A MODEL OF THE AXIOM OF DETERMINACY IN WHICH EVERY SET OF REALS IS UNIVERSALLY BAIRE (DRAFT)

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Abstract. The consistency of the theory $\mathsf{ZF}+\mathsf{AD}^+ +$ "every set of reals is universally Baire" is proved relative to $\mathsf{ZFC}+$ "there is a cardinal λ that is a limit of Woodin cardinals and of strong cardinals." The proof is based on the derived model construction, which was used by Woodin to show that the theory $\mathsf{ZF}+\mathsf{AD}^+ +$ "every set of reals is Suslin" is consistent relative to $\mathsf{ZFC}+$ "there is a cardinal λ that is a limit of Woodin cardinals and of $<\lambda$-strong cardinals." The Σ^2_1 reflection property of our model is proved using genericity iterations as in Neeman [7] and Steel [8].

1. Introduction

As usual, we refer to elements of the Baire space ω^ω as *reals*. A set of reals is *Suslin* if it is the projection of a tree on $\omega \times \mathrm{Ord}$. This generalizes the notion of analytic sets, which are projections of trees on $\omega \times \omega$. If the Axiom of Choice holds, then trivially every set of reals is $2^{\aleph_0}$-Suslin. A nontrivial strengthening is the notion of universal Baireness, introduced by Feng, Magidor, and Woodin [1]. Every analytic set of reals is universally Baire. Universally Baire sets of reals have regularity properties such as the Property of Baire and Lebesgue measurability, so if the Axiom of Choice holds, then not every set of reals is universally Baire.

Under the Axiom of Determinacy the picture is a bit different. The construction of Suslin representations for sets of reals is nontrivial and is central to the structure theory of models of AD. There are natural models of AD where not every set of reals is Suslin, such as $L(\mathbb{R})$ under the assumption that there are infinitely many Woodin cardinals with a measurable cardinal above them. From the stronger hypothesis of a cardinal λ that is a limit of Woodin cardinals and of $<\lambda$-strong cardinals, Woodin obtained a model of $\mathsf{AD}+$ "every set of reals is Suslin" via the derived model construction.

The usual derived model construction (see Steel [8, 9]) produces a model of AD satisfying $V = L(\wp(\mathbb{R}))$. However, no model of the form $V = L(A)$ where A is a set can satisfy the statement "every set of reals is universally Baire" or even the statement "every $\boldsymbol{\Sigma}^1_2$ set is universally Baire." This is because the latter statement is preserved by forcing as shown by Feng, Magidor, and Woodin [1, Corollary 3.2], whereas forcing with $\mathrm{Col}(\omega, A)$ over $L(A)$ produces a model with a $\boldsymbol{\Sigma}^1_2$ well-ordering of its reals.

We will modify the derived model construction (assuming a slightly stronger hypothesis) by adding appropriate structure above $\wp(\mathbb{R})$ to obtain a model of AD^+ + "every set is universally Baire."

Before we begin, we must take special care in defining the notion of universal Baireness because we will be using the definition in models that do not satisfy the Axiom of Choice.

Definition 1.1. If S and T are trees on $\omega \times \mathrm{Ord}$ and Z is any set, we say that the pair (S, T) is *Z-absolutely complementing* if $p[S] = \omega^\omega \setminus p[T]$ in every generic extension of V by the poset $\mathrm{Col}(\omega, Z)$.

Let A be a set of reals and let Z be a set. A *Z-absolutely complementing pair of trees for A* is a Z-absolutely complementing pair of trees (S, T) such that $A = p[S] = \omega^\omega \setminus p[T]$ in V.

We say that a set of reals A is *universally Baire* if for every set Z there is a Z-absolutely complementing pair of trees for A. The pointclass of universally Baire sets is denoted by uB.

Using a variation of Woodin's derived model construction, we will obtain the following result.

Theorem 1.2. *If the theory* ZFC + *"there is a cardinal that is a limit of Woodin cardinals and a limit of strong cardinals" is consistent, then so is the theory* $\mathsf{ZF} + \mathsf{AD}^+$ + *"every set is universally Baire."*

This result follows from a more specific result that we state in the next section (the Main Theorem.) Meanwhile, we make some remarks about universal Baireness.

Remark 1.3. The following variation of the definition is often useful, especially under the Axiom of Choice: If κ is an ordinal or $\kappa = \mathrm{Ord}$, we say that a set of reals A is *$<\kappa$-universally Baire* if for every ordinal $\alpha < \kappa$ there is an α-absolutely complementing pair of trees for A. The pointclass of $<\kappa$-universally Baire sets of reals is denoted by $\mathrm{uB}_{<\kappa}$.

Under the Axiom of Choice, if a set of reals A is $<\kappa$-universally Baire for some $\kappa < \mathrm{Ord}$ then this can be witnessed by a single pair (S, T) of trees that we call an *$<\kappa$-absolutely complementing* pair. Such $<\kappa$-absolutely complementing pairs can be formed by amalgamating smaller trees and they also arise naturally in the Martin–Solovay construction from κ-complete measures.

If a set of reals A is $<$Ord-universally Baire and the Axiom of Choice holds, then A is universally Baire as defined in Definition 1.1. This is because every generic extension is contained in a generic extension by a poset of the form $\mathrm{Col}(\omega, \alpha)$ where α is an ordinal. The authors do not know whether the Axiom of Choice is required for this implication.

For a universally Baire set of reals A there need not be a canonical choice of trees witnessing universal Baireness. Nevertheless, one can derive some canonical information:

Definition 1.4. Let Z be a set, let A be a universally Baire set of reals, and let $G \subset \mathrm{Col}(\omega, Z)$ be a V-generic filter. Then the *canonical expansion of A to $V[G]$* is defined by

$$A^{V[G]} = (p[S])^{V[G]} = (\omega^\omega \setminus p[T])^{V[G]}$$

for some (equivalently, for any) Z-absolutely complementing pair of trees (S, T) for A.

It is a standard fact that the canonical expansion does not depend on the choice of absolutely complementing pair.

2. Universally Baire sets in derived models

Theorem 1.2 will follow from a variation of the derived model theorem that we will state in this section. First we recall some standard notation from Steel [9].

Definition 2.1. Let λ be a limit of Woodin cardinals and let $G \subset \mathrm{Col}(\omega, {<}\lambda)$ be a V-generic filter. Define:

- $\mathbb{R}^*_G = \bigcup_{\xi<\lambda} \mathbb{R}^{V[G \restriction \xi]}$.
- $\mathrm{HC}^*_G = \bigcup_{\xi<\lambda} \mathrm{HC}^{V[G \restriction \xi]}$, where HC denotes the collection of hereditarily countable sets.
- Hom^*_G is the pointclass of sets $p[S] \cap \mathbb{R}^*_G$ for ${<}\lambda$-absolutely complementing pairs of trees (S, T) appearing in models $V[G \restriction \xi]$ where $\xi < \lambda$.

When the choice of the generic filter G is not important we will write $\mathbb{R}^*$, HC^*, and Hom^* for $\mathbb{R}^*_G$, HC^*_G, and Hom^*_G respectively.

The new ingredient that we add to the derived model construction is the following predicate.

Definition 2.2. In any model of ZF, let us define the predicate F to consist of all quadruples $(A, Z, p, \dot{x})$ such that

- $A \in \mathrm{uB}$, Z is a set, p is a condition in the collapse forcing $\mathrm{Col}(\omega, Z)$, $\dot{x}$ is a $\mathrm{Col}(\omega, Z)$-name for a real, and
- p forces $\dot{x}$ to be in the canonical expansion of A to the generic extension.

Now we can state the central result of the paper.

Main Theorem. *Let λ be a limit of Woodin cardinals and a limit of strong cardinals, let $G \subset \mathrm{Col}(\omega, {<}\lambda)$ be a V-generic filter, and define the model*

$$\mathcal{M} = (L^F(\mathbb{R}^*, \mathrm{Hom}^*))^{V(\mathbb{R}^*)}$$

where $\mathbb{R}^* = \mathbb{R}^*_G$ and $\mathrm{Hom}^* = \mathrm{Hom}^*_G$. Then we have

(1) $\mathcal{M} \models \mathsf{AD}^+$, and
(2) $\mathcal{M} \models$ "every set of reals is universally Baire."

