD} \) in \( J_\kappa(\mathbb{R}) \), or equivalently by admissibility, the assumption of \( \Delta^J_1 \) determinacy. In some cases, we will only need the hypothesis of \( \Delta^J_1 \) determinacy.

The following lemma is derived from an argument used by Kechris and Woodin to prove a determinacy transfer principle in \( L(\mathbb{R}) \) ([10, Thm. 1.8].) It is also similar to the proof of the Kechris–Solovay theorem [9, Thm. 3.1] that \( \Delta^1_2 \)-determinacy implies that \( \text{OD} \)-determinacy holds in \( L[x] \) for a Turing cone of reals \( x \).

In order to state the lemma we first make a definition. For a set of reals \( A \) and a Turing ideal\(^3\) \( \mathcal{I} \), we say that \( A \) is determined on \( \mathcal{I} \) if there is a strategy in \( \mathcal{I} \) (more precisely, coded by a real in \( \mathcal{I} \)) for the game \( G_A \) that defeats every real in \( \mathcal{I} \). In other words, there is a strategy \( \sigma \in \mathcal{I} \) such that for all reals \( x, y \in \mathcal{I} \) we have \( \sigma * y \in A \) (if \( \sigma \) is a strategy for Player I) and \( x * \sigma \notin A \) (if \( \sigma \) is a strategy for Player II.)

Lemma 2.10. Let \( J_\kappa(\mathbb{R}) \) be an admissible level of \( L(\mathbb{R}) \) that begins a gap and suppose that \( \Delta^J_1 \kappa(\mathbb{R}) \) is determined. Then there is a real \( t \) such that every \( \text{OD}^{<\kappa} \) set of reals is determined on every countable Turing ideal \( \mathcal{I} \) containing \( t \).

Proof. Suppose not. Let \( (A_\alpha : \alpha < \kappa) \) be a \( \Delta^J_1 \kappa(\mathbb{R}) \) sequence enumerating all \( \text{OD}^{<\kappa} \) sets of reals. For a real \( t \), define \( \alpha(t) \) as the least ordinal \( \alpha < \kappa \) such that there is a countable Turing ideal \( \mathcal{I} \) containing \( t \) and on which the set of reals \( A_\alpha \) is not determined. The function \( \mathbb{R} \to \kappa \) given by \( t \mapsto \alpha(t) \) is total and \( \Delta^J_1 \kappa(\mathbb{R}) \), so the following game is \( \Delta^J_1 \kappa(\mathbb{R}) \) and is therefore determined:

\[
\begin{array}{c}
\text{I} & x(0), z(0) & x(1), z(1) \ldots \\
\text{II} & y(0), s(0) & y(1), s(1), \ldots,
\end{array}
\]

\(^3\)A Turing ideal is a set of reals that is closed under recursive join \( \oplus \) and is downward closed under Turing reducibility.
where we say that Player I wins if and only if \( x \oplus y \in A_{\alpha(z \oplus s)} \). Here \( x \oplus y \) denotes the recursive join \((x(0), y(0), x(1), y(1), \ldots)\) of \( x \) and \( y \), and \( z \oplus s \) is defined similarly.

Assume that Player I has a winning strategy \( \sigma_G \) in the game \( G \). (The other case is similar.) Fix a real \( s \) coding \( \sigma_G \). By the definition of the ordinal \( \alpha(s) \) we may fix a countable Turing ideal \( \mathcal{I} \) containing the real \( s \) and on which \( A_{\alpha(s)} \) is not determined. Let \( \sigma \) be the strategy for Player I in the game \( G_{A_{\alpha(s)}} \) that is obtained from \( \sigma_G \) by pretending that the real \( s \) was played alongside \( y \) by Player II in the game \( G \), and ignoring the real \( z \) produced alongside \( x \) by Player I’s strategy \( \sigma_G \). That is, for every finite sequence of moves \( \bar{y} \in \omega^n \) we have

\[
\sigma(\bar{y}) = \bar{x} \iff \sigma_G(\bar{y}, s \upharpoonright n) = (\bar{x}, \bar{\varepsilon}) \text{ for some } \bar{\varepsilon} \in \omega^n.
\]

This strategy \( \sigma \) can be computed from the real \( s \) coding \( \sigma_G \), so it is in \( \mathcal{I} \). Therefore we may fix a real \( y \in \mathcal{I} \) with \( \sigma \ast y \notin A_{\alpha(s)} \). On the other hand, because \( \sigma_G \) is a winning strategy for Player I in \( G \) there is a real \( z \in \mathcal{I} \) such that \( \sigma \ast y \in A_{\alpha(z \oplus s)} \). We will derive a contradiction by showing that \( \alpha(z \oplus s) = \alpha(s) \). Every Turing ideal containing \( z \oplus s \) also contains \( s \), so \( \alpha(z \oplus s) \geq \alpha(s) \). But \( z \oplus s \in \mathcal{I} \), so \( \alpha(z \oplus s) \leq \alpha(s) \) by the minimization in the definition of \( \alpha \). This is a contradiction. \( \square \)

Now it is a simple matter to prove determinacy of the envelope (indeed almost trivial, but we have isolated this part of the argument because it seems to make essential use of DC\( _\mathbb{R} \) whereas the lemma might be useful in a broader context.)

**Proposition 2.11.** Let \( J_\kappa(\mathbb{R}) \) be an admissible level of \( L(\mathbb{R}) \) that begins a gap and suppose that \( \Delta^{J_\kappa(\mathbb{R})}_1 \) is determined. Then \( \Env(\Sigma^{J_\kappa(\mathbb{R})}_1) \) is determined.

**Proof.** By Lemma 2.10 we can take a real \( t \) such that every OD\( ^{<\kappa} \) set of reals is determined on every countable Turing ideal \( \mathcal{I} \) containing \( t \). Let \( A \) be a set of reals in \( \Env(\Sigma^{J_\kappa(\mathbb{R})}_1) \) and suppose toward a contradiction that \( A \) is not determined. Then by a Skolem hull argument using DC\( _\mathbb{R} \) there is a countable Turing ideal \( \mathcal{I} \) containing \( t \) and on which \( A \) is not determined. We can take a set of reals \( A' \in \text{OD}^{<\kappa} \) such that \( A \cap \mathcal{I} = A' \cap \mathcal{I} \), so the set \( A' \) is not determined on \( \mathcal{I} \) either, which is a contradiction. \( \square \)

We can use the determinacy of \( \Env(\Sigma^{J_\kappa(\mathbb{R})}_1) \) together with the canonical well-ordering of OD\( ^{<\kappa} \) to get a canonical well-ordering of \( \Env(\Sigma^{J_\kappa(\mathbb{R})}_1) \). A similar argument appears in the proof of Kunen’s theorem that, under AD, every (countably complete) measure on an ordinal \( \kappa < \Theta \) is ordinal-definable (see [7, Thm. 28.20].)

**Proposition 2.12.** Let \( J_\kappa(\mathbb{R}) \) be an admissible level of \( L(\mathbb{R}) \) that begins a gap and suppose that \( \Delta^{J_\kappa(\mathbb{R})}_1 \) is determined. Then every pointset in \( \Env(\Sigma^{J_\kappa(\mathbb{R})}_1) \) is ordinal-definable.
A (the argument for arbitrary product spaces is similar.) For a set of reals \( A \), let \( J \) be a pointset. A remark that for reals (subsets \( A \)) a local determinacy hypothesis with our definition of the envelope. We follow Martin's argument as given in Jackson [3, Lem. 3.12] for arbitrary the least (in the canonical \( \Delta^1<\kappa \) reals and \( \alpha \)) as coding, in some simple way, a pair \( (\sigma, A) \). Let \( C \) denote the set of reals \( y \) such that for some for some pointset \( A \in Env(\Sigma^1_{\kappa}(\mathbb{R})) \) and a real \( z \), define
\[
\alpha(z, A) = \text{least } \alpha \text{ such that } (\forall x \leq_T z) (x \in A \iff x \in A_\alpha).
\]
Clearly \( \alpha(z, A) \) only depends on the Turing degree of \( z \). Let \( R \) denote the relation \(<, =, \text{ or } > \). For sets of reals \( A', B' \in \Delta^1<\kappa \) the set \( \{ z \in \mathbb{R} : \alpha(z, A') R \alpha(z, B') \} \) is also in \( \Delta^1<\kappa \), so using the characterization of the envelope in Lemma 2.6 it is easy to check that for sets of reals \( A, B \in Env(\Sigma^1_{\kappa}(\mathbb{R})) \) the set \( \{ z \in \mathbb{R} : \alpha(z, A) R \alpha(z, B) \} \) is also in \( Env(\Sigma^1_{\kappa}(\mathbb{R})) \). This set is Turing invariant and by Proposition 2.11 we have enough determinacy to use the proof of Martin's cone theorem to show that it contains a Turing cone for exactly one of the three cases \( R \in \{<, =, > \} \). Therefore we can define a wellordering of \( Env(\Sigma^1_{\kappa}(\mathbb{R})) \) by letting \( A \leq B \) if and only if \( \alpha(z, A) \leq \alpha(z, B) \) for a Turing cone of reals \( z \). This is clearly a prewellordering. If \( A \neq B \) then \( \alpha(z, A) \neq \alpha(z, B) \) on a cone, so it is in fact a wellordering. \( \square \)

Next we will show that the envelope is closed under real quantification. We follow Martin's argument as given in Jackson [3, Lem. 3.12] for arbitrary boldface inductive-like pointclasses under AD, checking that it works under a local determinacy hypothesis with our definition of the envelope. We remark that for reals (subsets \( A \)) of the product space \( \mathcal{X} = \omega \) the argument is essentially that of [13, Lem. 4.1].

In order to prove closure under real quantification, we first extract a lemma from the argument. To prove the lemma we seem to need AD in \( J_{\kappa}(\mathbb{R}) \) (equivalently, \( \Delta^1_{\kappa}(\mathbb{R}) \) determinacy) and not just \( \Delta^1_{\kappa}(\mathbb{R}) \) determinacy.

**Lemma 2.13.** Let \( J_{\kappa}(\mathbb{R}) \) be an admissible level of \( L(\mathbb{R}) \) that begins a gap and suppose that \( J_{\kappa}(\mathbb{R}) \) satisfies AD. For any product space \( \mathcal{X} \) and any pointset \( A \subset \mathcal{X} \), the following statements are equivalent.

1. \( A \in Env(\Sigma^1_{\kappa}(\mathbb{R})) \).
2. For every countable set \( \sigma \subset \mathcal{X} \) there is a cone of Turing degrees \( d \) such that for some for some pointset \( A' \in OD^{<\kappa}(d) \) we have \( A \cap \sigma = A' \cap \sigma \).

**Proof.** The forward direction is trivial using the characterization of the envelope in Lemma 2.6.

For the converse, assume that condition (2) holds. We consider a real \( y \) as coding, in some simple way, a pair \( (\sigma_y, a_y) \) where \( \sigma_y \) is a countable set of reals and \( a_y \subset \sigma_y \). Given a real \( y \) and a Turing degree \( d \), we let \( A_{y,d} \) denote the least (in the canonical \( \Delta^1_{\kappa}(d) \) well-ordering) pointset \( A' \in OD^{<\kappa}(d) \) such that \( a_y = A' \cap \sigma_y \), if it exists.

Let \( C \) denote the set of reals \( y \) such that for a cone of Turing degrees \( d \) the pointset \( A_{y,d} \) is defined. Condition (2) says precisely that \( y \in C \) for all reals \( y \) such that \( a_y = A \cap \sigma_y \). We have \( C \in \Sigma^1_{\kappa}(\mathbb{R}) \) by admissibility.
Given two reals \( y, z \in C \), take a Turing degree \( d_0 \) such that for all Turing degrees \( d \geq_T d_0 \) the pointsets \( A_{y,d} \) and \( A_{z,d} \) are both defined. By \( \Delta_1^{J_\kappa(\mathbb{R})}(y, z, d_0) \) determinacy, which follows from the assumption of \( \AD \) in \( J_\kappa(\mathbb{R}) \), either \( A_{y,d} < A_{z,d} \) for a cone of \( d \), or \( A_{y,d} = A_{z,d} \) for a cone of \( d \), or \( A_{y,d} > A_{z,d} \) for a cone of \( d \).

