UNIVERSALLY BAIRE SETS AND GENERIC ABSOLUTENESS

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Abstract. We prove several equivalences and relative consistency results involving notions of generic absoluteness beyond Woodin's $(\undertilde{\mathbf{\Sigma}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness for a limit of Woodin cardinals λ. In particular, we prove that two-step $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below a measurable cardinal that is a limit of Woodin cardinals has high consistency strength, and that it is equivalent with the existence of trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas. The construction of these trees uses a general method for building an absolute complement for a given tree T assuming many "failures of covering" for the models $L(T, V_\alpha)$ below a measurable cardinal.

Introduction

Generic absoluteness principles assert that certain properties of the set-theoretic universe cannot be changed by the method of forcing. Some properties, such as the truth or falsity of the Continuum Hypothesis, can always be changed by forcing. Accordingly, one approach to formulating generic absoluteness principles is to consider properties of a limited complexity such as those corresponding to pointclasses in descriptive set theory: $\undertilde{\mathbf{\Sigma}}^1_2$, $\undertilde{\mathbf{\Sigma}}^1_3$, projective, and so on. (Another approach is to limit the class of allowed forcing notions. For a survey of results in this area, see [1].) Shoenfield's absoluteness theorem implies that $\undertilde{\mathbf{\Sigma}}^1_2$ statements are always generically absolute. Generic absoluteness principles for larger pointclasses tend to be equiconsistent with strong axioms of infinity, and they may also relate to the extent of the universally Baire sets.

For example, one-step $\undertilde{\mathbf{\Sigma}}^1_3$ generic absoluteness is shown in [3] to be equiconsistent with the existence of a Σ_2-reflecting cardinal and to be equivalent with the statement that every $\undertilde{\mathbf{\Delta}}^1_2$ set of reals is universally Baire. As another example, two-step $\undertilde{\mathbf{\Sigma}}^1_3$ generic absoluteness, which is the statement that one-step $\undertilde{\mathbf{\Sigma}}^1_3$ generic absoluteness holds in every forcing extension, is shown in [16] to be equivalent with the statement that every set has a sharp and in [3] to be equivalent with the statement that every $\undertilde{\mathbf{\Sigma}}^1_2$ set of reals is

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universally Baire. As a third example, projective generic absoluteness (either one-step or two-step) is equiconsistent with the existence of infinitely many strong cardinals, as shown by theorems of Woodin and Hauser [6]. A question raised in [3] asks whether projective generic absoluteness implies, or is implied by, the statement that every projective set is universally Baire. This question is still open.

A bit higher in the complexity hierarchy we reach an obstacle: The continuum hypothesis is a Σ^2_1 statement that cannot be generically absolute, so generic absoluteness principles at this level of complexity must be limited in some way. One approach is to consider a generic absoluteness principle that is conditioned on CH. This principle is consistent relative to large cardinals by a theorem of Woodin (see [8] for a proof.)

The approach that is more relevant to this paper is a restriction from Σ^2_1 to $(\Sigma^2_1)^\Gamma$ where Γ is a pointclass of "well-behaved" sets of reals (in particular, it should not contain well-orderings of $\mathbb{R}$.) The pointclass uB of universally Baire sets of reals and also its local version uB_λ are both examples of such pointclasses, and indeed Woodin has shown that if λ is a limit of Woodin cardinals then generic absoluteness holds for $(\undertilde{\mathbf{\Sigma}}^2_1)^{\mathrm{uB}_\lambda}$ statements with respect to generic extensions by posets of size less than λ.

In this paper we investigate generic absoluteness for pointclasses beyond $(\undertilde{\mathbf{\Sigma}}^2_1)^{\mathrm{uB}_\lambda}$, both in relation to strong axioms of infinity and determinacy and in relation to the extent of the universally Baire sets. First, we consider generic absoluteness for $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ statements. This level of complexity is interesting because, whereas $(\undertilde{\mathbf{\Sigma}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness follows from the modest large cardinal hypothesis that λ is a limit of Woodin cardinals, generic absoluteness for the slightly larger pointclass $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ is not known to follow from any large cardinal assumption whatsoever (although it can be forced from large cardinals.) Moreover, inner model theory suggests a possible reason that it should *not* follow from large cardinals.

In Section 2 we define the principle of one-step $\exists^{\mathbb{R}}(\undertilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below a limit of Woodin cardinals λ, get a consistency strength upper bound for it in terms of large cardinals, and get an equivalent characterization in terms of a closure property of the pointclass of λ-universally Baire sets. The problem of finding a consistency strength lower bound for this generic absoluteness principle remains open.