We now proceed to give the proof of the Main Theorem modulo some lemmas that will be proved in later sections. As in the proof of the derived model theorem, part 1 will be obtained as an immediate consequence of the following result.

Σ^2_1 **Reflection Lemma** (Lemma 5.1)**.** *Let λ be a limit of Woodin cardinals and a limit of strong cardinals, let $G \subset \mathrm{Col}(\omega, {<}\lambda)$ be a V-generic filter, and define the model $\mathcal{M} = (L^F(\mathbb{R}^*, \mathrm{Hom}^*))^{V(\mathbb{R}^*)}$ where $\mathbb{R}^* = \mathbb{R}^*_G$ and $\mathrm{Hom}^* = \mathrm{Hom}^*_G$. For every sentence φ, if there is a set of reals $A \in \mathcal{M}$ such that $(\mathrm{HC}^*, \in, A) \models \varphi$, then there is a set of reals $A \in \mathrm{Hom}^V_{<\lambda}$ such that $(\mathrm{HC}, \in, A) \models \varphi$.*

This implies part 1 of the Main Theorem because AD^+ can be reformulated as a Π^2_1 statement that holds in the pointclass $\mathrm{Hom}^V_{<\lambda}$, the pointclass of ${<}\lambda$-homogeneously Suslin sets of reals, which is determined. The determinacy of homogeneously Suslin sets was essentially proved by Martin in the course of showing determinacy for $\mathbf{\Pi}^1_1$ sets of reals. For background on homogeneously Suslin sets, see Steel [8, 9] or Larson [3]. For now we only remark that $\mathrm{Hom}^V_{<\lambda} = \mathrm{uB}^V_{<\lambda}$ because λ is a limit of Woodin cardinals.

The proof of the Σ^2_1 reflection property is the most technically demanding part of this paper. It will be given in Section 5 after some background and preliminary work in Sections 3 and 4. Our proof will be similar to that given by Steel [8] in the case of the derived model. Roughly speaking, the proof can be adapted to the model $\mathcal{M}$ because the information added by the predicate F is canonical.

In the remainder of this section, we outline a proof of part 2 of the Main Theorem under the assumption that part 1 holds.

Lemma 2.3. *Under the hypothesis of the Main Theorem, every set of reals in $\mathcal{M}$ is in* Hom^*.

Proof. Let A be a set of reals in $\mathcal{M}$. Then $L(A, \mathbb{R}^*) \models \mathsf{AD}^+$ by part 1, so A is in the derived model $D(V, \lambda)$. Here is it important that we use the "new" derived model $D(V, \lambda)$ as defined by Woodin [10, Thm. 31]. See Zhu [11] for an exposition of Woodin's proof of the new derived model theorem. Because λ is a limit of ${<}\lambda$-strong cardinals, the derived model $D(V, \lambda)$ satisfies "every set of reals is Suslin" (unpublished, but see Steel [9, §9] for a proof in the case of the "old" derived model.) Therefore A is Suslin and co-Suslin in $D(V, \lambda)$, and in particular in $V(\mathbb{R}^*)$, which implies $A \in \mathrm{Hom}^*$. □

Because λ is a limit of strong cardinals in the hypothesis of the Main Theorem, a standard argument shows that every set in Hom^* is ${<}\mathrm{Ord}$-universally Baire in the symmetric model $V(\mathbb{R}^*)$. This implies that Hom^* sets are (fully) universally Baire in $V(\mathbb{R}^*)$ by the following observation, which will be used later also.

Lemma 2.4. *Let λ be a cardinal and let $G \subset \mathrm{Col}(\omega, {<}\lambda)$ be a V-generic filter. Then in the symmetric model $V(\mathbb{R}^*_G)$, for every set Z there is an* *ordinal η such that every η-absolutely complementing pair of trees on $\omega \times \mathrm{Ord}$ is Z-absolutely complementing.*

Proof. Let $\eta \geq \lambda$ be large enough that there is a surjection $\eta \to Z$ in the choice model $V[G]$, and let (S, T) be an η-absolutely complementing pair of trees in $V(\mathbb{R}^*_G)$. Then because S and T are subsets of the ground model, a standard symmetry argument (see Steel [9, p. 307] for example) shows that $(S, T) \in V[G \upharpoonright \alpha]$ for some ordinal $\alpha < \lambda$. The pair (S, T) is also η-absolutely complementing in the model $V[G \upharpoonright \alpha]$ because this property is downward absolute.

Because $\eta \geq \lambda$, every generic extension of $V[G]$ by the poset $\mathrm{Col}(\omega, \eta)$ is also a generic extension of $V[G \upharpoonright \alpha]$ by the poset $\mathrm{Col}(\omega, \eta)$, so the pair (S, T) is η-absolutely complementing in $V[G]$. Because there is a surjection $\eta \to Z$, this implies that the pair (S, T) is Z-absolutely complementing in $V[G]$. This property is downward absolute, so the pair (S, T) is Z-absolutely complementing in $V(\mathbb{R}^*_G)$ as desired. □

To complete the proof the Main Theorem, it suffices to show that the universal Baireness of Hom^* sets in $V(\mathbb{R}^*)$ is "absorbed" by the model $\mathcal{M}$ via the F predicate:

Lemma (Lemma 6.1). *Let λ be a limit of Woodin cardinals and a limit of strong cardinals, let $G \subset \mathrm{Col}(\omega, {<}\lambda)$ be a V-generic filter, and define the model $\mathcal{M} = (L^F(\mathbb{R}^*, \mathrm{Hom}^*))^{V(\mathbb{R}^*)}$ where $\mathbb{R}^* = \mathbb{R}^*_G$ and $\mathrm{Hom}^* = \mathrm{Hom}^*_G$. Then every set of reals in Hom^* is universally Baire in $\mathcal{M}$.*

We will prove this lemma in Section 6 by showing that for every set $A \in \mathrm{Hom}^*$ the predicate F puts enough information into the model $\mathcal{M}$ to construct trees witnessing $A \in \mathrm{uB}^{\mathcal{M}}$. We remark that an alternative approach of coding arbitrarily chosen absolutely complementing pairs of trees into the F predicate directly is unlikely to work as then Σ^2_1 reflection is likely to fail.

3. $\mathbb{R}$-genericity iterations

As in Steel's stationary-tower-free proof of Σ^2_1 reflection for derived models [8], our proof of Σ^2_1 reflection for the model $\mathcal{M}$ of the Main Theorem will make fundamental use of the genericity iterations of Neeman [6], repeated ω many times to produce an $\mathbb{R}$-genericity iteration. See also Neeman [7] for a thorough account of the use of $\mathbb{R}$-genericity iterations to prove a special case of the derived model theorem.

The definitions we give below are more or less the usual ones; we include them here to set out the notation that we will use in the rest of the paper.