We can define a regular norm \( \varphi \) on \( C \) by \( \varphi(y) \leq \varphi(z) \) if \( A_{y,d} \leq A_{z,d} \) for a cone of Turing degrees \( d \). It is easy to verify that \( \varphi \) is a \( \Sigma_1^{J_\kappa(\mathbb{R})} \)-norm. Therefore \( \text{ran}(\varphi) \leq \kappa \) and a straightforward calculation using admissibility shows that (the graph of) the norm \( \varphi \) is \( \Delta_1^{J_\kappa(\mathbb{R})} \).

Now we can see that \( A \in \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})}) \). Let \( \sigma \subseteq \mathcal{X} \) be countable. Take \( y \in \mathbb{R} \) with \( \sigma_y = \sigma \) and \( a_y = A \cap \sigma_y \), so that \( y \in C \). Define the set \( A_y \subseteq \mathcal{X} \) by \( x \in A_y \iff x \in A_{y,d} \) for a cone of Turing degrees \( d \). (Note that we have sufficient Turing determinacy that either \( x \in A_{y,d} \) or \( x \notin A_{y,d} \) must hold for a cone of Turing degrees \( d \).)

The set \( A_y \) so defined does not depend on the particular choice of \( y \) but only on the ordinal \( \alpha = \varphi(y) \), so we can call it \( A_\alpha \). We have \( A_\alpha \cap \sigma = A \cap \sigma \) and the set \( A_\alpha \) is \( \Delta_1 \)-definable over \( J_\kappa(\mathbb{R}) \) from the parameter \( \alpha \), so \( A_\alpha \in \text{OD}^{<\kappa} \) as desired. \( \square \)

Now it is a simple matter to prove closure under real quantification.

**Proposition 2.14.** Let \( J_\kappa(\mathbb{R}) \) be an admissible level of \( L(\mathbb{R}) \) that begins a gap and suppose that \( J_\kappa(\mathbb{R}) \) satisfies \( \AD \). Then \( \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})}) \) is closed under real quantification.

**Proof.** We verify the closure under \( \exists \mathbb{R} \)—the case of \( \forall \mathbb{R} \) will then follow using closure under complementation. Fix a product space \( \mathcal{X} \) and consider a pointset \( A \subseteq \mathcal{X} \times \mathbb{R} \) with \( A \in \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})}) \). We define the projection \( B \subseteq \mathcal{X} \) by \( B = \exists \mathbb{R} A \). We will show that \( B \in \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})}) \) by verifying that it satisfies condition (2) of Lemma 2.13. Let \( \sigma \subseteq \mathcal{X} \) be a countable set.

Fixing any sufficiently large Turing degree \( d \), for all points \( x \in \sigma \) we have \( x \in B \iff \exists y \leq_T d ((x, y) \in A) \). Define the countable set \( \rho \subseteq \mathcal{X} \times \mathbb{R} \) by \( \rho = \sigma \times \{ y \in \mathbb{R} : y \leq_T d \} \). Take \( A' \in \text{OD}^{<\kappa} \) such that \( A \cap \rho = A' \cap \rho \). Defining the set \( B' \) by \( x \in B' \iff \exists y \leq_T d ((x, y) \in A') \), we have \( B' \in \text{OD}^{<\kappa}(d) \) and \( B \cap \sigma = B' \cap \sigma \) as desired. \( \square \)

Next we show that the envelope produces a “gap in scales.” As usual this result is well-known under \( \AD \) (see [3, Lem. 3.17 and remark following 3.18]) and we merely check that it works under our local determinacy hypothesis with our definition of the envelope. For the case where \( J_\kappa(\mathbb{R}) \) is the least admissible level of \( L(\mathbb{R}) \), so that \( \Sigma_1^{J_\kappa(\mathbb{R})} \) is the pointclass of inductive sets, this result is equivalent to Martin’s [13, Cor. 4.4].

**Lemma 2.15.** Let \( J_\kappa(\mathbb{R}) \) be an admissible level of \( L(\mathbb{R}) \) that begins a gap and suppose that \( J_\kappa(\mathbb{R}) \) satisfies \( \AD \). Then there is a \( \Pi_1^{J_\kappa(\mathbb{R})} \) pointset that admits no \( \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})}) \)-semiscale.
Proof. We claim that there is no $\text{Env}(\Sigma_1^J)$-semiscale on the $\Pi_1^J$ pointset $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \notin \text{OD}^\kappa(x)\}$.

First note that the relation $R$ is total: If there were a real $x$ such that every real was in $\text{OD}^\kappa(x)$ then we would get a $\Delta_1^J(x)$ well-ordering of the reals, which would be in $J^\kappa(\mathbb{R})$ by admissibility, contrary to our determinacy hypothesis.

Now suppose toward a contradiction that there is a $\text{Env}(\Sigma_1^J)$-semiscale on $R$. The usual construction of a uniformization from a semiscale (by taking leftmost branches of the tree of the semiscale) can be phrased in terms of the sequence of prewellorderings rather than in terms of the semiscale itself or its tree. The envelope is projectively closed, so it can carry out this construction and we get a uniformization of $R$ in $\text{Env}(\Sigma_1^J)$. But then using closure under $\exists R$ (or $\forall R$) we obtain a contradiction with Lemma 2.8. \qed

As we will see later, it is possible for a universal $\Pi_1^J$ set to have a semi-scale (and even a scale) each of whose prewellorderings is in $\text{Env}(\Sigma_1^J)$, even though the sequence of prewellorderings cannot be. Given such a semiscale, the sequence of prewellorderings itself must be Wadge-cofinal in the pointclass $\text{Env}(\Sigma_1^J)$:

**Proposition 2.16.** Let $J^\kappa(\mathbb{R})$ be an admissible level of $L(\mathbb{R})$ that begins a gap and suppose that $J^\kappa(\mathbb{R})$ satisfies AD. Let $\vec{\psi}$ be a semiscale on a universal $\Pi_1^J$ pointset such that for each $i < \omega$ the prewellordering $\leq_{\psi_i}$ is in $\text{Env}(\Sigma_1^J)$. Then every pointset in $\text{Env}(\Sigma_1^J)$ is continuously reducible to the prewellordering $\leq_{\psi_i}$ for some $i < \omega$.

Proof. If not, then by Wadge’s lemma there is a pointset in $\text{Env}(\Sigma_1^J)$ to which every prewellordering $\leq_{\psi_i}$ is continuously reducible, so the sequence of prewellorderings is itself in the envelope, contradicting Lemma 2.15. Here we use the determinacy of the envelope together with its basic closure properties to see that the relevant Wadge games are determined. \qed

### 3. Scales on co-inductive sets

The pointclass $\text{IND}$ of inductive sets is equal to $\Sigma_1^J$ where $J^\kappa(\mathbb{R})$ is the least admissible level of $L(\mathbb{R})$ (see [23, Cor. 2.3]) so the pointclass $\text{Env}(\text{IND})$ is defined by Definition 2.2. The boldface ambiguous part of IND is equal to $\text{HYP}$, the pointclass of boldface hyperprojective sets, and also to the class of pointsets in $J^\kappa(\mathbb{R})$. (For this section we may as well take these characterizations as the definitions of IND and $\text{HYP}$.)

If $\text{HYP}$ determinacy holds, then Moschovakis [17] showed that IND has the scale property: every inductive set has an inductive scale. The complexity of scales on co-inductive sets, on the other hand, can be measured in terms of the pointclass $\text{Env}(\text{IND})$. 
As in [13] we define $\Sigma^*_0$ to be the pointclass of sets $A \cup B$ where $A$ is inductive and $B$ is co-inductive. For $n < \omega$ let $\Pi^*_n$ be the class of complements of $\Sigma^*_n$ sets and let $\Sigma^*_{n+1}$ be the class of projections of $\Pi^*_n$ sets.

**Lemma 3.1.** Suppose that $\text{HYP}^\sim$ is determined. Then $\bigcup_{n<\omega} \Sigma^*_n \subset \text{Env}(\text{IND})$.

**Proof.** This follows from the fact that $\text{Env}(\text{IND})$ contains $\text{IND}$ and is closed under Boolean combinations and real quantification (Propositions 2.7 and 2.14.)

From this we easily obtain the following result.

**Theorem 3.2** (Martin [13, Lem. 4.1]). Suppose that $\text{HYP}^\sim$ is determined. Let $A \in \bigcup_{n<\omega} \Sigma^*_n$ be a set of integers (a “real.”) Then $A$ is an element of $C^\text{IND}$, the largest countable inductive set of reals.

**Proof.** If $A \in \bigcup_{n<\omega} \Sigma^*_n$, then $A \in \text{Env}(\text{IND})$. In other words $A \in \text{Env}(\Sigma^*_{J^\kappa(R)}(\kappa))$ where $J^\kappa(R)$ is the least admissible level of $L(R)$. Because $A \subset \omega$ this simply means that $A \in \text{OD}^{<\kappa}$. The class of $\text{OD}^{<\kappa}$ subsets of $\omega$ is $\Sigma^*_{J^\kappa(R)}$ and admits a $\Delta^*_{J^\kappa(R)}$ surjection from $\kappa$. The restriction of this $\kappa$-sequence to any ordinal $\alpha < \kappa$ has countable range by $\text{AD}$ in $J^\kappa(R)$, so the entire sequence has countable range by admissibility.

We also get a “determinacy transfer” theorem that is an instance of [10, Thm. 8.1].

**Theorem 3.3** (Kechris–Woodin). Suppose that $\text{HYP}^\sim$ is determined. Then the pointclass $\bigcup_{n<\omega} \Sigma^*_n$ is determined.

**Proof.** Relativizing the proof of Lemma 3.1 we have $\bigcup_{n<\omega} \Sigma^*_n \subset \text{Env}(\text{IND})$, so the conclusion follows by the determinacy of the envelope (Proposition 2.11.)

The determinacy of $\bigcup_{n<\omega} \Sigma^*_n$ is exactly the hypothesis used for Moschovakis’s construction ([18, Thm. 2.2]) of scales on co-inductive sets. We can use these scales to describe the extent of the envelope. The result is well-known under $\text{AD}$; we check that it holds under our partial determinacy hypothesis with our definition of the envelope.

**Proposition 3.4.** Suppose that $\text{HYP}^\sim$ is determined. Then we have $\text{Env}(\text{IND}) = \bigcup_{n<\omega} \Sigma^*_n$.

**Proof.** We have $\bigcup_{n<\omega} \Sigma^*_n \subset \text{Env}(\text{IND})$ by the relativization of Lemma 3.1, so by Moschovakis [18, Thm. 2.2] every co-inductive pointset admits a scale whose prewellorderings are all in $\bigcup_{n<\omega} \Sigma^*_n$, and therefore also in $\text{Env}(\text{IND})$. They must be Wadge-cofinal in $\text{Env}(\text{IND})$ by Proposition 2.16, so the desired equality holds.

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4See [8] for the notion of “largest countable $\Gamma$ set.”
4. Scales in $L(\mathbb{R})$

In this section we will see how the results of Section 3 can be generalized from IND to the other inductive-like pointclasses $\Sigma^J_\kappa(\mathbb{R})$ in $L(\mathbb{R})$. Assume that $\kappa$ begins a gap in $L(\mathbb{R})$.

If $J_\kappa(\mathbb{R})$ satisfies AD then by Steel [23, Thm. 2.1] every $\Sigma^J_1(\mathbb{R})$ set has a $\Sigma^J_1(\mathbb{R}) \cap \text{IND}$ to the other inductive-like pointclasses $\Sigma^J_\kappa(\mathbb{R})$. Assume that $\kappa$ begins a gap in $L(\mathbb{R})$.

If $J_\kappa(\mathbb{R})$ is not admissible (equivalently, if it does not satisfy $\Sigma_0$-collection) then by [23, Lem. 2.7] we have $\Pi^J_2(\mathbb{R}) = \exists^R \Sigma^J_1(\mathbb{R})$, $\Sigma^J_3(\mathbb{R}) = \exists^R \Pi^J_2(\mathbb{R})$ and so on, and if these pointclasses are determined then they all have the scale property by the second periodicity theorem of Moschovakis [19, Thm. 6C.3].

So we proceed to consider pointclasses of the form $\Sigma^J_1(\mathbb{R})$ where $J_\kappa(\mathbb{R})$ is an admissible level of $L(\mathbb{R})$. Such pointclasses are inductive-like. In this case the complexity of scales on $\Pi^J_1(\mathbb{R})$ sets can be measured in terms of $\text{Env}(\Sigma^J_1(\mathbb{R}))$.