In Section 3 we prove some lemmas for constructing absolute complements of trees. These lemmas do not require any facts about generic absoluteness or about the pointclass of uB_λ sets, and they may be read independently of the rest of the paper. Alternatively, the reader who wishes to focus on results pertaining to generic absoluteness may skip this section except for the statements of Lemmas 3.3 and 3.4.

In the remaining sections, we consider stronger principles of generic absoluteness and we derive consistency strength lower bounds for these principles in terms of strong axioms of determinacy. The method for doing this is outlined as follows. Given a limit λ of Woodin cardinals, we show that certain generic absoluteness principles imply strong axioms of determinacy in the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ associated to a V-generic filter on $\mathrm{Col}(\omega, {<}\lambda)$. It is convenient to express these strong axioms of determinacy in terms of the extent of the Suslin sets.

In Section 4 we prove the following result which gives a consistency strength lower bound for two-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below a limit λ of Woodin cardinals in the case that λ is measurable.

Theorem 0.1. *If λ is a measurable cardinal and a limit of Woodin cardinals, and two-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds with respect to generic extensions by posets of size less than λ, then the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies* AD + *"every $\utilde{\mathbf{\Pi}}^2_1$ set of reals is Suslin."*

Under $\mathsf{ZF} + \mathsf{AD}$ the statement "every $\utilde{\mathbf{\Pi}}^2_1$ set of reals is Suslin" is equivalent to the statement $\theta_0 < \Theta$ about the length of the Solovay sequence. In terms of large cardinals, it is equiconsistent with the "$\mathsf{AD} + \theta_0 < \Theta$ hypothesis," which says that ZFC holds and there is a limit λ of Woodin cardinals and a cardinal $\delta < \lambda$ that is ${<}\lambda$-strong. Note that the hypothesis "λ is a measurable cardinal and a limit of Woodin cardinals" alone, without any generic absoluteness assumption, is significantly weaker than the $\mathsf{AD} + \theta_0 < \Theta$ hypothesis.

In Section 5 we consider another generic absoluteness principle, and derive an even stronger lower bound for it in terms of strong axioms of determinacy:

Theorem 0.2. *If λ is a measurable cardinal and a limit of Woodin cardinals, and $L(\mathrm{uB}_\lambda, \mathbb{R}) \equiv L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ for every generic extension $V[g]$ by a poset of size less than λ, then the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies* $\mathsf{AD} + \mathsf{DC}$ + *"every set of reals is Suslin."*

The theory $\mathsf{ZF} + \mathsf{AD}$ + "every set of reals is Suslin" is equivalent to the theory $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ by work of Martin and Woodin. In terms of large cardinals, by work of Woodin and Steel it is equiconsistent with the "$\mathsf{AD}_{\mathbb{R}}$ hypothesis," which says that ZFC holds and there is a cardinal λ that is a limit of Woodin cardinals and of cardinals that are ${<}\lambda$-strong. For more information about these equivalences and equiconsistencies involving $\mathsf{AD}_{\mathbb{R}}$, see [15, §8]. In the context of $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$, the addition of DC contributes a bit more consistency strength. However, Theorem 0.2 is still very far from a proof of equiconsistency because the best known consistency strength upper bound for the hypothesis exceeds that of a supercompact cardinal.

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1. λ-universally Baire sets and $(\utilde{\mathbf{\Sigma}}^2_1)^{\mathrm{uB}_\lambda}$ sets

For our purposes, a *tree* is a tree on $\omega^k \times \mathrm{Ord}$ for some natural number k. That is, it is a collection of finite sequences of elements of $\omega^k \times \mathrm{Ord}$, closed under initial segments and ordered by reverse inclusion. Usually we assume that $k = 1$ for simplicity and leave the obvious generalizations to the reader. Also, we will freely abuse notation by treating a finite sequences of $(k+1)$-tuples from $\omega^k \times \mathrm{Ord}$ as a $(k+1)$-tuple of finite sequences. We follow the usual convention that elements of the Baire space ω^ω are called *reals* and the Baire space itself may be denoted by $\mathbb{R}$ when appropriate. Given a tree T on $\omega \times \mathrm{Ord}$ and a real x we define a tree T_x on Ord, called a *section* of T, by

$$T_x = \{s \in \mathrm{Ord}^{<\omega} : (x \restriction |s|, s) \in T\}.$$

The *projection* $p[T]$ of T is the set of reals defined by

$$p[T] = \{x \in \mathbb{R} : T_x \text{ is ill-founded}\}.$$

For any given real x the statement $x \in p[T]$ is generically absolute by the absoluteness of well-foundedness. Reals added by forcing may or may not be in $p[T]$.