Definition 3.1. Let P_0 be a countable model of a sufficient fragment of set theory and let $\bar{\lambda}$ be a limit of Woodin cardinals of P_0. An *$\mathbb{R}$-genericity iteration of P_0 at $\bar{\lambda}$* is a sequence in a generic extension of V by $\mathrm{Col}(\omega, \mathbb{R})$ consisting of:

- models P_j for $j \le \omega$ with $P_j \in V$ for $j < \omega$,
- elementary embeddings $i_{jk} : P_j \to P_k$ for $i \le j \le \omega$ with $i_{jk} \in V$ for $i \le j < \omega$,[1]
- Woodin cardinals $\delta_j < i_{0,j}(\bar{\lambda})$ of P_j for $j < \omega$, and
- for all $j < \omega$, a P_{j+1}-generic filter $g_{j+1} \in V$ for the poset $\mathrm{Col}(\omega, i_{j,j+1}(\delta_j))$,

such that

- the maps i_jk commute with each other: $i_{k\ell} \circ i_{jk} = i_{j\ell}$ for $j \le k \le \ell \le \omega$,
- the model P_ω and the maps $i_{j,\omega}$ are obtained as direct limits in the natural way,
- for $j < \omega$ we have $i_{j,j+1}(\delta_j) < \delta_{j+1}$,
- for $j < k \le \omega$ the map $i_{j+1,k}$ has critical point above $i_{j,j+1}(\delta_j)$, so it extends to a map $i^*_{j+1,k} : P_{j+1}[g_{j+1}] \to P_k[g_{j+1}]$,
- every real $x \in V$ is in $P_{j+1}[g_{j+1}]$ for some $j < \omega$,
- the Woodin cardinals $i_{j,\omega}(\delta_j)$ for $j < \omega$ are cofinal in $i_{0,\omega}(\bar{\lambda})$, and
- there is a P_ω-generic filter $g \subset \mathrm{Col}(\omega, {<}i_{0,\omega}(\bar{\lambda}))$ such that $\mathbb{R}^V = \mathbb{R}^*_g$, so that the symmetric submodel $P_\omega(\mathbb{R}^V)$ of $P_\omega[g]$ can be defined.

To ensure that the symmetric model $P_\omega(\mathbb{R}^V)$ resembles V in some sense, we will use $\mathbb{R}$-genericity iterations of countable hulls P_0 of V where the iteration maps factor into the uncollapse map of the hull:

Definition 3.2. Let λ be a limit of Woodin cardinals and let $\pi_0 : P_0 \to V$ be a sufficiently elementary embedding of a countable model P_0 into V such that $\lambda \in \mathrm{ran}(\pi_0)$.

An $\mathbb{R}$-genericity iteration of P_0 at $\pi_0^{-1}(\lambda)$, as in Definition 3.1, is π_0-*realizable* if there are maps $\pi_j : P_j \to V$ for $j \le \omega$ that commute with the iteration maps, meaning that $\pi_k \circ i_{jk} = \pi_j$ for $j \le k \le \omega$, and such that $\pi_j \in V$ for $j < \omega$.

For simplicity we will abuse notation by referring to a π_0-realizable genericity iteration of P_0 at $\pi_0^{-1}(\lambda)$ as a π_0-realizable genericity iteration of P_0 at λ; this should not cause any confusion because λ is a limit of Woodin cardinals and $\pi_0^{-1}(\lambda)$ is a countable ordinal.

For λ, P_0, and π_0 as above, a π_0-realizable $\mathbb{R}$-genericity iteration of P_0 at λ can be shown to exist in $V^{\mathrm{Col}(\omega,\mathbb{R})}$ by using Neeman's genericity iterations ω many times. At stage n we do a genericity iteration of the model P_n, obtaining an iteration tree of length ω, and use a result of Martin and Steel [5, Theorem 3.12] to obtain a cofinal branch whose associated embedding $i_{n,n+1}$ factors into π_n.

One sense in which the models given by realizable $\mathbb{R}$-genericity iterations resemble V is given by the following well-known observation. If C is a ${<}\lambda$-universally Baire set of reals and $\pi_0 : P_0 \to V$ is a sufficiently elementary

[1]The name $\mathbb{R}$-*genericity iteration* suggests that the maps i_{jk} are obtained as iteration maps, but for the purposes of this paper we do not require them to be.

countable hull of V with $(C,\lambda) \in \operatorname{ran}(\pi_0)$, then for every π_0-realizable genericity iteration $P_0 \to P_\omega$ at λ we have $C \in P_\omega(\mathbb{R}^V)$. This is because $\operatorname{ran}(\pi_0)$ contains a $<\lambda$-absolutely complementing pair of trees (S,T) for C and we have $p[\pi_\omega^{-1}(S)] \subset p[S]$ and $p[\pi_\omega^{-1}(T)] \subset p[T]$, so the canonical expansion of $\pi_\omega^{-1}(C)$ to $P_\omega(\mathbb{R}^V)$ is equal to C itself. This is a key point, so it bears repeating:

$$(\pi_\omega^{-1}(C))^{P_\omega(\mathbb{R}^V)} = C.$$

The following lemma is implicit in Neeman [7] and Steel [8]. Its proof uses Windszus's theorem to show that the set A in question is $<\lambda$-weakly homogeneously Suslin, and then uses the main theorem from the Martin–Steel proof of projective determinacy [4, Theorem 5.11] to show that it is $<\lambda$-homogeneously Suslin.

Lemma 3.3. *Let λ be a limit of Woodin cardinals and let C be a $<\lambda$-universally Baire set of reals. Let $\pi_0 : P_0 \to V$ be a sufficiently elementary countable hull of V with $(C,\lambda) \in \operatorname{ran}(\pi_0)$. Suppose that A is a set of reals in V and there is a formula ψ such that for every π_0-realizable $\mathbb{R}$-genericity iteration $P_0 \to P_\omega$ in $V^{\operatorname{Col}(\omega,\mathbb{R})}$ and every real $x \in V$ we have*

$$x \in A \iff P_\omega(\mathbb{R}^V) \models \psi[C,x].$$

Then $A \in \operatorname{Hom}_{<\lambda}$.

Note that the special case $A = C$ of the lemma shows that every $<\lambda$-universally Baire set of reals is $<\lambda$-homogeneously Suslin, a result whose original proof used the stationary tower in place of genericity iterations; see Larson [3, Theorem 3.3.8].

4. Absoluteness of the F predicate

Although the Main Theorem is only concerned with the predicate F as it is defined in the symmetric model $V(\mathbb{R}^*)$, the proof of Σ_1^2 reflection will need to consider the predicate as it is defined in other models (namely V and $\mathbb{R}$-genericity iterates of countable hulls of V) and show that the definition satisfies a certain absoluteness property between these models (Lemma 4.5 below.)

First we will need a lemma that shows that the universally Baire sets added generically over a countable model that embeds into V are closely related to universally Baire sets in V. We recall some basic facts about trees of measures (also known as homogeneity systems) that will be used to prove the lemma.

A *tree of measures* (on an ordinal γ, the choice of which will usually not concern us) is a family $(\mu_s : s \in \omega^{<\omega})$ of measures such that each measure μ_s concentrates on the set $\gamma^{|s|}$ and such that μ_t projects to μ_s whenever t extends s. Given a tree $\vec{\mu} = (\mu_s : s \in \omega^{<\omega})$ of κ-complete measures where

κ is an uncountable cardinal, for every real x we let $\vec{\mu}_x$ denote the tower of measures $(\mu_{x\restriction n} : n < \omega)$. We define the set of reals

$$\mathbf{S}_{\vec{\mu}} = \{x \in \omega^\omega : \vec{\mu}_x \text{ is well-founded}\}.$$

A set of reals A is *κ-homogeneous* if $A = \mathbf{S}_{\vec{\mu}}$ for some tree $\vec{\mu}$ of κ-complete measures.

By an observation of Woodin (see Steel [9, Proposition 2.5],) every tree $\vec{\mu}$ of κ-complete measures on γ is a κ-homogeneity system for some tree T on $\omega \times \gamma$, meaning that each measure μ_s concentrates on the set $T \cap \gamma^{|s|}$ and for every real $x \in p[T]$ the tower $\vec{\mu}_x$ is well-founded. For every real $x \notin p[T]$ then the tower $\vec{\mu}_x$ must be ill-founded, so $S_{\vec{\mu}} = p[T]$.

By the Levy–Solovay theorem, for every generic extension $V[g]$ of V by a poset of size less than κ, every κ-complete $\mu \in V$ induces a corresponding measure $\hat{\mu}$ in $V[g]$ and the ultrapower of the ordinals by μ in V agrees with the ultrapower of the ordinals by $\hat{\mu}$ in $V[g]$. (Henceforth, we will denote both measures by μ where it will not cause confusion.) So we can also define the set $\mathbf{S}_{\vec{\mu}}$ in small generic extensions, and we have $(\mathbf{S}_{\vec{\mu}})^{V[g]} \cap V = (\mathbf{S}_{\vec{\mu}})^V$.