We will need a bit more of the fine structure of $L(\mathbb{R})$ as found, for example, in [23, §1]. For an ordinal $\beta$ we take the expression $\rho^J_\beta(\mathbb{R}) = \mathbb{R}$, which is read as “the $n$th projection of $J_\beta(\mathbb{R})$ is $\mathbb{R}$,” to mean that there is a $\Sigma^J_n(\mathbb{R})$ set of reals that is not in $J_\beta(\mathbb{R})$. This is equivalent to the existence of a $\Sigma^J_n(\mathbb{R})$ partial surjection $\mathbb{R} \rightarrow J_\beta(\mathbb{R})$. We remark that arbitrary parameters from $J_\beta(\mathbb{R})$, not only reals, are allowed in the definitions of $\Sigma^J_n(\mathbb{R})$ pointsets.

Under certain circumstances the following lemma allows us to replace quantification over $J_\beta(\mathbb{R})$ with quantification over $\mathbb{R}$. The argument is standard but we include it here for the reader’s convenience.

**Lemma 4.1.** Let $\beta$ be an ordinal and let $n$ be an integer such that $\rho^J_\beta(\mathbb{R}) = \mathbb{R}$. Then for every sufficiently large finite set of ordinals $s \in [\omega^\beta]^{<\omega}$ we have the following equalities of pointclasses:

$$\begin{align*}
\Sigma^J_{n+1}(\mathbb{R})(s) &= \exists^R (\Sigma^J_n(\mathbb{R})(s) \land \Pi^J_n(\mathbb{R})(s)) \\
\Sigma^J_{k+1}(\mathbb{R})(s) &= \exists^R \Pi^J_k(\mathbb{R})(s), \quad \text{for all } k > n.
\end{align*}$$

**Proof.** The right-to-left inclusions are trivial, so it remains to show the left-to-right inclusions. Because $\rho^J_\beta(\mathbb{R}) = \mathbb{R}$, there is a $\Sigma^J_n(\mathbb{R})$ partial surjection $\mathbb{R} \rightarrow J_\beta(\mathbb{R})$. There is always a $\Sigma^J_1(\mathbb{R})$ surjection $[\omega^\beta]^{<\omega} \times \mathbb{R} \rightarrow J_\beta(\mathbb{R})$, so we may take the parameters defining our partial surjection $\mathbb{R} \rightarrow J_\beta(\mathbb{R})$ to be reals and ordinals. Let $s \in [\omega^\beta]^{<\omega}$ be a finite set of ordinals that is sufficiently large that there is a $\Sigma^J_n(\mathbb{R})$ partial surjection $\mathbb{R} \rightarrow J_\beta(\mathbb{R})$ that is definable from $s$ and a real $z$, say by $\theta(-, -, s, z)$ where $\theta$ is a $\Sigma_n$ formula. Then for any formula $\phi$ we have $\exists X \in J_\beta(\mathbb{R}) \phi(X, v)$ if and only if there are

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5By $\Sigma \land \Pi$ we denote the pointclass consisting of sets of the form $A \cap B$ with $A \in \Sigma$ and $B \in \Pi$. 

Letting \( R \in \mathbb{R} \) such that
\[
\exists X \in J_\beta(\mathbb{R}) \theta(x, X, s, z) \& \\
\forall X \in J_\beta(\mathbb{R}) (\theta(x, X, s, z) \implies \phi(X, v)).
\]

Letting \( \phi \) be \( \Pi_n \) or \( \Pi_k \) with \( k > n \), we obtain the desired result. \( \square \)

Assuming now that \( J_\kappa(\mathbb{R}) \) satisfies AD and is admissible and \( \kappa \) begins a gap, we would like to describe \( \text{Env}(\Sigma^1_n(\mathbb{R})) \) in terms of the Jensen hierarchy. To do this we need to use a reflection property introduced by Steel in \([23]\).

**Definition 4.2.** The gap \([\kappa, \beta]\) in \( L(\mathbb{R}) \) is **strong** if \( J_\kappa(\mathbb{R}) \) is admissible, and letting \( n < \omega \) be least such that \( \rho_n(J_\beta(\mathbb{R})) = \mathbb{R} \), every \( \Sigma_\alpha \)-type realized in \( J_\beta(\mathbb{R}) \) is realized in \( J_\alpha(\mathbb{R}) \) for some \( \alpha < \beta \) (and therefore for some \( \alpha < \kappa \) by the definition of a gap.) Otherwise we say the gap is **weak**.

The first example of a strong gap is \([\kappa, \kappa] \) where \( J_\kappa(\mathbb{R}) \) is the first admissible level of \( L(\mathbb{R}) \); as mentioned previously, the pointclass \( \Sigma^1_1(\mathbb{R}) \) is equal to IND in this case. We allow \( \beta = 6^{L(\mathbb{R})} \) in the definition of a weak gap \([\kappa, \beta]\).

**Lemma 4.3.** Suppose that \( J_\kappa(\mathbb{R}) \) is an admissible level of \( L(\mathbb{R}) \) satisfying AD and \([\kappa, \beta^*] \) is a gap. Let \( \beta = \beta^* \) if the gap is weak and \( \beta = \beta^* + 1 \) if the gap is strong. Then \( \mathrm{OD}^{<\beta} \subset \text{Env}(\Sigma^1_n(\mathbb{R})) \).

**Proof.** Fix a product space \( X' \).

If the gap \([\kappa, \beta^*] \) is weak, let \( A \subset X' \) be a pointset with \( A \in \mathrm{OD}^{<\beta} \) and let \( \sigma \subset X' \) be a countable set. The property of being ordinal-definable over a level of the Jensen hierarchy is \( \Sigma_1 \), so applying the definition of a gap to some real parameter coding the pair \((\sigma, A \cap \sigma)\) we get a pointset \( A' \subset X' \) with \( A \in \mathrm{OD}^{<\kappa} \) and \( A' \cap \sigma = A \cap \sigma \). Therefore \( A \in \text{Env}(\Sigma^1_n(\mathbb{R})) \) by the characterization of the envelope in Lemma 2.6.

If the gap \([\kappa, \beta^*] \) is strong, so that \( \mathrm{OD}^{<\beta} = \mathrm{OD}^{\beta^*} \), let \( n \) be the least integer such that \( \rho_n(J_{\beta^*}(\mathbb{R})) = \mathbb{R} \). Then by Lemma 4.1 the \( \mathrm{OD}^{\beta^*} \) pointsets are generated via Boolean combinations and real quantification from pointsets that are \( \Sigma^1_n(J_{\beta^*}(\mathbb{R})) \) in ordinal parameters. (This is a generalization of the way the pointclasses \( \Sigma^i_n \) for \( i < \omega \) are generated from IND.) The envelope is closed under Boolean combinations and real quantification, so it is enough to show that it contains the pointset \( A \subset X' \) given by

\[
x \in A \iff J_{\beta^*}(\mathbb{R}) \models \theta[x, s]
\]

where \( s \in [\omega^\beta]^\omega \) and \( \theta \) is a \( \Sigma_n \) formula.

Given a countable set \( \sigma \subset X' \); take a real \( z \) coding \( \sigma \). By the definition of a strong gap the \( \Sigma^i_n \)-type of \((z, s)\) in \( J_{\beta^*}(\mathbb{R}) \) is realized in some level \( J_\alpha(\mathbb{R}) \) with \( \alpha < \kappa \). The real \( z \) is determined by its type, so there is a finite sequence of ordinals \( \bar{s} \in [\omega \alpha]^\omega \) such that the \( \Sigma^i_n \)-type of \((z, \bar{s})\) in \( J_\alpha(\mathbb{R}) \) is equal to the \( \Sigma^i_n \)-type of \((z, s)\) in \( J_{\beta^*}(\mathbb{R}) \). Then we have \( A \cap \sigma = A' \cap \sigma \) where \( A' \in \mathrm{OD}^{<\kappa} \).
is the subset of $\mathcal{X}$ defined by $x \in A' \iff J_\alpha(\mathbb{R}) \models \theta[x, \bar{s}]$. This shows that $A \in \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})})$.

We obtain an immediate corollary regarding local ordinal-definability of reals. (The weak gap case is trivial because the property “the set of integers $A$ is ordinal-definable over one of my levels” is $\Sigma_1$, but we include it for simplicity.)

**Theorem 4.4** (Martin). Suppose that $J_\kappa(\mathbb{R})$ is an admissible level of $L(\mathbb{R})$ satisfying $\text{AD}$ and $[\kappa, \beta^*]$ is a gap. Let $\beta = \beta^*$ if the gap is weak and $\beta = \beta^* + 1$ if the gap is strong. Let $A \in \text{OD}^{<\beta}$ be a set of integers (a “real.”) Then $A \in \text{OD}^{<\kappa}$.

**Proof.** By Lemma 4.3 we have $A \in \text{Env}(\Sigma_1^{\tilde{J}_\kappa(\mathbb{R})})$. Because the space $\omega$ itself is a countable pointset this means $A \in \text{OD}^{<\kappa}$ by the characterization of the envelope in Lemma 2.6.

For every ordinal $\beta$, every pointset $A \in J_\beta(\mathbb{R})$ is definable from parameters over some level $J_\alpha(\mathbb{R})$ with $\alpha < \beta$, and using a definable surjection $[\omega]^{<\omega} \times \mathbb{R} \to J_\alpha(\mathbb{R})$ we see that $A \in \text{OD}^{<\beta}(x)$ for some real $x$. So by relativizing Lemma 4.3 to arbitrary reals we immediately obtain the following determinacy transfer result. (The weak gap case is trivial because $\text{AD}$ can be expressed as a $\Pi_1$ statement, but we include it for simplicity.)

**Theorem 4.5** (Kechris–Woodin [10]). Suppose that $J_\kappa(\mathbb{R})$ is an admissible level of $L(\mathbb{R})$ satisfying $\text{AD}$ and $[\kappa, \beta^*]$ is a gap. Let $\beta = \beta^*$ if the gap is weak and $\beta = \beta^* + 1$ if the gap is strong. Then $J_\beta(\mathbb{R})$ satisfies $\text{AD}$.

**Proof.** Given a set of reals $A \in J_\beta(\mathbb{R})$, we have $A \in \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})})$ by relativizing Lemma 4.3, so $A$ is determined by Proposition 2.11.

This determinacy transfer allows us to use Steel’s construction of scales in $L(\mathbb{R})$ in [23] to construct scales on $\Pi_1^{J_\kappa(\mathbb{R})}$ sets. We can then determine the envelope of the pointclass $\Sigma_1^{J_\kappa(\mathbb{R})}$ exactly. The proof from $\text{AD}$ is well-known and we merely check that it can be adapted to yield a proof from our local determinacy hypothesis using our definition of the envelope.

**Proposition 4.6.** Suppose that $J_\kappa(\mathbb{R})$ is an admissible level of $L(\mathbb{R})$ satisfying $\text{AD}$ and $[\kappa, \beta^*]$ is a gap with $\beta^* < \Theta^{L(\mathbb{R})}$. Let $\beta = \beta^*$ if the gap is weak and $\beta = \beta^* + 1$ if the gap is strong. Then $\text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})}) = J_\beta(\mathbb{R}) \cap \wp(\mathbb{R})$.

**Proof.** We have $J_\beta(\mathbb{R}) \cap \wp(\mathbb{R}) \subset \text{Env}(\Sigma_1^{J_\kappa(\mathbb{R})})$ by relativizing Lemma 4.3. It remains to show the reverse inclusion. Because $J_\beta(\mathbb{R}) \models \text{AD}$, it follows from the scale construction in [23] (as noted in [21]) that every set of reals in

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6This result is the natural adaptation of Martin [13, Lem. 4.1] to Steel’s notion of a strong gap. Steel [21, Thm. 1.14] attributes it to Martin.
$J_\beta(\mathbb{R})$ has a scale whose prewellorderings are all in $J_\beta(\mathbb{R})$.\footnote{It is also possible to carry out the argument using results stated explicitly in [23]. To do this, let $n$ be least such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$ (which exists because $\beta < \Theta^L(\mathbb{R})$ and is equal to 1 in the strong gap case) and use the scale property for the pointclass $\Sigma_1^{J_\beta(\mathbb{R})}$ together with the fact that this pointclass is the collection of countable unions of sets in $J_\beta(\mathbb{R})$.} Note that $\kappa < \beta$ (because every admissible gap of the form $[\kappa, \kappa]$ is strong.) So in particular, a universal $\Pi_1^\Sigma_1^{J_\beta(\mathbb{R})}$ set has a scale $\vec{\psi}$ such that each prewellordering $\leq \psi_i$ is in $J_\beta(\mathbb{R})$ and hence also in $\text{Env}(\Sigma_1^{J_\beta(\mathbb{R})})$. By Proposition 2.16 the sequence of prewellorderings is Wadge-cofinal in the envelope, so $\text{Env}(\Sigma_1^{J_\beta(\mathbb{R})}) \subset J_\beta(\mathbb{R})$ as desired. \hfill \Box

5. Companions

In order to generalize the definitions and results of Section 2 to arbitrary boldface inductive-like pointclasses it is helpful, though perhaps not strictly necessary, to use the notion of the companion of a pointclass introduced by Moschovakis in [16].