An equivalent definition of the projection $p[T]$ is to let $[T]$ be the set of infinite branches of T, which is a closed subset of $\omega^\omega \times \mathrm{Ord}^\omega$, and to let $p[T]$ be its projection onto the first coordinate.

A set of reals A is *Suslin* if it is the projection of a tree. That is, $A = p[T]$ for some tree T on $\omega \times \mathrm{Ord}$. The pointclass of Suslin sets is a natural generalization of the pointclass of $\boldsymbol{\Sigma}^1_1$ (analytic) sets, which have the form $p[T]$ for trees T on $\omega \times \omega$. Suslin sets form an important object of study in the context of the Axiom of Determinacy. However, under the Axiom of Choice every set of reals is trivially Suslin as witnessed by a tree T on $\omega \times 2^{\aleph_0}$. Accordingly in the context of AC one has to add some conditions in order to get an interesting definition.

A pair of trees $(T, \tilde{T})$ on $\omega \times \mathrm{Ord}$ is λ-*absolutely complementing*, where λ is a cardinal, if in every forcing extension by a poset of size less than λ the trees project to complements: $p[T] = \mathbb{R} \setminus p[\tilde{T}]$. The condition $p[T] \cap p[\tilde{T}] = \emptyset$ is generically absolute by the absoluteness of well-foundedness, but the condition $p[T] \cup p[\tilde{T}] = \mathbb{R}$ may fail to be generically absolute. (And for the trivial Suslin representations given by the Axiom of Choice it is not generically absolute.)

A set of reals A is λ-*universally Baire* if there is a λ-absolutely complementing pair of trees $(T, \tilde{T})$ such that $A = p[T]$. The pointclass uB_λ is defined to consist of the sets of reals that are λ-universally Baire.[1] Note that if λ is a limit cardinal and a set of reals A is κ-universally Baire for all cardinals $\kappa < \lambda$ then it is λ-universally Baire; this can be seen by "amalgamating" a transfinite sequence of trees into a single tree whose projection is the union of their projections. A set of reals is *universally Baire* (uB) if it is

[1] We follow the notational convention of [8] and [15] but not of the original paper [3]. What we call λ-universally Baire would be called $<\lambda$-universally Baire according to the original convention because it involves generic extensions by posets of size less than λ.

λ-universally Baire for all cardinals λ. The notion of universally Baire sets of reals was introduced in [3] as a generalization of the property of Baire.

Given a set of reals $A \in \mathrm{uB}_\lambda$ and a generic extension $V[g]$ of V by a poset of size less than λ, there is a *canonical extension* $A^{V[g]} \subset \mathbb{R}^{V[g]}$ of A defined by $A^{V[g]} = p[T]^{V[g]}$ for any λ-absolutely complementing pair of trees $(T, \tilde{T}) \in V$ with $A = p[T]^V$. This extension of A does not depend on the choice of λ-absolutely complementing trees.

Important examples of λ-universally Baire sets are given by the Martin–Solovay tree construction, which shows that every $<\!\lambda$-homogeneously Suslin set of reals is λ-universally Baire. Conversely, if λ is a limit of Woodin cardinals then by the Martin–Steel theorem and a theorem of Woodin (see [8, Theorem 3.3.13]) every λ-universally Baire set of reals is $<\!\lambda$-homogeneously Suslin, so we have an equality of pointclasses: $\mathrm{uB}_\lambda = \mathrm{Hom}_{<\lambda}$. In turn, by a theorem of Martin every homogeneously Suslin set of reals is determined—that is, that one player or the other has a winning strategy in the game whose payoffs are according to that set.