Given a tree $\vec{\mu}$ of κ-complete measures, the Martin–Solovay tree $\mathrm{ms}(\vec{\mu})$ is the tree of attempts to build a real x and a sequence of ordinals witnessing the ill-foundedness of the corresponding tower $\vec{\mu}_x$ (see Larson [3] for a precise definition.) The Martin–Solovay tree has the key property that

$$\mathbf{S}_{\vec{\mu}} = \omega^\omega \setminus p[\mathrm{ms}(\vec{\mu})].$$

The definition of the Martin–Solovay tree is absolute to small generic extensions $V[g]$, so we have $\mathbf{S}_{\vec{\mu}} = \omega^\omega \setminus p[\mathrm{ms}(\vec{\mu})]$ in $V[g]$ as well. Another property of the Martin–Solovay tree that we will use is the fact that for every $n < \omega$ the subtree $\mathrm{ms}(\vec{\mu}) \restriction n$ consisting of the first n levels of $\mathrm{ms}(\vec{\mu})$ is determined by finitely many measures of $\vec{\mu}$.

For any tree T on γ for which the tree of κ-complete measures $\vec{\mu}$ is a homogeneity system, the pair $(T, \mathrm{ms}(\vec{\mu}))$ is $<\kappa$-absolutely complementing. This implies that every κ-homogeneous set of reals $A = \mathbf{S}_{\vec{\mu}}$ is $<\kappa$-universally Baire and that whenever $V[g]$ is a generic extension of V by a poset of size less than κ we have $A^{V[g]} = (\mathbf{S}_{\vec{\mu}})^{V[g]}$. In other words, the canonical extension of A given by κ-homogeneity agrees with that given by $<\kappa$-universal Baireness.

If κ is a cardinal and δ_0 and δ_1 are Woodin cardinals with $\kappa < \delta_0 < \delta_1$, then every δ_1^+-universally Baire set of reals is κ-homogeneously Suslin by theorems of Woodin and Martin–Steel (see Steel [9] or Larson [3] for a discussion of this result.) In particular, if λ is a limit of Woodin cardinals then every $<\lambda$-universally Baire set of reals is $<\lambda$-homogenously Suslin. (The converse is always true by the Martin–Solovay construction.)

In the presence of Woodin cardinals, the pointclass of homogeneously Suslin sets is closed under complementation in a strong sense. Namely, if δ is a Woodin cardinal, Y is a set of δ^+-complete measures with $|Y| < \delta$, and $\kappa < \delta$ is a cardinal, then by Steel [8, Lemma 2.1] there is a “tower-flipping” function f that associates to every finite tower $(\rho_0, \ldots, \rho_{n-1})$ of measures from Y a finite tower $f((\rho_0, \ldots, \rho_{n-1}))$ of κ-complete measures with the following properties. First, f respects extensions of finite towers, so that for every infinite tower $\vec{\rho}$ of measures from Y it continuously associates an infinite tower $\bigcup_{n<\omega} f(\vec{\rho} \restriction n)$ of κ-complete measures. Second, this associated tower $\bigcup_{n<\omega} f(\vec{\rho} \restriction n)$ is well-founded if and only if the given tower $\vec{\rho}$ is ill-founded. Third, this property of the tower-flipping function continues to hold in all generic extensions of size less than κ.

The following lemma shows that in the presence of a Woodin cardinal, the Martin–Solovay construction can be used to give absolutely complementing trees of a nice form that we will later use to prove an absoluteness result for the F predicate. A similar idea appears in the proof of Steel [8, Theorem 2.2, Subclaim 1.1].

Lemma 4.1. *Let $\pi : P \to V$ be a sufficiently elementary embedding from a countable transitive set P. Let $\bar{\alpha}$ be a cardinal of P and let $\bar{\delta}$ be a Woodin cardinal of P with $\bar{\alpha} < \bar{\delta}$. Let $g \subset \mathrm{Col}(\omega, \bar{\alpha})$ be a P-generic filter in V and let A be a $\bar{\delta}^+$-homogeneously Suslin set of reals in $P[g]$.*

Then for every cardinal $\bar{\kappa}$ of P with $\bar{\alpha} < \bar{\kappa} < \bar{\delta}$ there is a $<\bar{\kappa}$-absolutely complementing pair of trees $(\bar{S}, \bar{T}) \in P[g]$ for A such that for every $n < \omega$ the restrictions $\bar{S} \restriction n$ and $\bar{T} \restriction n$ to finite levels[2] are in P, and defining the trees $S = \bigcup_{n<\omega} \pi(\bar{S} \restriction n)$ and $T = \bigcup_{n<\omega} \pi(\bar{T} \restriction n)$, the pair $(S, T) \in V$ is $<\pi(\bar{\kappa})$-absolutely complementing.

Proof. Let $A = \mathbf{S}_{\vec{\mu}}$ where $\vec{\mu}$ is a tree of $\bar{\delta}^+$-complete measures in $P[g]$. By the Levy–Solovay theorem each measure μ_s is induced by the $\bar{\delta}^+$-complete measure $\mu_s \cap V \in V$, which we will also denote by μ_s. In P we can take a set of $\bar{\delta}^+$-complete measures Y of size $\bar{\alpha}$ such that every measure μ_s of $\vec{\mu}$ is in Y (or more precisely, is induced by a measure in Y.)

Given a cardinal $\bar{\kappa}$ of P with $\bar{\alpha} < \bar{\kappa} < \bar{\delta}$, let $f \in P$ be a tower-flipping function that continuously associates to every tower of measures from Y a tower of $\bar{\kappa}$-complete measures. In $P[g]$, define the tree of $\bar{\kappa}$-complete measures $\vec{\nu} = (\nu_s : s \in \omega^{<\omega})$ that is continuously associated to $\vec{\mu}$ by f in the sense that $(\nu_{s\restriction n} : n \leq |s|) = f((\mu_{s\restriction n} : n \leq |s|))$ for every finite sequence $s \in \omega^{<\omega}$. So in $P[g]$ we have $\mathbf{S}_{\vec{\nu}} = \omega^\omega \setminus \mathbf{S}_{\vec{\mu}}$ by the tower-flipping property of f. Moreover, $\pi(f)$ is a tower-flipping function in V by the elementarity of π, so we have $\mathbf{S}_{\pi``\vec{\nu}} = \omega^\omega \setminus \mathbf{S}_{\pi``\vec{\mu}}$ in V.

Now the Martin–Solovay construction gives us the trees we want. More precisely, in $P[g]$ we define the Martin–Solovay trees $\bar{S} = \mathrm{ms}(\vec{\mu})$ and $\bar{T} = \mathrm{ms}(\vec{\nu})$, and in V we define the Martin–Solovay trees $S = \mathrm{ms}(\pi``\vec{\mu})$ and $T = \mathrm{ms}(\pi``\vec{\nu})$. Because the first n levels of the Martin–Solovay tree are determined by finitely many measures in the ground model we have $(\bar{S} \restriction n, \bar{T} \restriction n) \in P$ for all $n < \omega$. For the same reason, we have $S = \bigcup_{n<\omega} \pi(\bar{S} \restriction n)$ and $T = \bigcup_{n<\omega} \pi(\bar{T} \restriction n)$.

[2]For a tree T and a natural number n we let $T \restriction n$ be the subset of T consisting of nodes of length less than n

In the model $P[g]$ the pair $(\bar{S}, \bar{T})$ is $<\bar{\kappa}$-absolutely complementing: this follows from the fact that $p[\bar{S}] = \omega^\omega \setminus p[\bar{T}]$ (in $P[g]$) and the fact that $\bar{S}$ and $\bar{T}$, being Martin–Solovay trees, are each $<\bar{\kappa}$-absolutely complemented. A similar argument shows that in V the pair (S, T) is $<\pi(\bar{\kappa})$-absolutely complementing, as desired. □

Using a strong cardinal, we can strengthen the lemma to give any desired degree of absolute complementation.