Recall that a structure $\mathcal{M} = (M; \in, \vec{R})$, where $\vec{R} = (R_1, \ldots, R_n)$ is a finite sequence of relations on $M$, is called admissible if the set $M$ is transitive, nonempty, closed under pairing and union, and the structure $\mathcal{M}$ satisfies the $\Delta_0$-separation and $\Delta_0$-collection axiom schemas (for formulas mentioning $R_1, \ldots, R_n$). Note that an admissible structure must in fact satisfy the ostensibly stronger axiom schemas of $\Delta_1$-separation and $\Sigma_1$-collection.

Definition 5.1 (Moschovakis [16, §9]). A companion of a boldface inductive-like pointclass $\Gamma$ is an admissible structure $\mathcal{M} = (M; \in, \vec{R})$, with $\vec{R} \in M$, satisfying the two conditions

Projectability: there is a $\Delta_1^\text{ad}$ partial surjection $\mathbb{R} \to M$\footnote{A Spector class on $\mathbb{R}$ in the terminology of [16].}

Resolvability: there is a $\Delta_1^\text{ad}$ sequence $(M_\alpha : \alpha < \text{Ord}^M)$ of subsets of $M$ whose union is equal to $M$

and such that $\Gamma$ is the pointclass of all $\Sigma_1^\text{ad}$ pointsets.

In our notions of $\Sigma_1^\text{ad}$ (and $\Delta_1^\text{ad}$) here we mean that any set in $\mathcal{M}$ is allowed as a parameter. Companions exist:

Theorem 5.2 (Moschovakis [16, Thm. 9E.1]). Let $\Gamma$ be a boldface inductive-like pointclass. Then $\Gamma$ has a companion.

For any companion $\mathcal{M} = (M; \in, \vec{R})$ of the pointclass $\Gamma$, the underlying set $M$ consists of all sets that can be coded by well-founded trees in $\Delta_\Gamma$. The pointsets in the set $M$ are exactly the $\Delta\Gamma$ pointsets and $\text{Ord} \cap M$ is equal to the prewellordering ordinal $\kappa$ of $\Gamma$. The set $M$ can also be characterized as the smallest admissible set such that $\Delta_\Gamma \subset M$.

\footnote{By this we mean that the graph of the partial surjection is $\Delta_1^\text{ad}$, which implies that the domain is $\Sigma_1^\text{ad}$.}
Remark 5.3. A companion of the pointclass $\Sigma^J_1(\mathbb{R})$, where $J_\kappa(\mathbb{R})$ is admissible and $\kappa$ begins a gap in $L(\mathbb{R})$, is given by $(J_\kappa(\mathbb{R}); \in, 0)$. Resolvability follows from the fact that the sequence of levels $(J_\kappa(\mathbb{R}) : \alpha < \kappa)$ is $\Delta^J_1(\mathbb{R})$. For projectability note that the fact that $\kappa$ begins a gap implies that there is a $\Sigma^J_1(\mathbb{R})$ partial surjection $F : \mathbb{R} \to J_\kappa(\mathbb{R})$ (see [23, Lem. 1.11]). Then we can define a $\Delta^J_1(\mathbb{R})$ partial surjection $G$ such that $G(x) = a$ if and only if the least level of $J_\kappa(\mathbb{R})$ satisfying the $\Sigma_1$ fact $a \in \text{ran}(F)$ also satisfies the $\Sigma_1$ fact $F(x) = a$.

Remark 5.4. Assume that $\text{AD}^+$ holds, there is a largest Suslin cardinal $\kappa$, and $\kappa$ is a member of the Solovay sequence ($\theta_\alpha : \alpha < \Omega$). Define the pointclass $\Gamma = S(\kappa)$ and let $\mathcal{M} = (M; \in, R)$ be a companion of $\Gamma$. Then although the underlying set $M$ of the companion is definable, the companion structure $\mathcal{M}$ itself is not definable, nor even ordinal-definable from any element of $M$. This example shows that the predicates in $\tilde{R}$ may contribute essential information to the companion structure, in contrast to the case $\Gamma = \Sigma^J_1(\mathbb{R})$.

When we are only interested in boldface $\Sigma_1$ (and $\Delta_1$) definability over a companion, we may speak loosely of “the” companion of $\Gamma$ because of the following theorem.

**Theorem 5.5** (Moschovakis [16, Thm. 9E.3]). Let $\Gamma$ be a boldface inductive-like pointclass. If $\mathcal{M} = (M; \in, \tilde{R})$ and $\mathcal{M}' = (M'; \in, \tilde{R}')$ are companions of $\Gamma$, then $M = M'$ and moreover the $\Sigma_1^\mathcal{M}$ subsets of $M$ are exactly the $\Sigma_1^\mathcal{M}'$ subsets of $M'$.

By contrast, the notions of lightface $\Sigma_1^\mathcal{M}$ and $\Delta_1^\mathcal{M}$ definability may depend on the choice of companion $\mathcal{M}$ for $\Gamma$.

Next we define a type of companion for which projectability and resolvability are witnessed by distinguished predicates. This will give us well-behaved notions of lightface definability over the companion. We will also require the resolution sequence to have some nice properties that facilitate an analogy with the sequence of levels $(J_\alpha(\mathbb{R}) : \alpha < \kappa)$ of $J_\kappa(\mathbb{R})$.

**Definition 5.6.** Let $\Gamma$ be a boldface inductive-like pointclass and let $\kappa$ be the prewellordering ordinal of $\Gamma$. We say that a companion $\mathcal{M} = (M; \in, R_0, R_1, \ldots)$ of $\Gamma$ is good if the predicate $R_0$ is a partial surjection $\mathbb{R} \to M$ and the predicate $R_1$ is a sequence $(M_\alpha : \alpha < \kappa)$ satisfying $\bigcup_{\alpha < \kappa} M_\alpha = M$ along with the following additional properties.

- $M_\alpha$ is transitive for all $\alpha < \kappa$,
- $\mathbb{R} \subseteq M_0$, and
- $M_\alpha \subseteq M_\beta$ whenever $\alpha < \beta < \kappa$.

Given a good companion $\mathcal{M}$ of $\Gamma$ and an ordinal $\alpha < \kappa$, the $\alpha^{th}$ level of $\mathcal{M}$ is the structure $\mathcal{M}_\alpha = (M_\alpha; \in, R_0 \cap M_\alpha, R_1 \cap M_\alpha, \ldots)$ where $M_\alpha$ is the $\alpha^{th}$ set in the distinguished resolution sequence $R_1$ of $\mathcal{M}$. 
By definition every companion has a $\Delta^\mathbb{R}_1$ partial surjection $\mathbb{R} \to M$ and a $\Delta^\mathbb{R}_1$ resolution sequence $(M_\alpha : \alpha < \kappa)$. Moreover, the sets $M_\alpha$ in this sequence can be enlarged to get a $\Delta^\mathbb{R}_1$ resolution sequence $(M'_\alpha : \alpha < \kappa)$ satisfying the “good” conditions above. Then we can obtain a good companion by expanding the companion by predicates for this partial surjection and resolution sequence. Note that adding $\Delta^\mathbb{R}_1$ relations as predicates to $M$ preserves admissibility (we have $\Delta_0$-separation and $\Delta_0$-collection with respect to the new predicates) and does not change the notion of $\Sigma^\mathbb{R}_1$-definability over $M$.

Note that if $M$ is a good companion for $\Gamma$ then using the distinguished partial surjection $\mathbb{R} \to M$ we see that every $\Gamma$ set (equivalently, every $\Sigma^\mathbb{R}_1$ set) is $\Sigma^\mathbb{R}_1(x)$ for some real $x$.

6. The envelope in the general case

In this section we generalize the definitions and results of Section 4 from boldface inductive-like pointclasses $\Sigma^J_\kappa(R)$ to arbitrary boldface inductive-like pointclasses $\Gamma$. Because $J_\kappa(R)$ is a companion of $\Sigma^J_\kappa(R)$, a natural generalization of Definition 2.2 is the following one.

**Definition 6.1.** Let $\Gamma$ be a boldface inductive-like pointclass and let $\kappa$ be the prewellordering ordinal of $\Gamma$. Let $M$ denote the companion of $\Gamma$. For a pointset $A$, we say $A \in \text{Env}(\Gamma)$ if $A \in (A_\alpha : \alpha < \kappa)$ for some $\Delta^\mathbb{R}_1$ sequence of pointsets ($A_\alpha : \alpha < \kappa$).

Recall that the pointsets in $M$ are exactly the $\Delta_\Gamma$ pointsets. Moreover, the definability condition ($A_\alpha : \alpha < \kappa$) is equivalent to the condition that the sequences ($A_\alpha : \alpha < \kappa$) and ($\neg A_\alpha : \alpha < \kappa$) are both $\Gamma$ in the codes relative to some/every $\Gamma$-norm onto $\kappa$. If AD holds then every sequence of $\Delta_\Gamma$ sets satisfies this definability condition by Moschovakis’s Coding Lemma. Therefore, under the assumption of AD the pointclass $\text{Env}(\Gamma)$ is equal to Martin’s pointclass $\Lambda(\Delta^\mathbb{R}_1, \kappa)$ (see Definition 1.4.)

**Remark 6.2.** We caution the reader that the notation “Env” and the term “envelope” are taken from Steel [22], where they were given a somewhat different definition. Jackson [4] proved that, in the intended context of AD, the pointclass $\Lambda(\Delta^\mathbb{R}_1, \kappa)$ is equal to the pointclass that Steel calls $\text{Env}(\Gamma)$ (which is a boldface pointclass defined from a lightface inductive-like pointclass $\Gamma$. Consequently our pointclass $\text{Env}(\Gamma)$ as defined in Definition 6.1 above is equal to both other pointclasses under AD. The author does not

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10 Let $U$ be a universal $\Gamma$ set and use the coding lemma ([19, 7D.5]) to get a $\Gamma$ choice set for the function $\alpha \mapsto \{(x, y) \in \mathbb{R} \times \mathbb{R} : A_\alpha = U_x = \neg U_y \}$.  
11 The pointclass $\Lambda(\Delta^\mathbb{R}_1, \kappa)$ is also simply equal to $\Lambda(\Gamma, \kappa)$—see [3]. However, we will not need to use this fact directly.
know whether our pointclass Env(Γ) can be proved equal to Steel’s pointclass Env(Γ) from only $\text{ZF} + \text{DC}_R + \text{Det}(\Delta_1^R)$.

Next we define some notions of ordinal-definability over a companion structure $\mathcal{M}$ that are analogous to the notions $\text{OD}^\alpha$ and $\text{OD}^{<\kappa}$ from Definition 2.3.

**Definition 6.3.** Let $\Gamma$ be a boldface inductive-like pointclass and let $\kappa$ be the prewellordering ordinal of $\Gamma$. Let $\mathcal{M}$ be a good companion of $\Gamma$. Let $A$ be a pointset, and let $p$ be a parameter in $\mathcal{M}$. For an ordinal $\alpha < \kappa$, we say

- $A \in \text{OD}^{\alpha,\mathcal{M}}(p)$ if $A$ is first-order definable over the level $\mathcal{M}_\alpha$ of $\mathcal{M}$ from $p$ and ordinal parameters.
- $A \in \text{OD}^{<\kappa,\mathcal{M}}(p)$ if $A \in \text{OD}^{\alpha,\mathcal{M}}(p)$ for some $\alpha < \kappa$.

The definitions of $\text{OD}^{\alpha,\mathcal{M}}$ and $\text{OD}^{<\kappa,\mathcal{M}}$ depend on the particular choice of good companion $\mathcal{M}$ for $\Gamma$, so they are mostly useful for lemmas where we temporarily fix a choice of good companion to do some complexity calculation. The following characterization of $\text{OD}^{<\kappa,\mathcal{M}}$ will also be useful.