In fact, from the hypothesis that λ is a limit of Woodin cardinals one can get not only the determinacy of uB_λ sets, but an entire model of $\mathsf{ZF} + \mathsf{AD}$ via Woodin's "derived model" construction. This construction will be useful to us as a way of bringing together the uB_λ sets existing in various generic extensions under a single umbrella. We define the following standard notation:

Definition 1.1. Let λ be a limit of Woodin cardinals and let $G \subset \mathrm{Col}(\omega, <\!\lambda)$ be a V-generic filter. Then we let

$$\mathbb{R}^*_G = \bigcup_{\alpha<\lambda} \mathbb{R}^{V[G\restriction\alpha]}, \qquad \mathrm{HC}^*_G = \bigcup_{\alpha<\lambda} \mathrm{HC}^{V[G\restriction\alpha]}, \text{ and}$$

$$A^*_G = \bigcup_{\alpha<\lambda} A^{V[G\restriction\alpha]}, \text{ if } A \in \mathrm{uB}_\lambda.$$

Finally, we define the pointclass Hom^*_G consisting of all subsets of $\mathbb{R}^*_G$ of the form A^*_G where $A \in (\mathrm{uB}_\lambda)^{V[G\restriction\alpha]}$ for some $\alpha < \lambda$. We remark that Hom^*_G might just as aptly been called uB^*_G.

The following theorem is a special case of Woodin's derived model theorem. For a proof of this special case, see [15]. The axiom AD^+ in the theorem is a technical strengthening of AD that holds in all known models of AD.

Theorem 1.2 (Woodin)**.** *Let λ be a limit of Woodin cardinals and let $G \subset \mathrm{Col}(\omega, <\!\lambda)$ be a V-generic filter. Then the model $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)$ satisfies AD^+.*

Remark 1.3. The theory of the model $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)$ does not depend on the choice of generic filter G because the Levy collapse forcing poset $\mathrm{Col}(\omega, <\!\lambda)$ is homogeneous. Therefore when appropriate we will omit G from the notation and refer to "the" model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$.

Remark 1.4. In the past Theorem 1.2 has been called the *derived model theorem* and the model $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)$ has been called the *derived model* of V at λ by G. More recently the term "derived model" usually refers to a model $D(V, \lambda, G)$ which in many cases properly contains $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)$, and the "derived model theorem" refers to the theorem (also due to Woodin) that this larger model satisfies AD^+. For the statement and proof of the derived model theorem in its full generality, see [17].

The advantage of the newer definition is that every model of $\mathsf{AD}^+ + V = L(\wp(\mathbb{R}))$ can be realized as $D(M, \lambda, G)$ for some model M, some limit λ of Woodin cardinals of M, and some M-generic filter $\mathrm{Col}(\omega, {<}\lambda)$. However, the models of the form $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)$ will be sufficient for our purposes in this paper.

The following generic absoluteness theorem will be used fundamentally throughout the paper. For a proof, see [15].

Theorem 1.5 (Woodin). *Let λ be a limit of Woodin cardinals. For every real x, every set of reals $A \in \mathrm{uB}_\lambda$, every formula $\varphi(v)$ in the language of set theory expanded by two new predicate symbols, and every generic extension $V[g]$ of G by a poset of size less than λ which is absorbed into a generic extension $V[G]$ by $\mathrm{Col}(\omega, {<}\lambda)$, the following equivalences hold:*

$$\begin{aligned} & \exists B \in \mathrm{uB}^V_\lambda\ (\mathrm{HC}^V; \in, A, B) \models \varphi[x] \\ \iff & \exists B \in \mathrm{uB}^{V[g]}_\lambda\ (\mathrm{HC}^{V[g]}; \in, A^{V[g]}, B) \models \varphi[x] \\ \iff & \exists B \in \mathrm{Hom}^*_G\ (\mathrm{HC}^*_G; \in, A^*_G, B) \models \varphi[x] \\ \iff & \exists B \in L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)\ (\mathrm{HC}^*_G; \in, A^*_G, B) \models \varphi[x]. \end{aligned}$$

To understand the possible uses of Theorem 1.5, it may be instructive to consider the following special cases:

- If $A = \{x\}$, we get generic absoluteness for various restricted notions of $\Sigma^2_1(x)$. In particular we get $(\Sigma^2_1(x))^{\mathrm{uB}_\lambda}$ generic absoluteness between V and $V[g]$.
- If φ does not mention B, we get generic absoluteness for statements that are projective in A and its expansions $A^{V[g]}$ and A^* respectively.

The main consequence of AD^+ in the derived model that we will need is given by the following theorem, also proved in [15].