Lemma 4.2. *Let $\pi : P \to V$ be a sufficiently elementary embedding from a countable transitive set P. Let $\bar{\alpha}$ be a cardinal of P and let $\bar{\delta}$ be a Woodin cardinal of P with $\bar{\alpha} < \bar{\delta}$. Let $g \subset \mathrm{Col}(\omega, \bar{\alpha})$ be a P-generic filter in V and let A be a $\bar{\delta}^+$-homogeneously Suslin set of reals in $P[g]$.*

Let $\bar{\kappa}$ and $\bar{\eta}$ be cardinals of P with $\bar{\alpha} < \bar{\kappa} < \bar{\delta} < \bar{\eta}$ and such that $\bar{\kappa}$ is $\bar{\eta}$-strong in P. Then there is an $\bar{\eta}$-absolutely complementing pair of trees $(\bar{S}, \bar{T}) \in P[g]$ for A such that for every $n < \omega$ the restrictions $\bar{S} \restriction n$ and $\bar{T} \restriction n$ to finite levels are in P, and defining the trees $S = \bigcup_{n<\omega} \pi(\bar{S} \restriction n)$ and $T = \bigcup_{n<\omega} \pi(\bar{T} \restriction n)$, the pair $(S, T) \in V$ is $\pi(\bar{\eta})$-absolutely complementing.

Proof. By Lemma 4.1 there is a $<\bar{\kappa}$-absolutely complementing pair of trees $(\bar{S}_0, \bar{T}_0)$ in $P[g]$ for A such that for every $n < \omega$ the restrictions $\bar{S}_0 \restriction n$ and $\bar{T}_0 \restriction n$ to finite levels are in P, and defining the trees $S_0 = \bigcup_{n<\omega} \pi(\bar{S}_0 \restriction n)$ and $T_0 = \bigcup_{n<\omega} \pi(\bar{T}_0 \restriction n)$, the pair $(S_0, T_0) \in V$ is $<\pi(\bar{\kappa})$-absolutely complementing.

In P, let $\bar{E}$ be an extender witnessing that $\bar{\kappa}$ is $\bar{\eta}$-strong. The ultrapower map $j_{\bar{E}} : P \to \mathrm{Ult}(P, \bar{E})$ extends canonically to a map $P[g] \to \mathrm{Ult}(P[g], \bar{E})$, which we will also denote by $j_{\bar{E}}$. Also in V the extender $E = \pi(\bar{E})$ witnesses that $\pi(\bar{\kappa})$ is $\pi(\bar{\eta})$-strong, and we have an ultrapower map $j_E : V \to \mathrm{Ult}(V, E)$.

Letting $\bar{S} = j_{\bar{E}}(\bar{S}_0)$ and $\bar{T} = j_{\bar{E}}(\bar{T}_0)$, the pair of trees $(\bar{S}, \bar{T}) \in P[g]$ is an $\bar{\eta}$-absolutely complementing pair for A. Similarly, letting $S = j_E(S_0)$ and $T = j_E(T_0)$, the pair of trees $(S, T) \in V$ is a $\pi(\bar{\eta})$-absolutely complementing pair of trees. Because $\pi \circ j_{\bar{E}} = j_E \circ \pi$, we have $S = \bigcup_{n<\omega} \pi(\bar{S} \restriction n)$ and $T = \bigcup_{n<\omega} \pi(\bar{T} \restriction n)$ as desired. □

Applying Lemma 4.2 to $\mathbb{R}$-genericity iterations yields the following result. Part 1 below can also be derived as an immediate consequence of Steel [8, Subclaim 1.1], which applies more generally to any limit λ of Woodin cardinals. However, in our situation we can use the fact that λ is a limit of λ-strong cardinals to give a proof of part 1 using Lemma 4.2 instead. Part 2 below will then follow from a similar argument using the fact that λ is a limit of (fully) strong cardinals.

Lemma 4.3. *Let λ be a limit of Woodin cardinals and a limit of strong cardinals. Let $\pi_0 : P_0 \to V$ be a sufficiently elementary embedding from a countable transitive set P_0 with $\lambda \in \mathrm{ran}(\pi_0)$. Let P_ω be obtained from P_0 by a π_0-realizable $\mathbb{R}$-genericity iteration at λ. Let $A \in \mathrm{uB}^{P_\omega(\mathbb{R}^V)}$ be a set of reals. Then we have*

(1) $A \in \mathrm{uB}^V$*, and*
(2) for every set $Z \in P_\omega(\mathbb{R}^V) \cap V$*, every condition* $p \in \mathrm{Col}(\omega, Z)$*, and every* $\mathrm{Col}(\omega, Z)$*-name* $\dot{x} \in P_\omega(\mathbb{R}^V) \cap V$ *for a real, the statement "p forces* $\dot{x}$ *to be in the canonical expansion of A" is absolute between* $P_\omega(\mathbb{R}^V)$ *and* V*.*

Proof. In terms of the notation for genericity iterations given in Definition 3.1, the set $A \in \mathrm{uB}^{P_\omega(\mathbb{R}^V)}$ comes from a set $A_j \in \mathrm{uB}^{P_j[g_j]}$ appearing at some stage $j < \omega$ in the sense that $A = (i^*_{j,\omega}(A_j))^{P_\omega(\mathbb{R}^V)}$ where the map $i^*_{j,\omega} : P_j[g_j] \to P_\omega[g_j]$ is defined as the canonical extension of the map $i_{j,\omega} : P_j \to P_\omega$. Define $P = P_j$, $\pi = \pi_j$, $g = g_j$, $A = A_j$. Let $\bar{\alpha} = \delta_j$, let $\bar{\kappa}$ be the least strong cardinal of P_j above δ_j, and let $\bar{\delta}$ be the least Woodin cardinal of P_j above $\bar{\kappa}$.

To prove part 1, apply Lemma 4.2 with $\bar{\eta} = \pi_j^{-1}(\lambda)$. This gives us a pair of $\pi_j^{-1}(\lambda)$-absolutely complementing trees $(\bar{S}, \bar{T}) \in P_j[g_j]$ for A_j such that for every $n < \omega$ the restrictions $\bar{S} \restriction n$ and $\bar{T} \restriction n$ to finite levels are in P_j, and defining the trees $S = \bigcup_{n<\omega} \pi_j(\bar{S} \restriction n)$ and $T = \bigcup_{n<\omega} \pi_j(\bar{T} \restriction n)$, the pair (S, T) is λ-absolutely complementing.

The realization map π_ω takes branches of $i^*_{j,\omega}(\bar{S})$ and $i^*_{j,\omega}(\bar{T})$ pointwise to branches of S and T respectively. More precisely, we have $\pi_\omega``(i^*_{j,\omega}(\bar{S})) \subset S$ and $\pi_\omega``(i^*_{j,\omega}(\bar{T})) \subset T$, so the projections of these trees satisfy the inclusions $p[i^*_{j,\omega}(\bar{S})] \subset p[S]$ and $p[i^*_{j,\omega}(\bar{T})] \subset p[T]$. Because $A = p[i^*_{j,\omega}(\bar{S})] = \omega^\omega \setminus p[i^*_{j,\omega}(\bar{T})]$, these two inclusions imply that $A = p[S] = \omega^\omega \setminus p[T]$. Therefore A is in uB^V_λ, which is equal to uB^V.