**Lemma 6.4.** Let $\Gamma$ be a boldface inductive-like pointclass and let $\kappa$ be the prewellordering ordinal of $\Gamma$. Let $\mathcal{M}$ be a good companion of $\Gamma$. Let $A$ be a pointset. Then the following statements are equivalent:

- $A \in \text{OD}^{<\kappa,\mathcal{M}}$.
- $A \in \Delta^\mathcal{M}_1(\alpha)$ for some finite sequence of ordinal parameters $s \in \kappa^{<\omega}$.

**Proof.** See Lemma 2.4. $\Box$

The following result (analogous to Lemma 2.5) now follows easily.

**Lemma 6.5.** Let $\Gamma$ be a boldface inductive-like pointclass and let $\mathcal{M}$ be a good companion of $\Gamma$. Then there is a $\Delta^\mathcal{M}_1$ sequence enumerating all $\text{OD}^{<\kappa,\mathcal{M}}$ pointsets.

The following characterization of the envelope (a generalization of Lemma 2.6) is immediate from the relativizations of Lemmas 6.4 and 6.5 to parameters $x \in \mathbb{R}$.

**Lemma 6.6.** Let $\Gamma$ be a boldface inductive-like pointclass and let $\kappa$ be the prewellordering ordinal of $\Gamma$. Let $\mathcal{M}$ be a good companion of $\Gamma$. Let $\mathcal{X}$ be a product space. For every pointset $A \subset \mathcal{X}$ the following statements are equivalent.

- $A \in \text{Env}(\Gamma)$.
- There is a real $x$ such that for every countable set $\sigma \subset \mathcal{X}$ there is a pointset $A' \in \text{OD}^{<\kappa,\mathcal{M}}(x)$ with $A \cap \sigma = A' \cap \sigma$.

We may establish properties of $\text{Env}(\Gamma)$ along the lines of Section 2, using $\Gamma$, $\mathcal{M}$, and $\text{OD}^{<\kappa,\mathcal{M}}$ in place of $\Sigma^1_{\alpha}(\mathbb{R})$, $J_\alpha(\mathbb{R})$, and $\text{OD}^{<\kappa}$ respectively. In most cases, we omit the proofs because they are completely analogous.
Proposition 6.7. Let $\Gamma$ be an inductive-like pointclass. Then $\text{Env}(\Gamma)$ contains $\Gamma$ and is closed under Boolean combinations, number quantification, and continuous reducibility.

Proof. See Proposition 2.7. \hfill \Box

Lemma 6.8. Let $\Gamma$ be an inductive-like pointclass and let $\kappa$ be the prewell-ordering ordinal of $\Gamma$. Let $\mathcal{M}$ be a good companion of $\Gamma$. If $A \subset \mathbb{R} \times \omega$ is a pointset whose sections satisfy $A_x \notin \text{OD}^{<\kappa}_{\sharp}(x)$ for all reals $x$, then $A \notin \text{Env}(\Gamma)$.

Proof. See Lemma 2.8. \hfill \Box

Next we establish some properties of $\text{Env}(\Gamma)$ that follow from the assumption of $\Delta_\Gamma$ determinacy.

Lemma 6.9. Let $\Gamma$ be a boldface inductive-like pointclass and suppose that $\Delta_\Gamma$ is determined. Let $\mathcal{M}$ be a good companion of $\Gamma$. Then there is a real $t \in \mathbb{R}$ such that every $\text{OD}^{<\kappa}_{\sharp}$ set of reals is determined on every countable Turing ideal containing $t$.

Proof. See Lemma 2.10. \hfill \Box

Proposition 6.10. Let $\Gamma$ be a boldface inductive-like pointclass and suppose that $\Delta_\Gamma$ is determined. Then $\text{Env}(\Gamma)$ is determined.

Proof. By the relativization of Lemma 6.9, using the argument of Proposition 2.11. \hfill \Box

Proposition 6.11. Let $\Gamma$ be a boldface inductive-like pointclass and suppose that $\Delta_\Gamma$ is determined. Then every pointset in $\text{Env}(\Gamma)$ is ordinal-definable from a pointset in $\Gamma$.

Proof. Let $\mathcal{M}$ be a good companion of $\Gamma$. Let $A \in \text{Env}(\Gamma)$, say $A \in \langle A_\alpha : \alpha < \kappa \rangle$ where the sequence of pointsets $(A_\alpha : \alpha < \kappa)$ is $\Delta_1^{\mathcal{M}}(x)$. Then the argument of Proposition 2.12 shows that $A$ is ordinal definable from $\mathcal{M}$ and $x$. Note that the companion $\mathcal{M}$ is definable from a $\Gamma$ pointset by projectability. \hfill \Box

To show that the envelope is closed under real quantification, we will need the following lemma.

Lemma 6.12. Let $\Gamma$ be a boldface inductive-like pointclass and suppose that $\Delta_\Gamma$ is determined. Let $\mathcal{M}$ be a good companion of $\Gamma$. Let $\mathcal{X}$ be a product space. For every pointset $A \subset \mathcal{X}$ the following statements are equivalent.

- $A \in \text{Env}(\Gamma)$.
- There is a real $x$ such that for every countable set $\sigma \subset \mathcal{X}$ there is a cone of Turing degrees $d$ such that for some pointset $A' \in \text{OD}^{<\kappa}_{\sharp}(x, d)$ we have $A \cap \sigma = A' \cap \sigma$.

Proof. See Lemma 2.13. \hfill \Box
Proposition 6.13. Let $\Gamma$ be a boldface inductive-like pointclass. If $\Delta_\Gamma$ is determined, then $\text{Env}(\Gamma)$ is closed under real quantification.


Next we show that the envelope produces a “gap in scales.” As usual this result is well-known under $\text{AD}$ and we merely check that it works under our local determinacy hypothesis with our definition of the envelope.

Lemma 6.14. Let $\Gamma$ be a boldface inductive-like pointclass. If $\Delta_\Gamma$ is determined, then there is a $\tilde{\Gamma}$ pointset that admits no $\text{Env}(\Gamma)$-semiscale.

Proof. Fixing a good companion $\mathcal{M}$ of $\Gamma$ the relation $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \notin \text{OD}^{<\kappa-\mathcal{M}}(x)\}$ is $\Pi^1_1$, hence $\tilde{\Gamma}$, and admits no $\text{Env}(\Gamma)$-semiscale by Lemma 6.8 and the argument of Lemma 2.15. □

Given such a semiscale, the sequence of prewellorderings must be Wadge-cofinal in the pointclass $\text{Env}(\Gamma)$:

Proposition 6.15. Let $\Gamma$ be an inductive-like pointclass and suppose that $\Delta_\Gamma$ is determined. Let $\tilde{\psi}$ be a semiscale on a universal $\tilde{\Gamma}$ pointset such that for each $i < \omega$ the prewellordering $\leq_{\psi_i}$ is in $\text{Env}(\Gamma)$. Then every pointset in $\text{Env}(\Gamma)$ is continuously reducible to the prewellordering $\leq_{\psi_i}$ for some $i < \omega$.

Proof. By Lemma 6.14, using the argument of Proposition 2.16. □

7. Putative semiscales on $\tilde{\Gamma}$ sets

Let $\Gamma$ be a boldface inductive-like pointclass and assume that $\Delta_\Gamma$ is determined. In many important cases this determinacy hypothesis implies that $\Gamma$ has the scale property; for example, in the case $\Gamma = \text{IND} \ (\text{Moschovakis}[17])$ and in the more general case $\Gamma = \Sigma^J_{1}(\mathbb{R})$ where $J_\kappa(\mathbb{R})$ is admissible and $\kappa$ begins a gap in $L(\mathbb{R})$ (Steel [23]). Henceforth we assume that $\Gamma$ has the scale property.

In some cases, as discussed in Sections 3 and 4, we also get scales on $\tilde{\Gamma}$ sets whose prewellorderings are Wadge-cofinal in $\text{Env}(\Gamma)$. However in other cases we cannot. For example if $V = L(\mathbb{R})$ and $\Gamma = \Sigma^L_{1}(\mathbb{R})$ then $\tilde{\Gamma}$ sets admit no scales at all, as observed by Kechris and Solovay (see [14]). More generally if $\text{AD}^+$ holds and there is a largest Suslin cardinal $\kappa$, then the pointclass $\Gamma = S(\kappa)$ is boldface inductive-like and $\tilde{\Gamma}$ sets do not admit scales.

Remark 7.1. Although the existence of a scale on the set of reals $A$ is equivalent to the existence of a semiscale on $A$, this equivalence is not local: Given a semiscale on a $\tilde{\Gamma}$ set $A$ whose norms are $\text{Env}(\Gamma)$-norms, the scale obtained from leftmost branches of the tree of the semiscale is not known in general to have the property that its norms are $\text{Env}(\Gamma)$-norms (see [3, Rmk. 3.16].) In this section we focus on semiscales because they will suffice for the following
applications. However we note that a similar but more involved construction, involving ideas from Jackson [5], would give us scales instead (see [25, §4.1].)

The following lemma will be used to verify that certain prewellorderings we construct on $\Gamma$ sets are in Env($\Gamma$).

**Lemma 7.2.** Let $\Gamma$ be a boldface inductive-like pointclass and let $\kappa$ be the prewellordering ordinal of $\Gamma$. Let $\varphi$ be a $\Gamma$-scale on a $\Gamma$ set $U$ and let $T$ be the tree of $\varphi$, so $T$ is a tree on $\omega \times \kappa$. Let $\leq$ be a prewellordering of the $\Gamma$ set $W = \mathbb{R} \setminus U$. Suppose that for every countable set of reals $\sigma$ there is a nonempty finite sequence of ordinals $s \in \kappa^{<\omega} \setminus \{\emptyset\}$ such that
\[ \forall x, y \in W \cap \sigma \ (x \leq y \iff \text{rank}_{T_x}(s) \leq \text{rank}_{T_y}(s)) \].\footnote{Here we say that the rank function takes the value zero for nodes not in the tree.}

Then the prewellordering $\leq$ is in Env($\Gamma$).

**Proof.** Let $\mathcal{M}$ be a good companion of $\Gamma$. Take a real parameter $t$ such that $U \in \Sigma^\mathcal{M}_1(t)$ and $\varphi$ is a $\Sigma^\mathcal{M}_1(t)$-scale. Let $\sigma$ be a countable set of reals. To show that the prewellordering $\leq$ is in Env($\Gamma$) using the characterization of the envelope given in Lemma 6.6, it suffices to find an OD $\varphi$ and an OD $\gamma$ such that $W = W' \cap \sigma$ and $\leq \cap (\sigma \times \sigma) = \leq' \cap (\sigma \times \sigma)$.

By our hypothesis, we can take a nonempty finite sequence of ordinals $s \in \kappa^{<\omega} \setminus \{\emptyset\}$ such that
\[ \forall x, y \in W \cap \sigma \ (x \leq y \iff \text{rank}_{T_x}(s) \leq \text{rank}_{T_y}(s)) \].

Take an ordinal $\alpha < \kappa$ that is sufficiently large that $s(0) \leq \alpha$ and $\varphi_0(x) \leq \alpha$ for all reals $x \in U \cap \sigma$, and define the set $U' = \{x \in U : \varphi_0(x) \leq \alpha\}$. We have $U \cap \sigma = U' \cap \sigma$. Restricting the norms of the scale $\varphi$ to the set $U'$ gives a scale on $U'$. The tree of this restricted scale $\varphi' = \varphi | U'$ is equal to the subtree $T'$ of $T$ defined by
\[ T' = \{(n_0, \ldots, n_{i-1}), (\xi_0, \ldots, \xi_{i-1}) \in T : \xi_0 \leq \alpha\}. \]

Define the set $W' = \mathbb{R} \setminus U' = \mathbb{R} \setminus p[T']$, which satisfies $W \cap \sigma = W' \cap \sigma$.

Any node $((n_0, \ldots, n_{i-1}), (\xi_0, \ldots, \xi_{i-1})) \in T$ such that $((\xi_0, \ldots, \xi_{i-1})$ extends $s$ must satisfy $\xi_0 = s(0) \leq \alpha$, so any such node is also in $T'$. So for every real $x$ (not just for every real $x \in \sigma$, although this would be enough for our argument) we have $s \in T_x \iff s \in T'_x$, and either $T_x$ and $T'_x$ are both ill-founded below $s$ or else they are both well-founded below $s$ and $\text{rank}_{T_x}(s) = \text{rank}_{T'_x}(s)$. Therefore we have $\leq \cap (\sigma \times \sigma) = \leq' \cap (\sigma \times \sigma)$ where $\leq'$ is the prewellordering of $W'$ defined by
\[ x \leq' y \iff \text{rank}_{T'_x}(s) \leq \text{rank}_{T'_y}(s) \].