Theorem 1.6 (Woodin). AD^+ *implies that every Σ^2_1 set of reals is the projection of a definable tree T on $\omega \times \mathrm{Ord}$.*

The tree in Theorem 1.6 comes from the *scale property* of Σ^2_1, but we will not need this sharper notion. We will often use the following immediate corollary of Theorems 1.5 and 1.6. Note that if λ is a limit of Woodin cardinals and T is a tree given by applying Theorem 1.6 in the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ then we have $T \in V$ by the homogeneity of the forcing $\mathrm{Col}(\omega, {<}\lambda)$.

Corollary 1.7 (Trees for $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ formulas)**.** *Let λ be a limit of Woodin cardinals. For every formula $\varphi(v)$ in the language of set theory expanded by a new predicate symbol, there is a tree $T_\varphi \in V$ such that for every generic extension $V[g]$ of V by a poset of size less than λ and every real $x \in V[g]$ we have*

$$x \in p[T_\varphi] \iff \exists B \in \mathrm{uB}_\lambda^{V[g]}\ (\mathrm{HC}^{V[g]}; \in, B) \models \varphi[x].$$

We remark that these trees can be used to get $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness between V and $V[g]$ by a standard argument using the absoluteness of well-foundedness, just as Shoenfield trees can be used to get Σ^1_2 generic absolutess. In the case of $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness it is simpler to prove the absoluteness directly, using the stationary tower, than to build the trees T_φ of Corollary 1.7. However, these trees T_φ will still be quite useful for other purposes.

The following theorem can be considered as a basis theorem for the pointclass Σ^2_1. For a proof, see [15, Lemma 7.2].

Theorem 1.8 (Woodin)**.** AD^+ *implies that every true Σ^2_1 statement has a witness that is a Δ^2_1 set of reals.*

Woodin's basis theorem easily generalizes to say that for every real x, every true $\Sigma^2_1(x)$ statement has a $\Delta^2_1(x)$ witness, uniformly in x in the following sense. The author no doubt is not the first to take note of this generalization, but does not know of a reference for it.

Lemma 1.9. *Assume AD^+ and let $\varphi(v)$ be a formula in the language of set theory expanded by a new unary predicate symbol. Consider the Σ^2_1 set S defined by*

$$y \in S \iff \exists B \subset \mathbb{R}\ (\mathrm{HC}; \in, B) \models \varphi[y].$$

Then to each real $y \in S$ we can assign a witness $B(y) \subset \mathbb{R}$ satisfying $(\mathrm{HC}; \in, B(y)) \models \varphi[y]$ *in such a way that the binary relations*

$$\{(y, z) \in \mathbb{R} \times \mathbb{R} : y \in S \ \ \&\ \ z \in B(y)\}$$
$$\{(y, z) \in \mathbb{R} \times \mathbb{R} : y \in S \ \ \&\ \ z \notin B(y)\}$$

are both Σ^2_1. In other words we have $B(y) \in \Delta^2_1(y)$ for each real $y \in S$, uniformly in y.

Proof. This follows easily from the main ingredient from the proof of Woodin's basis theorem (see [15, Lemma 7.2].) Namely the fact that for every real y, if there is a set of reals B such that $(\mathrm{HC}; \in, B) \models \varphi[y]$, then there is such a set B with the additional property that $B \in \mathrm{OD}_y^{L(A,\mathbb{R})}$ for some set of reals A. Note that the model $L(A, \mathbb{R})$ depends only on the Wadge rank of A and not on A itself. Minimizing this Wadge rank and then minimizing the witness B in the canonical well-ordering of $\mathrm{OD}_y^{L(A,\mathbb{R})}$ sets, a straightforward computation shows that the witness $B(y)$ we obtain in this way is $\Delta^2_1(y)$ uniformly in y. $\square$

In particular the above lemma applies to the pointclass Σ^2_1 of the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ where λ is a limit of Woodin cardinals. Next we will obtain a version of the lemma in terms of the pointclass $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$.

Lemma 1.10. *Let λ be a limit of Woodin cardinals and let $\varphi(v)$ be a formula in the language of set theory expanded by a new unary predicate symbol. Consider the $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ set S defined by*

$$y \in S \iff \exists B \in \mathrm{uB}_\lambda\ (\mathrm{HC}; \in, B) \models \varphi[y].$$

Then to each real $y \in S$ we can assign a witness $B(y) \in \mathrm{uB}_\lambda$ satisfying $(\mathrm{HC}; \in, B(y)) \models \varphi[y]$ in such a way that the binary relations

$$\{(y, z) \in \mathbb{R} \times \mathbb{R} : y \in S \ \ \&\ \ z \in B(y)\}$$
$$\{(y, z) \in \mathbb{R} \times \mathbb{R} : y \in S \ \ \&\ \ z \notin B(y)\}$$

are both $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$. In other words we have $B(y) \in (\Delta^2_1(y))^{\mathrm{uB}_\lambda}$ for each real $y \in S$, uniformly in y.