For part 2, note that our genericity iteration was formed in some generic extension of V by the poset $\mathrm{Col}(\omega, \mathbb{R})$, and the resulting model $P_\omega(\mathbb{R}^V)$ is countable there. In particular the set Z is countable there. So in a generic extension of V by the poset $\mathrm{Col}(\omega, \mathbb{R}) \times \mathrm{Col}(\omega, \omega)$, which is equivalent to $\mathrm{Col}(\omega, \mathbb{R})$, we can take a filter $H \subset \mathrm{Col}(\omega, Z)$ that contains the condition p and is both V-generic and $P_\omega(\mathbb{R}^V)$-generic. Let $x = \dot{x}_H$. We want to see that the two ways of expanding the set A, given by $A \in \mathrm{uB}^{P_\omega(\mathbb{R}^V)}$ and $A \in \mathrm{uB}^V$ respectively, agree on whether x is a member.[3]

By Lemma 2.4, in the model $P_\omega(\mathbb{R}^V)$ there is an ordinal $\eta_\omega \geq \pi_\omega^{-1}(\lambda)$ such that every η_ω-absolutely complementing set of trees is Z-absolutely complementing. Increasing j if necessary, we may assume that $\eta_\omega = i_{j,\omega}(\bar{\eta})$ for some ordinal $\bar{\eta} \in P_j$.

As in the proof of part 1, we can apply Lemma 4.2 to get a pair of $\bar{\eta}$-absolutely complementing trees $(\bar{S}, \bar{T}) \in P_j[g_j]$ for A_j such that for every $n < \omega$ the restrictions $\bar{S} \restriction n$ and $\bar{T} \restriction n$ to finite levels are in P_j, and defining the trees $S = \bigcup_{n<\omega} \pi_j(\bar{S} \restriction n)$ and $T = \bigcup_{n<\omega} \pi_j(\bar{T} \restriction n)$, the pair (S, T) is η-absolutely complementing where $\eta = \pi_j(\bar{\eta})$. In particular the pair

[3]This is trivial if $P_\omega(\mathbb{R}^V) \in V$, but probably the condition $P_\omega(\mathbb{R}^V) \in V$ may fail in general, which is why we seem to need Lemma 4.2.

(S,T) is $\mathbb{R}$-absolutely complementing. Moreover the pair $(i^*_{j,\omega}(\bar{S}), i^*_{j,\omega}(\bar{T}))$ is η_ω-absolutely complementing in $P_\omega[g_j]$, or equivalently in $P_\omega(\mathbb{R}^V)$, and so it follows from our choice of η_ω that it is Z-absolutely complmenting in $P_\omega(\mathbb{R}^V)$.

We can use the pair of trees $(i^*_{j,\omega}(\bar{S}), i^*_{j,\omega}(\bar{T}))$ to decide membership of x in the canonical expansion of A from the point of view of $P_\omega(\mathbb{R}^V)$, and use the pair of trees (S,T) to decide membership of x in the canonical expansion of A from the point of view of V. As in the proof of part 1, we get the same answer in both cases because the realization map π_ω takes branches of $i^*_{j,\omega}(\bar{S})$ pointwise to branches of S, and takes branches of $i^*_{j,\omega}(\bar{T})$ pointwise to branches of T. □

Lemma 4.3 can be used to show that

$$(L^F(\mathbb{R}, \mathrm{uB}))^{P_\omega(\mathbb{R}^V)} \subset (L^F(\mathbb{R}, \mathrm{uB}))^V.$$

Rather than proving this inclusion, we prove a stronger statement that is phrased in terms of the following definition.

Definition 4.4. In any model of ZF satisfying Wadge determinacy for the pointclass uB, and for any ordinal α less than or equal to the Wadge rank of the pointclass uB, define the pointclass

$$\mathrm{uB} \restriction \alpha = \{A \in \mathrm{uB} : |A|_{\mathrm{W}} < \alpha\},$$

and define the predicate F_α to consist of all quadruples $(A, Z, p, \dot{x})$ such that

- $A \in \mathrm{uB} \restriction \alpha$, Z is a set, p is a condition in the collapse forcing $\mathrm{Col}(\omega, Z)$, $\dot{x}$ is a $\mathrm{Col}(\omega, Z)$-name for a real, and
- p forces $\dot{x}$ to be in the canonical expansion of A to the generic extension.

For example, if α is the Wadge rank of the pointclass uB, then we simply have $\mathrm{uB} \restriction \alpha = \mathrm{uB}$ and $F_\alpha = F$. In terms of this definition we can state an immediate consequence of Lemma 4.3:

Lemma 4.5. *Let λ be a limit of Woodin cardinals and a limit of strong cardinals. Let $\pi_0 : P_0 \to V$ be a sufficiently elementary embedding from a countable transitive set P_0 with $\lambda \in \mathrm{ran}(\pi_0)$. Let P_ω be obtained from P_0 by a π_0-realizable $\mathbb{R}$-genericity iteration at λ. Then we have*

$$(L^F(\mathbb{R}, \mathrm{uB}))^{P_\omega(\mathbb{R}^V)} = (L^{F_\alpha}_\beta(\mathbb{R}, \mathrm{uB} \restriction \alpha))^V,$$

where α is the Wadge rank of the pointclass $\mathrm{uB}^{P_\omega(\mathbb{R}^V)}$ and β is the ordinal height of the model P_ω.

5. Proof of Σ^2_1 reflection

In this section we prove the following Σ^2_1 reflection result. The proof resembles the stationary-tower-free proof of Σ^2_1 reflection for the derived model given by Steel [8].

Lemma 5.1 (Σ^2_1 Reflection)**.** *Let λ be a limit of Woodin cardinals and a limit of strong cardinals, let $G \subset \mathrm{Col}(\omega, {<}\lambda)$ be a V-generic filter, and define the model $\mathcal{M} = (L^F(\mathbb{R}^*, \mathrm{Hom}^*))^{V(\mathbb{R}^*)}$ where $\mathbb{R}^* = \mathbb{R}^*_G$ and $\mathrm{Hom}^* = \mathrm{Hom}^*_G$. For every sentence φ, if there is a set of reals $A \in \mathcal{M}$ such that $(\mathrm{HC}^*, \in, A) \models \varphi$, then there is a set of reals $A \in \mathrm{Hom}^V_{<\lambda}$ such that $(\mathrm{HC}, \in, A) \models \varphi$.*

Recall that $\mathrm{Hom}^* = \mathrm{uB}^{V(\mathbb{R}^*)}$ because λ is a limit of strong cardinals, so we have an alternative characterization of $\mathcal{M}$ that will be useful in this section:

$$\mathcal{M} = (L^F(\mathbb{R}^*, \mathrm{uB}))^{V(\mathbb{R}^*)}.$$

Assume that the model $\mathcal{M}$ has a φ-witness. The idea of the proof is to take a countable hull of V containing a φ-witness, and then to do an $\mathbb{R}$-genericity iteration of the hull to get a φ-witness that is a subset of $\mathbb{R}^V$. More specifically, by the elementarity of the map $\pi_\omega : P_\omega \to V$ (and the definability of forcing) there is a φ-witness in the model $(L^F(\mathbb{R}, \mathrm{uB}))^{P_\omega(\mathbb{R}^V)}$, which is equal to the model $(L^{F_\alpha}_\beta(\mathbb{R}, \mathrm{uB} \restriction \alpha))^V$ by Lemma 4.5 where α is the Wadge rank of the pointclass $\mathrm{uB}^{P_\omega(\mathbb{R}^V)}$ and β is the ordinal height of the model P_ω. In particular, there is a φ-witness in the model $(L^F(\mathbb{R}, \mathrm{uB}))^V$.

We will show that the "least" φ-witness arising in this manner is in $\mathrm{Hom}_{<\lambda}$. To do this, we will consider different countable hulls P_0 and different genericity iterations of these hulls giving rise to possibly different models $(L^F(\mathbb{R}, \mathrm{uB}))^{P_\omega(\mathbb{R}^V)}$. We will see that the "least" φ-witness will be present in all of these models and will admit a "uniform definition" there in the sense of Lemma 3.3.

We consider two cases involving two different definitions of "least" φ-witness. The cases are exhaustive but not mutually exclusive; if they overlap, either one can be used.

Case 1. Assume that for stationary many (or just cofinally many sufficiently elementary) countable hulls $\pi_0 : P_0 \to V$ with $\lambda \in \mathrm{ran}(\pi_0)$, every π_0-realizable $\mathbb{R}$-genericity iteration $P_0 \to P_\omega$ at λ has the property that $\mathrm{uB}^{P_\omega(\mathbb{R}^V)} = \mathrm{uB}^V$.