It suffices to show that the set $W'$ and its prewellordering $\leq'$ have the desired complexity, namely that they are both in OD $<\kappa$.$\gamma$ (t). Because the
norm $\varphi_0$ is a $\Sigma^m_1(t)$-norm we have $U', W' \in \Delta^m_1(t, \alpha)$ and in particular we have $U', W' \in \mathcal{M}$. Moreover the restricted scale $\varphi'$ on $U'$ is a $\Delta^m_1(t, \alpha)$-scale and we have $\varphi' \in \mathcal{M}$. Because $\mathcal{M}$ is admissible it can compute the tree $T'$ of this scale; that is, we have $T' \in \mathcal{M}$ and $\{T'\} \in \Delta^m_1(t, \alpha)$. In particular the restricted scale $\varphi'$ on $U'$ is a $\Delta^m_1(t, \alpha)$-scale and we have $\varphi' \in \mathcal{M}$. Moreover the ordinals appearing in $T'$ are bounded in $\kappa$, even though the definition of $T'$ required this only for the ordinals appearing in the first coordinate. Moreover $\mathcal{M}$ can compute rank functions, so the prewellordering $\leq'$ is in $\Delta^m_1(t, \alpha)$. Therefore it is in $\text{OD}^{<\kappa, \mathcal{M}(t)}$ as desired. \qed

We now turn to some elementary observations about semiscales that do not depend on our descriptive-set-theoretic context (with $\Gamma$ an inductive-like pointclass, etc.) It will be convenient to make the following definition.

**Definition 7.3.** Let $W$ be a set of reals and let $C$ be a countable collection of pointsets. We say that $C$ forms a semiscale on $W$ if, whenever $\{x_k : k < \omega\}$ is a sequence of reals in $W$ converging to a real $x$, and such that for every element $\leq$ of $C$ that is a prewellordering of $W$ the $\leq$-equivalence class of $x_k$ is eventually constant as $k \to \omega$, we have $x \in W$.

The definition of “forms a semiscale on $W$” is fairly loose in that the collection $C$ may contain elements that are not prewellorderings of $W$. Also, although it is required to be countable, it is not required to come with a particular enumeration in order type $\omega$.

Nevertheless if $C$ forms a semiscale on $W$ then we do indeed obtain a semiscale $\psi$ on $W$ by taking any enumeration $(\leq_i : i < \omega)$ of $C$ and defining the norm $\psi_i : W \to \text{Ord}$ by letting $\psi_i(x)$ be the rank of $x$ in the prewellordering $\leq_i$. If one enumeration of $C$ yields a semiscale in this manner then so does any other enumeration.

**Remark 7.4.** In the intended application of Definition 7.3 we will have $W \in \tilde{\Gamma}$ and $C \subset \text{Env}(\tilde{\Gamma})$. In particular, we will be able to form a semiscale out of prewellorderings that are shown to be in $\text{Env}(\tilde{\Gamma})$ using Lemma 7.2.

Let $T$ be the tree of a $\tilde{\Gamma}$-scale on a universal $\tilde{\Gamma}$ set. If $\text{AD}_R$ holds then by a theorem of Martin ([15, Thm. 3.1]) $T$ is weakly homogeneous and a standard argument (see [3, Lem. 3.15]) gives a semiscale on the $\tilde{\Gamma}$ set $W = \mathbb{R} \setminus p[T]$ whose prewellorderings are all in $\text{Env}(\tilde{\Gamma})$. In order to weaken the hypothesis of Martin’s theorem from $\text{AD}_R$ to $\text{AD} + \text{Env}(\tilde{\Gamma}) \neq \varphi(\mathbb{R})$, Woodin ([27]) defined a game where one player plays measures from a given countable set of measures (considered as a putative homogeneity system.) Below we describe a game that is essentially equivalent to Woodin’s game except that this player instead plays prewellorderings from a given countable set of prewellorderings (considered as a putative semiscale.) The reason for modifying the game in this way is that in some situations, such as that of Section 9, the relevant measures may not exist.\footnote{Another approach, taken by the author in [25, Ch. 3], is to consider putative homogeneity systems consisting of partial measures rather than total measures.}
Definition 7.5. Let $C$ be a countable collection of pointsets and let $T$ be a tree on $\omega \times \kappa$ for some ordinal $\kappa$. Define the following game between the two players B (for “branch”) and S (for “semiscale”):

$$(G^C_T) \quad \begin{array}{c|cccc}
\text{B} & x(0), f(0) & g(0) & x(1), f(1) & g(1) & \cdots \\
\text{S} & \leq_0 & \leq_1
\end{array}$$

Rules. For all $n < \omega$:

- $x(n) \in \omega$, $f(n) \in \kappa$, $\leq_n \in C$, and $g(n) \in \text{Ord}$.
- $((x(0), \ldots, x(n)), (f(0), \ldots, f(n))) \in T$.
- $\leq_n$ is a prewellordering of $\mathbb{R} \setminus p[T]$.
- $((x(0), \ldots, x(n)), (g(0), \ldots, g(n))) \in T_{\bar{\psi}}$ where $T_{\bar{\psi}}$ is the tree of the putative semiscale $\bar{\psi}$ on $\mathbb{R} \setminus p[T]$ that Player S is building.\(^\dagger\)

The first player to deviate from these rules loses, and if both players follow the rules for all $\omega$ moves then we say that Player B wins, so the game is a closed game for Player B.

The significance of the games $G^C_T$ comes from the following lemma.

Lemma 7.6. Let $C$ be a countable collection of pointsets and let $T$ be a tree on $\omega \times \kappa$ for some ordinal $\kappa$. Then Player B has a winning strategy in the game $G^C_T$ if and only if $C$ does not form a semiscale on the set of reals $\mathbb{R} \setminus p[T]$.

Proof. Fix an enumeration $(\leq_i : i < \omega)$ of all of the pointsets in $C$ that are prewellorderings of $\mathbb{R} \setminus p[T]$ and let $\bar{\psi}$ denote the corresponding putative semiscale on $\mathbb{R} \setminus p[T]$.

Now suppose that Player B has a winning strategy. By feeding the sequence of prewellorderings $(\leq_i : i < \omega)$ into the strategy, we obtain a real $x$ that is in the projection of $T$ as witnessed by $f \in \kappa^\omega$ but is also in the projection of the tree of the putative semiscale $\bar{\psi}$ as witnessed by $g \in \text{Ord}^\omega$. Therefore $\bar{\psi}$ is not a semiscale on $\mathbb{R} \setminus p[T]$.

For the converse, if the putative semiscale $\bar{\psi}$ on $\mathbb{R} \setminus p[T]$ is not a semiscale, then this is witnessed by a convergent sequence of reals $x_k \rightarrow x \pmod{\bar{\psi}}$ such that $x \in p[T]$ and $x_k \notin p[T]$ for all $k < \omega$. Let $f \in \kappa^\omega$ witness $x \in p[T]$. Then Player B has a winning strategy in $G^C_T$: Regardless of the moves of Player S we play out the coordinates of $x$ and $f$ one-by-one as $x(n)$ and $f(n)$, and in response to a prewellordering $\leq$ played by Player S on turn $n$ we let $g(n)$ be the eventual ordinal rank, as $k \rightarrow \omega$, of the real $x_k$ in that prewellordering $\leq$.

\(\square\)

Note that the game $G^C_T$ is a closed game on a wellordered set, so by the Gale–Stewart theorem it is determined and we could rephrase the lemma as “Player S has a winning strategy if and only if $C$ forms a semiscale.” Moreover Player B’s moves are ordinals, so by the proof of the Gale–Stewart

\(^\dagger\)More precisely, there is a real $x_n \in \mathbb{R} \setminus p[T]$ such that for all $i \leq n$ we have $x_n(i) = x(i)$ and $g(i)$ is the rank of $x_n$ in the prewellordering $\leq_i$. 

theorem, if there is a winning strategy for Player B then there is a canonical winning strategy $F^C_T$. The main point of considering these games is that these strategies $F^C_T$ give us a canonical way to generate failures of the semiscale property in the cases where it fails.

8. Application to divergent models of $\text{AD}^+$

We will use Proposition 6.10 (determinacy of the envelope) and Lemma 7.6 (criterion for the existence of semiscales,) to give a simple proof of the following theorem on divergent models of $\text{AD}^+$. The original proof, which is unpublished, used a different argument involving Sacks forcing, similar to the proof of the derived model theorem (see [28].) We say that models $M_0$ and $M_1$ of $\text{AD}^+$ are divergent if neither $M_0 \cap \wp(R)$ nor $M_1 \cap \wp(R)$ contains the other. By Wadge’s Lemma, this implies that no model of determinacy contains both $M_0$ and $M_1$.

**Theorem 8.1** (Woodin). If $M_0$ and $M_1$ are divergent models of $\text{AD}^+$ with $\mathbb{R} \cup \text{Ord} \subset M_0, M_1$, then the model $M = L(M_0 \cap M_1 \cap \wp(R))$ satisfies $\text{AD}^+ + \text{DC} + \text{"every set of reals is Suslin."}$

Notice that $M \cap \wp(R) = M_0 \cap M_1 \cap \wp(R)$ because both $M_0$ and $M_1$ are closed under constructibility. Also, $\text{AD}^+$ holds in $M$ because it is downward absolute to any transitive model that contains all the reals.

By a standard argument the pointclass $M \cap \wp(R)$ is closed under countable sequences in $V$ (in the sense of any natural coding.) This is because by divergence and Wadge’s lemma, both $M_0$ and $M_1$ have surjections $\mathbb{R} \rightarrow M \cap \wp(R)$. A countable sequence from $M \cap \wp(R)$ is coded by a real relative to each surjection, and both models contain all the reals, so the sequence is in both models. Therefore $\text{cof}(\Theta^M) > \omega$, which implies that $\text{DC}$ holds in $M$ by Solovay [20].

To prove the theorem, it remains to show that $M$ satisfies the statement “every set of reals is Suslin.” Under $\text{AD}^+$ the set of Suslin cardinals is closed below $\Theta$ (see [11, Thm. 7.2]) so it suffices to show that $M$ has no largest Suslin cardinal. Let $\kappa$ be a Suslin cardinal of $M$. By the Coding Lemma the models $M_0$, $M_1$, and $M$ have the same subsets of $\kappa$. Trees on $\omega \times \kappa$ are essentially subsets of $\kappa$, so the models $M_0$, $M_1$, and $M$ have the same $\kappa$-Suslin sets of reals. We define the pointclass $\Gamma$ by

$$\Gamma = S(\kappa)^{M_0} = S(\kappa)^{M_1} = S(\kappa)^{M}.$$  

Recall that under $\text{AD}$, for any Suslin cardinal $\kappa$ the pointclass $S(\kappa)$ is $\mathbb{R}$-parameterized, closed under $\exists^\mathbb{R}$ and Wadge reducibility, and has the scale property. If $\Gamma$ is not closed under $\forall^\mathbb{R}$ then we can get scales beyond $\Gamma$ (namely on $\forall^\mathbb{R}\overline{\Gamma}$) by Moschovakis’s second periodicity theorem, in which case the model $M$ has a Suslin cardinal greater than $\kappa$ as desired.

So we now assume that $\Gamma$ is closed under $\forall^\mathbb{R}$, in which case $\Gamma$ is a boldface inductive-like pointclass. Moreover its prewellordering ordinal is $\kappa$ itself.
Notice that Env(Γ̃)M₀ and Env(Γ̃)M₁ are both contained in Env(Γ) because membership in Env(Γ) is absolute between transitive models containing Γ. All three envelopes are determined by Proposition 6.10 and are closed under continuous reducibility and Boolean combinations, so by Wadge’s Lemma we may assume without loss of generality that

$$ Env(Γ̃)M₀ \subset Env(Γ̃)M₁. $$

(This observation is the new ingredient in the proof.)

It follows that Env(Γ̃)M₀ ≠ ℘(R) ∩ M₀ because otherwise M₀ and M₁ would not be divergent. Therefore, working in the model M₀ we may use the theorem stated below to get a semiscale ψ̃ ∈ M₀ on a universal ˇΓ̃ set whose prewellorderings are in the pointclass Env(Γ̃)M₀.

Assuming the existence of such a semiscale, it is a simple matter to complete the proof. Because Env(Γ̃)M₀ is contained in Env(Γ̃)M₁ it is contained in the intersection model M, so the prewellorderings of ψ̃ are all in M. By closure of M ∩ ℘(R) under countable sequences, the semiscale ψ̃ itself is in M. So in M our universal Γ̃ set is Suslin. It cannot be κ-Suslin in M because it is not in Γ̃, so M must have a Suslin cardinal above κ as desired.

**Theorem 8.2** (Woodin [27]). Assume AD. Let Γ be a boldface inductive-like pointclass. If Env(Γ) ≠ ℘(R) then every Γ̃ set has a semiscale whose prewellorderings are in Env(Γ).

**Remark 8.3.** As previously mentioned, a similar result was first proved by Martin under the hypothesis of AD. Jackson [5] showed that “semiscale” can be strengthened to “scale” in the conclusion of Theorem 8.2.

For the sake of completeness, and also to illustrate ideas which will be used later in Section 9, we will give a proof of Theorem 8.2 using the games GC_T of Section 7.

**Proof of Theorem 8.2.** If Env(Γ) ≠ ℘(R) then by Wadge’s lemma there is a surjection π : R → Env(Γ). So we have a fine, countably complete measure U on ℘ω₁(Env(Γ)), namely the pushforward of Martin’s cone measure on ℘ω₁(R). The existence of such a measure is the only consequence of AD beyond ∆Γ determinacy that we will need for the proof.

Let κ be the prewellordering ordinal of Γ and let T on ω × κ be the tree of a Γ-scale ψ on a universal Γ̃ set. It suffices to show that there is a semiscale ψ̃ on the universal Γ̃ set R \ p[T], each of whose prewellorderings is in Env(Γ). Assume toward a contradiction that there is no such semiscale. Then by Lemma 7.6, Player B has a winning strategy in the game GC_T for every countable set C ⊂ Env(Γ). For such a set C we let FC denote the canonical winning strategy for Player B in the game GC_T.

We define a sequence of prewellorderings (≤n : n < ω) of R \ p[T], each of which is a member of Env(Γ), by recursion on n. We will obtain a contradiction by showing that for U-almost every set C ⊂ ℘ω₁(Env(Γ)) this
sequence of prewellorderings is a legal play for Player S in the game $G^C_T$ that defeats Player B’s “winning” strategy $F^C$.

Let $n < \omega$. Once we have defined the prewellorderings $\leq_0, \ldots, \leq_{n-1}$, we have these prewellorderings as elements of $C$ for $\mathcal{U}$-almost every set $C$ by fineness, so we may regard them as moves for Player S in the game $G^C_T$ and the strategy $F^C$ gives us moves

$$x^C(0), f^C(0), g^C(0), \ldots, x^C(n), f^C(n)$$

for Player B in response. That is,

$$B \mid x^C(0), f^C(0) \quad g^C(0) \quad \ldots \quad x^C(n), f^C(n)$$

$\leq_0$ is the corresponding partial run of the game $G^C_T$ according to the strategy $F^C$. Define the nonempty finite sequence of ordinals

$$s_n^C = f^C(0), \ldots, f^C(n).$$

Then we can define the prewellordering $\leq_n$ of $\mathbb{R} \setminus p[T]$ by

$$y \leq_n z \iff \forall_{\mathcal{U}} \mathcal{C} \left( \text{rank}_{T_y}(s_n^C) \leq \text{rank}_{T_z}(s_n^C) \right).$$

Here (as elsewhere) we use the convention that assigns the rank zero to nodes that are not in the tree. This prewellordering $\leq_n$ is easily seen to be in $\text{Env}(\Gamma)$ by Lemma 7.2 and the countable completeness of $\mathcal{U}$.

Once we have our sequence of prewellorderings $(\leq_n : n < \omega)$, let $\vec{\psi}$ denote the corresponding putative semiscale on the set $\mathbb{R} \setminus p[T]$. Because the measure $\mathcal{U}$ is fine and countably complete we have $\{\leq_n : n < \omega\} \subset C$ for $\mathcal{U}$-almost every set $C \in \wp_{\omega_1}(\text{Env}(\Gamma))$. For such a set $C$, the sequence $(\leq_n : n < \omega)$ is a legal play for Player S in the game $G^C_T$.

Using countable completeness twice, we obtain a real $x$ that is played as $x^C$ for almost all sets $C$. Momentarily considering any set $C$ that is in all the measure-one sets considered above, we see that $x \in p[T]$ (as witnessed by the branch $f^C$) and also that $x$ is in the projection of the tree of $\vec{\psi}$ (as witnessed by the branch $g^C$). So $x$ is a counterexample to the semiscale property for $\vec{\psi}$ in the sense that $x \in p[T]$ and we can find a sequence of reals $(x_k : k < \omega)$ with $x_k \notin p[T]$ for all $k < \omega$ and with $x_k \rightarrow x \pmod{\vec{\psi}}$.

Using countable completeness and the definition of the prewellorderings $\leq_n$, we can take a set $C$ in all the measure-one sets considered above and with the additional property that for every $n < \omega$ the ordinal rank of the node $s_n^C$ in the tree $T_{x_k}$ is eventually constant as $k \rightarrow \omega$. Let

$$h(n) = \lim_{k \rightarrow \omega} \text{rank}_{T_{x_k}}(s_n^C).$$

For any $n < \omega$ we have $s_n^C \in T_x$ by the rules for Player B, so we have $s_n^C \in T_{x_k}$ for all sufficiently large $k$ because $x_k \rightarrow x$. That is, eventually we are not in the trivial case where we have defined the rank to be zero because the node is not in the tree. Therefore, for sufficiently large $k$ the ordinals $h(n)$ and $h(n+1)$ are the ordinal ranks of the node $s_n^C$ and its successor $s_{n+1}^C$. 
in a well-founded tree, so \( h(n) > h(n + 1) \) and we get a strictly decreasing sequence of ordinals, a contradiction. ∎

9. APPLICATION TO DERIVED MODELS

If \( \delta \) is a supercompact cardinal and \( G \subseteq \text{Col}(\omega, < \delta) \) is a \( V \)-generic filter, then the symmetric submodel \( V(R^{V[G]}) \) of the generic extension \( V[G] \) satisfies “every tree on \( \omega \times \text{Ord} \) is weakly homogeneous” and therefore satisfies “every Suslin set of reals is co-Suslin” (Woodin; see [15, Thm. 3.2].) This implies that the derived model at \( \delta \) satisfies “every Suslin set of reals is co-Suslin,” which, because of \( \text{AD}^+ \), is equivalent to “every set of reals is Suslin.”

If one wishes only to obtain a derived model satisfying “every set of reals is Suslin” it suffices to assume the weaker large cardinal hypothesis that \( \delta \) is a limit of Woodin cardinals and of \( < \delta \)-strong cardinals (see [24] for a proof in the case of the “old” derived model.)

In this section we weaken the hypothesis of supercompactness in a different direction, namely to the hypothesis of indestructible weak compactness. This is a “weakening” in the sense that indestructible weak compactness can be forced from a supercompact cardinal using the Laver preparation [12] (because it follows from indestructible supercompactness) and it is unknown whether this relative consistency result can be reversed, although this is conjectured in [1]. The least weakly compact cardinal can be indestructibly weakly compact ([2, Thm. 3.11],) so indestructibly weakly compact cardinals have very weak reflection properties.

**Theorem 9.1.** Assume that \( \text{ZFC} \) holds, \( \delta \) is a limit of Woodin cardinals, and \( \delta \) is weakly compact after forcing with \( \text{Col}(\delta, \delta^+) \). Then the derived model \( D(V, \delta) \) satisfies \( \text{AD}^+ + \text{DC} + \) “every set of reals is Suslin.”

**Remark 9.2.** In terms of relative consistency, Theorem 9.1 was anticipated by [6, Theorem 0.4], which uses inner model theory to give a a model of large cardinal axioms of slightly higher consistency strength than \( \text{AD}^+ + \text{DC} + \) “every set of reals is Suslin” from the same degree of indestructible weak compactness. This inner model theoretic approach also has the advantage that it does not need the assumption that \( \delta \) is a limit of Woodin cardinals, whereas Theorem 9.1 needs the Woodin cardinals in order to make use of the derived model theorem. (A hybrid “descriptive inner model theoretic” proof should be possible as well, using the core model induction to build the model of determinacy instead of using the derived model theorem.)

However, any attempt to use inner model theory or descriptive inner model theory to prove that every set of reals—or even every \( \Pi^1_2 \)-set of reals—is Suslin in the derived model \( D(V, \delta) \) itself rather than in the derived model of a mouse or in a submodel of \( D(V, \delta) \) would probably require a full solution to the mouse set conjecture.
Before proving Theorem 9.1 we give some background on derived models. For more information, see [24]. Assume that \( \delta \) is a limit of Woodin cardinals. Let \( G \subseteq \text{Col}(\omega, \delta) \) be a \( V \)-generic filter and define the set of reals

\[
\mathbb{R}^*_G = \bigcup_{\alpha < \delta} \mathbb{R} \cap V[G \upharpoonright \alpha].
\]

In the context of the theorem we have \( \mathbb{R}^*_G = \mathbb{R}^{V[G]} \) because \( \delta \) is inaccessible, but the notation \( \mathbb{R}^*_G \) is well-established for the general case so we will use it. We say that a tree \( T \) is \( \delta \)-absolutely complemented if there is a tree \( \tilde{T} \) such that \( p[\tilde{T}] = \mathbb{R} \setminus p[T] \) in every generic extension of \( V \) by a poset of size less than \( \delta \). Let \( uB_\delta \) denote the pointclass of \( \delta \)-absolutely Baire sets of reals; that is, the sets of reals given by projections \( p[T] \) for \( \delta \)-absolutely complemented trees \( T \). Because \( \delta \) is a limit of Woodin cardinals this is equal to the pointclass \( \text{Hom}_{<\delta} \) of \( <\delta \)-homogeneously Suslin sets of reals by work of Martin, Solovay, Steel, and Woodin.

Absolutely complementing trees give us a canonical way to extend universally Baire sets of reals to generic extensions (which agrees with the canonical extensions given by homogeneity systems.) Accordingly we define the pointclass

\[
\text{Hom}^*_G = \{ p[T] \cap \mathbb{R}^*_G : \exists \alpha < \delta \ (T \in V[G \upharpoonright \alpha] \land V[G \upharpoonright \alpha] \models \text{“} T \text{ is } \delta \text{-absolutely complementing”} \}. \]

Note that \( \text{Hom}^*_G \) is the pointclass of sets that are both Suslin and co-Suslin in the symmetric submodel \( V(\mathbb{R}^*_G) \) of \( V[G] \).

Woodin’s derived model theorem in its most general form [26, Thm. 31] (see [28] for a proof) says that there is a transitive model \( D(V, \delta, G) \), called the derived model of \( V \) at \( \delta \) by \( G \) that is maximal under inclusion subject to the conditions

- \( D(V, \delta, G) \models \text{ZF} + \text{AD}^+ + V = L(\wp(\mathbb{R})) \) and
- \( L(\mathbb{R}^*_G) \subset D(V, \delta, G) \subset V(\mathbb{R}^*_G) \).

Moreover, Woodin showed that \( \text{Hom}^*_G \subset D(V, \delta, G) \). The pointclass \( \text{Hom}^*_G \) consists exactly of the sets of reals that are both Suslin and co-Suslin in \( D(V, \delta, G) \); one direction follows from a theorem of Steel (applied in all intermediate models \( V[G \upharpoonright \alpha] \)) which says that the pointclass \( \text{Hom}_{<\delta} \) has the scale property. If \( \delta \) is regular (and therefore inaccessible) then the pointclass \( \text{Hom}^*_G \) has uncountable Wadge cofinality, so the derived model \( D(V, \delta, G) \) satisfies DC.

Because the forcing \( \text{Col}(\omega, \delta) \) is homogeneous, the theory of the derived model does not depend on \( G \). Accordingly, we will drop \( G \) from the notation and speak of “the” derived model \( D(V, \delta) \) when appropriate.

Remark 9.3. The term “derived model” (or sometimes “old derived model”) has also been used to refer to the submodel \( L(\mathbb{R}^*, \text{Hom}^*) \) of \( D(V, \delta) \). Because \( \text{AD}^+ \) is downward absolute to transitive models containing all the reals, the “old” derived model theorem, which says that \( L(\mathbb{R}^*, \text{Hom}^*) \) satisfies...
AD⁺, can be seen as a special case of the derived model theorem. The models $L(\mathbb{R}^\ast, \text{Hom}^\ast)$ and $D(V, \delta)$ agree to the extent that in each model the pointclass of sets that are Suslin and co-Suslin is exactly $\text{Hom}^\ast$. However, it is possible to have $L(\mathbb{R}^\ast, \text{Hom}^\ast) \subsetneq D(V, \delta)$ because the model $D(V, \delta)$ may contain sets of reals that are not constructible from its Suslin co-Suslin sets.