In this case, let β be the least ordinal such that the model $(L^F_\beta(\mathbb{R}, \mathrm{uB}))^V$ has a φ-witness. Take a set of reals $C \in \mathrm{uB}^V$ such that the model $(L^F_\beta(\mathbb{R}, \mathrm{uB}))^V$ has a φ-witness that is ordinal-definable from the parameter C and the predicate F, and let $A \in (L^F_\beta(\mathbb{R}, \mathrm{uB}))^V$ be the least such φ-witness in the canonical well-ordering.

Take a countable hull $\pi_0 : P_0 \to V$ such that $(C, \lambda) \in \mathrm{ran}(\pi_0)$ and for every π_0-realizable $\mathbb{R}$-genericity iteration $P_0 \to P_\omega$ at λ we have $\mathrm{uB}^{P_\omega(\mathbb{R}^V)} = \mathrm{uB}^V$. For every such $\mathbb{R}$-genericity iteration we have $\beta \in P_\omega$ by the minimality of β; otherwise the model $(L^F(\mathbb{R}, \mathrm{uB}))^{P_\omega(\mathbb{R}^V)}$ would not be tall enough to reach a φ-witness, violating the elementarity of π_ω. Moreover, we recall that the set C is in $P_\omega(\mathbb{R}^V)$ because it is the canonical expansion of the set $\pi_\omega^{-1}(C) \in P_\omega$.

For every π_0-realizable $\mathbb{R}$-genericity iteration $P_0 \to P_\omega$ at λ, the ordinal β is definable in $P_\omega(\mathbb{R}^V)$ as the least ordinal such that the model

$(L_\beta^F(\mathbb{R}, \mathrm{uB}))^{P_\omega(\mathbb{R}^V)}$ contains a φ-witness, and in turn the least φ-witness A is definable from C in the model $(L_\beta^F(\mathbb{R}, \mathrm{uB}))^{P_\omega(\mathbb{R}^V)}$ by the same definition as in the model $(L_\beta^F(\mathbb{R}, \mathrm{uB}))^V$, which is the same model. This implies that $A \in \mathrm{Hom}_{<\lambda}$ by Lemma 3.3 as desired.

Case 2. Assume that for stationary many (or just cofinally many sufficiently elementary) countable hulls $\pi_0 : P_0 \to V$ with $\lambda \in \mathrm{ran}(\pi_0)$, there is a π_0-realizable $\mathbb{R}$-genericity iteration $P_0 \to P_\omega$ at λ such that $\mathrm{uB}^{P_\omega(\mathbb{R}^V)} \subsetneq \mathrm{uB}^V$.

In this case we will need the following claim, which says that we can put an upper bound on the pointclass necessary to construct a φ-witness.

Claim. *Under the assumption of Case 2, there is a set of reals $D \in \mathrm{uB}^V$ such that, letting D^* denote the canonical expansion $D^{V(\mathbb{R}^*)}$ and letting α^* denote the Wadge rank of D^* in $V(\mathbb{R}^*)$, the model*

$$(L^{F_{\alpha^*}}(\mathbb{R}^*, \mathrm{uB} \restriction \alpha^*))^{V(\mathbb{R}^*)}$$

contains a φ-witness.

Proof. Take a sufficiently elementary countable hull $\pi_0 : P_0 \to V$ with $\lambda \in \mathrm{ran}(\pi_0)$ and take a π_0-realizable $\mathbb{R}$-genericity iteration $P_0 \to P_\omega$ at λ such that $\mathrm{uB}^{P_\omega(\mathbb{R}^V)} \subsetneq \mathrm{uB}^V$. Then by Lemma 4.5 and the elementarity of the realization map $\pi_\omega : P_\omega \to V$ there is a φ-witness in the model $(L_\beta^{F_\alpha}(\mathbb{R}, \mathrm{uB} \restriction \alpha))^V$ where α is the Wadge rank of the pointclass $\mathrm{uB}^{P_\omega(\mathbb{R}^V)}$ and $\beta = \mathrm{Ord}^{P_\omega}$.

Take a set of reals $D \in \mathrm{uB}^V$ of Wadge rank α, take a sufficiently elementary countable hull $\pi_0' : P_0' \to V$ with $(D, \lambda) \in \mathrm{ran}(\pi_0')$, and take a π_0'-realizable $\mathbb{R}$-genericity iteration $P_0' \to P_\omega'$ at λ such that $\mathrm{uB}^{P_\omega'(\mathbb{R}^V)} \subsetneq \mathrm{uB}^V$.

The model $P_\omega'(\mathbb{R}^V)$ sees the set D as the canonical expansion of $\pi_\omega^{-1}(D)$ and it sees the pointclass $\mathrm{uB}^V \restriction \alpha$ as its collection of uB sets of Wadge rank less than that of D. If we have $\mathrm{Ord}^{P_\omega} \leq \mathrm{Ord}^{P_\omega'}$ then the model $P_\omega'(\mathbb{R}^V)$ is tall enough to see that a φ-witness can be built from $\mathrm{uB}^V \restriction \alpha$ using the F predicate, and the claim follows from the elementarity of the realization map $\pi_\omega' : P_\omega' \to V$.

On the other hand, if $\mathrm{Ord}^{P_\omega} > \mathrm{Ord}^{P_\omega'}$ then we can repeat the argument with the hull $\pi_0' : P_0' \to V$ in place of the hull $\pi_0 : P_0 \to V$. (This is why we made sure to take the second hull with $\mathrm{uB}^{P_\omega'(\mathbb{R}^V)} \subsetneq \mathrm{uB}^V$ also.) We can repeat the argument any finite number of times, so either the claim holds for some set $D \in \mathrm{uB}^V$, or we get an infinite decreasing sequence of ordinals $\mathrm{Ord}^{P_\omega} > \mathrm{Ord}^{P_\omega'} > \mathrm{Ord}^{P_\omega''} > \cdots$, a contradiction. □

Now we can proceed as in Case 1. Let $D \in \mathrm{uB}^V$ be a set as in the claim. Taking any countable hull $\pi_0 : P_0 \to V$ with $D \in \mathrm{ran}(\pi_0)$ and taking any genericity iterate $P_0 \to P_\omega$ of this hull, by the elementarity of the factor map π_ω and the proof of Lemma 4.5 we see that there is a φ-witness in the model $(L_\beta^{F_\alpha}(\mathbb{R}, \mathrm{uB} \restriction \alpha))^V$ where $\alpha = |D|_{\mathrm{W}}$ and β is some ordinal in

P_ω. Let β be the least ordinal such that there is a φ-witness in the model $(L_\beta^{F_\alpha}(\mathbb{R}, \mathrm{uB} \restriction \alpha))^V$.

Take a set of reals $C \in \mathrm{uB}^V \restriction \alpha$ such that the model $(L_\beta^{F_\alpha}(\mathbb{R}, \mathrm{uB} \restriction \alpha))^V$ has a φ-witness that is ordinal-definable from the parameter C and the predicate F, and let $A \in (L_\beta^{F_\alpha}(\mathbb{R}, \mathrm{uB} \restriction \alpha))^V$ be the least such φ-witness in the canonical well-ordering.

Take a countable hull $\pi_0 : P_0 \to V$ with $(\lambda, C, D) \in \mathrm{ran}(\pi_0)$. For every $\mathbb{R}$-genericity iteration $P_0 \to P_\omega$ we have $\beta \in P_\omega$ by the minimality of β; otherwise the model $(L^{F_\alpha}(\mathbb{R}, \mathrm{uB} \restriction \alpha))^{P_\omega(\mathbb{R}^V)}$ would not be tall enough to reach a φ-witness, violating the elementarity of π_ω.