If the derived model $D(V, \delta)$ satisfies “every set of reals is Suslin,” then so does the submodel $L(\mathbb{R}^\ast, \text{Hom}^\ast)$, and similarly for DC, so Theorem 9.1 implies the corresponding result for the model $L(\mathbb{R}^\ast, \text{Hom}^\ast)$.

Under AD⁺ the set of Suslin cardinals is closed below $\Theta$, so to show that $D(V, \delta)$ satisfies “every set of reals is Suslin” it suffices to show that it has no largest Suslin cardinal. Let $\kappa$ be a Suslin cardinal of $D(V, \delta)$. Then the pointclass $\Gamma = S(\kappa)^{D(V, \delta)}$ is $\mathbb{R}$-parameterized, closed under $\exists \mathbb{R}$ and continuous reducibility, and has the scale property. If $\Gamma$ is not closed under $\forall \mathbb{R}$ then we can get scales beyond $\Gamma$ (namely on $\forall \mathbb{R} \Gamma$) by second periodicity, in which case the model $M$ has a Suslin cardinal greater than $\kappa$ as desired.

Now assume, as in the hypothesis of Theorem 9.1, that our limit of Woodin cardinals $\delta$ is weakly compact after forcing with $\text{Col}(\delta, \delta^+)$. Forcing with $\text{Col}(\delta, \delta^+) \subset V$ does not change $\mathbb{R}^\ast$ or $\text{Hom}^\ast$ because it does not change $V_\delta$. Moreover, forcing with $\text{Col}(\delta, \delta^+) V$ over the model $V[G]$ does not change $\text{Env}(\Gamma)$. It does not shrink $\text{Env}(\Gamma)$ because the membership of a set of reals in $\text{Env}(\Gamma)$ is absolute between transitive models of set theory containing $\Gamma$ and having the same reals. On the other hand, it does not enlarge $\Gamma$ because the forcing is homogeneous and every set of reals in $\text{Env}(\Gamma)$ is ordinal-definable from a set in $\Gamma$ by Proposition 6.11.

The cardinality of $\text{Env}(\Gamma)$ in $V[G]$ is at most $\delta^+ = \omega_2 = \mathfrak{c}^+$ by Wadge’s Lemma because $\text{Env}(\Gamma)$ is a determined pointclass by Proposition 6.10. So passing from $V$ to $V^{\text{Col}(\delta, \delta^+)}$ we may assume that $\delta$ is weakly compact and that $\text{Env}(\Gamma)$ has cardinality $\delta = \omega_1 = \mathfrak{c}$ in $V[G]$. We now use the following lemma to construct a semiscale on a universal $\Gamma$ set.

**Lemma 9.4.** Assume that ZFC holds, $\delta$ is weakly compact, and $G \subset \text{Col}(\omega, <\delta)$ is a V-generic filter. Let $\Gamma \in V(\mathbb{R}_G^\ast)$ be a boldface inductive-like pointclass such that $\Delta_\Gamma$ is determined. If $\text{Env}(\Gamma)$ has size $\delta = \omega_1$ in $V[G]$, then every pointset in $\Gamma$ has a semiscale in $V(\mathbb{R}_G^\ast)$ whose prewellorderings are all in $\text{Env}(\Gamma)$.

Assuming that the lemma holds, so that every $\Gamma$ pointset has a semiscale in $V(\mathbb{R}_G^\ast)$, we can easily complete the proof of the theorem. (We remark that the fact that the prewellorderings are in $\text{Env}(\Gamma)$ will not be needed, although it would be needed for an argument that used the core model induction
rather than the derived model theorem.) By the standard construction of the tree of a semiscale, every set of reals in \( \Gamma \) is co-Suslin as well as Suslin in \( V(\mathbb{R}^*) \). This means that \( \Gamma \subset \text{Hom}^* \), or equivalently that every set of reals in \( \Gamma \) is Suslin and co-Suslin in \( D(V, \delta) \). But a universal \( \Gamma \) set of reals cannot be \( \kappa \)-Suslin in \( D(V, \delta) \), so \( D(V, \delta) \) must have a Suslin cardinal above \( \kappa \), as desired. It remains to prove the lemma.

**Proof of Lemma 9.4.** By Proposition 6.11 every set of reals in \( \text{Env}(\Gamma) \) is ordinal-definable from a set of reals in \( \Gamma \). By our hypothesis on the size of \( \text{Env}(\Gamma) \) we may arrange that \( \text{Env}(\Gamma) \in M(\mathbb{R}^*) \). This means that \( \Gamma \subset \text{Hom}^* \), and also that \( M[G] \models |\text{Env}(\Gamma)| = \omega_1 \).

Because \( \delta \) is weakly compact we can take an elementary embedding \( j : M \rightarrow N \) with \( N \) transitive and \( \text{crit}(j) = \delta \). Now we can take a \( V[G] \)-generic filter \( H \subset \text{Col}(\omega,<j(\delta)) \) with \( G \subset H \) and extend \( j \) to an elementary embedding

\[
\hat{j} : M[G] \rightarrow N[H]
\]

in the usual way, by sending \( \tau \in G \) to \( j(\tau)_H \) for every forcing term \( \tau \). The pointwise image of the envelope,

\[
C = \hat{j}(\text{Env}(\Gamma)),
\]

is a countable subset of \( \text{Env}(\hat{j}(\Gamma)) \) in \( N[H] \).

Let \( \kappa \) be the prewellordering ordinal of \( \Gamma \) and let \( T \) on \( \omega \times \kappa \) be the tree of a \( \Gamma \)-scale on a universal \( \Gamma \) set. Let

\[
U = p[T] \cap \mathbb{R}^* = p[T] \cap \mathbb{R}^{V[G]}.
\]

We have \( T, U \in M(\mathbb{R}^*) \). It suffices to show that there is a semiscale on the \( \Gamma \) set \( \mathbb{R}^* \setminus U \) that is in \( M(\mathbb{R}^*) \) and has the property that each of its prewellorderings is in \( \text{Env}(\Gamma) \).

Because \( \text{Env}(\Gamma) \in M(\mathbb{R}^*) \) and \( \delta \) is inaccessible it follows that every countable sequence in \( M[G] \) of sets in \( \text{Env}(\Gamma) \) is also in \( M(\mathbb{R}^*) \). So by the elementarity of \( \hat{j} \) it will suffice to show that the prewellorderings of \( \hat{j}(\mathbb{R}^* \setminus U) \) in \( C \) form, when enumerated in order type \( \omega \), a semiscale on \( \hat{j}(\mathbb{R}^* \setminus U) \). Assume toward a contradiction that they do not form such a semiscale. Then by Lemma 7.6, Player B has a winning strategy in the game \( G^C_{\hat{j}(T)} \) as defined in \( N[H] \).

Let \( F \) denote the canonical winning strategy for Player B in the game \( G^C_{\hat{j}(T)} \). We define a sequence \( (\leq n : n < \omega) \) of prewellorderings of \( \mathbb{R}^* \setminus U \), each in \( \text{Env}(\Gamma) \), by recursion on \( n \). Once we have defined the prewellorderings \( \leq_0, \ldots, \leq_{n-1} \in \text{Env}(\Gamma) \), so that we have \( \hat{j}(\leq_0), \ldots, \hat{j}(\leq_{n-1}) \in C \), let

\[
\begin{array}{c|cccc}
B & x(0), f(0) & g(0) & \cdots & x(n), f(n) \\
S & j(\leq_0) \\
\end{array}
\]
be the corresponding partial run of the game $G^C_{j(T)}$ according to the strategy $F$. Define the finite sequence of ordinals
$$s_n = (f(0), \ldots, f(n)) \in j(\kappa)^n$$
and define the prewellordering $\leq_n$ of $\mathbb{R}^* \setminus U$ by
$$y \leq_n z \iff \text{rank}_{j(T)_y}(s_n) \leq \text{rank}_{j(T)_x}(s_n).$$
(We use the convention that assigns the rank zero to nodes that are not in the tree.) Using Lemma 7.2 and the elementary of $\hat{\omega}$, it is easy to check that the prewellordering $\leq_n$ is indeed in $\text{Env}(\mathcal{U})$.

Once we have the sequence of moves $(\leq_n : n < \omega)$, we may continue with the argument. Because $\text{Env}(\mathcal{U}) \in M(\mathbb{R}^*)$ the restriction $j \upharpoonright \text{Env}(\mathcal{U})$ depends only on the set $\mathbb{R}^{V[H]}$ and not on the generic filter $H$. So by the homogeneity of the forcing and the definability of the canonical winning strategy $F$, both the real $x$ and the sequence of moves $(\leq_n : n < \omega)$ are in $V[G]$. Because the model $M$ is closed under $<\delta$-sequences in $V$, its generic extension $M[G]$ is closed under countable sequences in $V[G]$, and we have $x \in M[G]$ and $(\leq_n : n < \omega) \in M[G]$.

Letting $\psi$ be the putative semiscale on the set $\mathbb{R}^* \setminus U$ corresponding to the sequence of prewellorderings $(\leq_n : n < \omega)$, we have $\vec{\psi} \in M[G]$ as well. In the model $N[H]$ the real $x$ witnesses that $j(\vec{\psi})$, which is the putative semiscale from the sequence of prewellorderings $(j(\leq_n) : n < \omega)$, is not a semiscale on the set $j(\mathbb{R}^* \setminus U)$. This is because $x$ was played by the strategy $F$ in response to this sequence of prewellorderings, and $F$ is a winning strategy for Player B.

Therefore by the elementarity of $j$, in $M[G]$ the real $x$ witnesses that $\vec{\psi}$ is not a semiscale on the set $\mathbb{R}^* \setminus U$, meaning that we have a sequence of reals $(x_k : k < \omega)$ with $x_k \in \mathbb{R}^* \setminus U$ for each $k < \omega$ and with $x_k \rightarrow x$ (mod $\vec{\psi}$), but $x \in U$. By the definition of $\vec{\psi}$ this means that for every $n < \omega$ the ordinal rank of the node $s_n$ in the tree $j(T)_{x_k}$ is eventually constant as $k \rightarrow \omega$, so we can define
$$h(n) = \lim_{k \rightarrow \omega} \text{rank}_{j(T)_{x_k}}(s_n).$$
For any $n$, note that we have $s_n \in j(T)_{x_k}$ by the rules for Player B and we have $x_k \rightarrow x$, so we have $s_n \in j(T)_{x_k}$ for all sufficiently large $k$. That is, eventually we are not in the trivial case where the rank is defined to be zero because the node is not in the tree. Therefore, for sufficiently large $k$ the ordinals $h(n)$ and $h(n + 1)$ are the ordinal ranks of the node $s_n$ and its successor $s_{n+1}$ in a well-founded tree, so $h(n) > h(n + 1)$ and we get a strictly decreasing sequence of ordinals. This contradiction completes the proof of the lemma.

The reader can easily verify that the proofs of Lemma 9.4 and Theorem 9.1 respectively can be adapted to yield the following results. See [25,
Prop. 3.8.1] for a proof using a more complicated method involving partial measures instead of prewellorderings.\footnote{The result was stated and proved there for the “old” derived model $L(Hom^*, \mathbb{R}^*)$, but this difference is inessential to the proof.}

**Lemma 9.5.** Assume that ZFC holds, $\delta$ is a cardinal that is $\delta^+$-strongly compact, and $G \subset \text{Col}(\omega, <\delta)$ is a $V$-generic filter. Let $\Gamma \in V(\mathbb{R}^*_G)$ be a boldface inductive-like pointclass such that $\Delta_\Gamma$ is determined. Then every set in $\hat{\Gamma}$ has a semiscale in $V(\mathbb{R}^*_G)$ whose prewellorderings are all in $\text{Env}(\Gamma)$.

**Theorem 9.6.** Assume that ZFC holds, $\delta$ is a limit of Woodin cardinals, and $\delta$ is $\delta^+$-strongly compact. Then the derived model $D(V, \delta)$ satisfies $\text{AD}^+ + \text{DC} + \text{“every set of reals is Suslin.”}$

**References**


[27] W. Hugh Woodin. Private communication, August 2012.