For every such genericity iteration, the ordinal α is definable from the set D in $P_\omega(\mathbb{R}^V)$ as its Wadge rank, the ordinal β is definable from α in $P_\omega(\mathbb{R}^V)$ as the least ordinal such that the model $(L_\beta^{F_\alpha}(\mathbb{R}, \mathrm{uB} \restriction \alpha))^{P_\omega(\mathbb{R}^V)}$ contains a φ-witness, and in turn the least φ-witness A is definable from C in the model $(L_\beta^{F_\alpha}(\mathbb{R}, \mathrm{uB} \restriction \alpha))^{P_\omega(\mathbb{R}^V)}$ by the same definition as in the model $(L_\beta^{F_\alpha}(\mathbb{R}, \mathrm{uB} \restriction \alpha))^V$,which is the same model. This implies that $A \in \mathrm{Hom}_{<\lambda}$ by Lemma 3.3 as desired.

6. Building trees from the F predicate

In this section we will prove the last remaining component required for the proof of the main theorem:

Lemma 6.1. *Let λ be a limit of Woodin cardinals and a limit of strong cardinals, let $G \subset \mathrm{Col}(\omega, {<}\lambda)$ be a V-generic filter, and define the model $\mathcal{M} = (L^F(\mathbb{R}^*, \mathrm{Hom}^*))^{V(\mathbb{R}^*)}$ where $\mathbb{R}^* = \mathbb{R}^*_G$ and $\mathrm{Hom}^* = \mathrm{Hom}^*_G$. Then every set of reals in Hom^* is universally Baire in $\mathcal{M}$.*

Let $A^* \in \mathrm{Hom}^*$ and let Z be a set in $\mathcal{M}$. By Lemma 2.4 and the discussion preceding it, there is a Z-absolutely complementing pair of trees for A^* in $V(\mathbb{R}^*)$. We will use information that is fed into $\mathcal{M}$ by the predicate F to construct the desired $\mathrm{Col}(\omega, Z)$-absolutely complementing trees for A^* in $\mathcal{M}$.

For convenience, we may assume by passing to a small forcing extension of V that $A^* = A^{V(\mathbb{R}^*)}$ for some set of reals $A \in \mathrm{Hom}^V_{<\lambda}$.

We begin by noting that, letting τ be the standard $\mathrm{Col}(\omega, Z)$-name in $V(\mathbb{R}^*)$ for the canonical expansion of A^*, the restriction $\tau \cap \mathcal{M}$ is in $\mathcal{M}$ because it is coded into the predicate F. So the model $\mathcal{M}$ knows a way to expand the set A^* to its generic extensions. However, this is not enough—we need to see that this way of expanding is given by a pair of Z-absolutely complementing trees in $\mathcal{M}$. To do this, we will use not only information about how to expand A^*, but also information about how to expand semiscales on A^* and its complement.

A thorough introduction to semiscales is given by Kechris and Moschovakis [2]. A key fact is that for any set of reals B, there is a semiscale on B if and only if there is a tree on $\omega \times \mathrm{Ord}$ projecting to B (that is, B is

Suslin.) The advantage of working with semiscales rather than Suslin representations is that they can be dealt with in terms of descriptive-set-theoretic complexity.

By Steel [9, Theorem 5.3], every $\mathrm{Hom}_{<\lambda}$ set has a $\mathrm{Hom}_{<\lambda}$-semiscale,[4] meaning that the sequence of prewellorderings of the semiscale is coded by a $\mathrm{Hom}_{<\lambda}$ set of reals. Let $\vec{\varphi}$ and $\vec{\psi}$ be $\mathrm{Hom}_{<\lambda}$-semiscales on A and $\omega^\omega \setminus A$ respectively, and define the corresponding code sets

$$R_0 = \{(n,x,y) \in \omega \times \mathbb{R} \times \mathbb{R} : \varphi_n(x) \leq \varphi_n(y)\}$$
$$R_1 = \{(n,x,y) \in \omega \times \mathbb{R} \times \mathbb{R} : \psi_n(x) \leq \psi_n(y)\}.$$

Because the corresponding code sets R_0 and R_1 are $<\lambda$-homogeneously Suslin, they are universally Baire. The canonical expansions $R_0^* = R_0^{V(\mathbb{R}^*)}$ and $R_1^* = R_1^{V(\mathbb{R}^*)}$ are in $\mathcal{M}$ because $\mathrm{Hom}^* \subset \mathcal{M}$.

Now let $H \subset \mathrm{Col}(\omega, Z)$ be a $V(\mathbb{R}^*)$-generic filter, so in particular H is $\mathcal{M}$-generic. We claim that

$$(\mathrm{HC}; \in, A, R_0, R_1) \prec (\mathrm{HC}^*; \in, A^*, R_0^*, R_1^*)$$
$$\prec (\mathrm{HC}^{V(\mathbb{R}^*)[H]}; \in, A^{V(\mathbb{R}^*)[H]}, R_0^{V(\mathbb{R}^*)[H]}, R_1^{V(\mathbb{R}^*)[H]}).$$

The first part of the claim is due to Woodin; for a proof, see Steel [9, Theorem 3.6]. The second part is proved by a straightforward adaptation of Woodin's argument, as follows:

Because λ is a limit of Woodin cardinals, for every projective formula about the sets A, R_0, and R_1 there is a tree in V that projects to the corresponding set of reals in every generic extension by a poset of size less than λ; moreover, because λ is also a limit of strong cardinals, there are trees in V that work for arbitrarily large generic extensions. Therefore the claim follows by the absoluteness of well-foundedness and the Tarski–Vaught criterion for elementarity.

The elementarity supplied by the claim implies that the canonical expansions $R_0^{V(\mathbb{R}^*)[H]}$ and $R_1^{V(\mathbb{R}^*)[H]}$ are sequences of prewellorderings that code semiscales on the expanded sets $A^{V(\mathbb{R}^*)[H]}$ and $(\omega^\omega \setminus A)^{V(\mathbb{R}^*)[H]}$ respectively in the generic extension $V(\mathbb{R}^*)[H]$.

Letting ρ_0 and ρ_1 be the standard $\mathrm{Col}(\omega, Z)$-names in $V(\mathbb{R}^*)$ for the canonical expansions of R_0^* and R_1^* respectively, the restrictions $\rho_0 \cap \mathcal{M}$ and $\rho_1 \cap \mathcal{M}$ are in $\mathcal{M}$ because they are coded into the predicate F. Therefore the restrictions $R_0^{V(\mathbb{R}^*)[H]} \cap \mathcal{M}[H]$ and $R_1^{V(\mathbb{R}^*)[H]} \cap \mathcal{M}[H]$ are in the model $\mathcal{M}[H]$. They code semiscales on the set $A^{V(\mathbb{R}^*)[H]} \cap \mathcal{M}[H]$ and its complement respectively because this property is preserved by restrictions.

[4]In fact Steel's theorem gives a $\mathrm{Hom}_{<\lambda}$-scale, but we will not need the lower semicontinuity property of scales.

In $\mathcal{M}[H]$, let T_0 and T_1 be the trees of the semiscales coded by the sequences of prewellorderings $R_0^{V(\mathbb{R}^*)[H]} \cap \mathcal{M}[H]$ and $R_1^{V(\mathbb{R}^*)[H]} \cap \mathcal{M}[H]$ respectively, so we have

$$A^{V(\mathbb{R}^*)[H]} \cap \mathcal{M}[H] = (p[T_0])^{\mathcal{M}[H]} = (\omega^\omega \setminus p[T_1])^{\mathcal{M}[H]}.$$

These semiscales, and therefore their trees also, depend only on the model $\mathcal{M}[H]$ and not on the choice of the generic filter H. In particular the trees are unchanged by finite variations of H,[5] which implies that they are definable in $\mathcal{M}$ from the names $\rho_0 \cap \mathcal{M}$ and $\rho_1 \cap \mathcal{M}$ respectively in terms of what is forced by the empty condition. Therefore the pair of trees (T_0, T_1) is in $\mathcal{M}$, and it is a Z-absolutely complementing pair for A as desired.

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[5]Here we say that H' is a finite variation of H if the functions $\bigcup H$ and $\bigcup H'$ differ only in finitely many places.