Proof. This is easily seen to follow from Lemma 1.9 using the generic absoluteness between V and the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ that is provided by Theorem 1.5. Note that if y is a real in V, then for any $\Delta^2_1(y)$ set of reals $A \in L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ the restriction $A \cap V$ is in uB_λ as well as in $(\Delta^2_1(y))^{\mathrm{uB}_\lambda}$ because of the λ-absolutely complementing trees provided by Corollary 1.7. $\square$

2. One-step $\exists^{\mathbb{R}}(\boldsymbol{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness

Next we will consider principles of generic absoluteness for pointclasses beyond $(\boldsymbol{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$. First we define a "lightface" (effective) version.

Definition 2.1. Let λ be a limit of Woodin cardinals. Then $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ *generic absoluteness below* λ is the statement that for every formula $\varphi(v)$ in the language of set theory expanded by a new unary predicate symbol, and for every extension $V[g]$ of V by a poset of size less than λ, we have

$$\exists y \in \mathbb{R}^V\ \forall B \in \mathrm{uB}^V_\lambda\ (\mathrm{HC}^V; \in, B) \models \varphi[y]$$
$$\iff \exists y \in \mathbb{R}^{V[g]}\ \forall B \in \mathrm{uB}^{V[g]}_\lambda\ (\mathrm{HC}^{V[g]}; \in, B) \models \varphi[y].$$

Remark 2.2. The canonical inner model M_ω for ω many Woodin cardinals does not satisfy $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below its limit of Woodin cardinals λ, as mentioned in [15, Remark 5.4]. This is because it satisfies the $\forall^{\mathbb{R}}(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ statement "every real is an a mouse with a uB_λ iteration strategy," but this statement fails in the generic extension to add a Cohen real, and indeed it fails in every generic extension by a poset of size less than λ that adds a real.

This remark applies not only to M_ω but also to some other mice satisfying stronger large cardinal axioms, as shown in [14]. It is an open question whether $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below λ is implied by any large cardinal hypothesis on λ.

Remark 2.3. The upward direction (from V to $V[g]$) of $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness is automatic from $(\utilde{\mathbf{\Sigma}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness; more specifically, from $(\Sigma^2_1(y))^{\mathrm{uB}_\lambda}$ generic absoluteness where y is a real witnessing that the $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ statement holds in V.

In the terminology of Hamkins and Löwe in [4], the existence of a real y witnessing an $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ statment is a *button* from the point of view of V_λ. That is, once "pushed" (made true) by forcing, it cannot be "unpushed" (made false) by any further forcing. Therefore $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness is a special case of the maximality principle MP defined by Hamkins in [5]. More precisely, if λ is a limit of Woodin cardinals and $V_\lambda \models \mathsf{MP}$ then $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below λ.

The consistency of $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness can be established by a compactness argument. The argument is based on the argument for the consistency of the maximality principle MP in [5]; we just have to localize it to V_λ and check that the hypothesis "λ is a limit of Woodin cardinals" is preserved, which we do below for the convenience of the reader.

Proposition 2.4. *If the theory* ZFC + *"there are infinitely many Woodin cardinals" is consistent, then so is* ZFC + *"λ is a limit of Woodin cardinals and $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below λ."*

Proof. Let M be a model of ZFC + "there are infinitely many Woodin cardinals" and let $\lambda^M \in M$ be a limit of Woodin cardinals of M. Let T be the theory in the language of set theory expanded by a constant symbol λ consisting of the ZFC axioms, the assertion that λ is a limit of Woodin cardinals, and for each formula $\varphi(v)$, the assertion "if the statement $\exists y \in \mathbb{R}\ \forall B \in \mathrm{uB}_\lambda\ (\mathrm{HC}; \in, B) \models \varphi[y]$ holds in some generic extension of V by a poset of size less than λ, then it holds already in V." The theory T implies that $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below λ, so it remains to check that it is a consistent theory.

Indeed, given any finite subset $T_0 \subset T$ let $\varphi_0(v), \ldots, \varphi_{n-1}(v)$ enumerate all the formulas $\varphi(v)$ that are mentioned in T_0 and have the property that the $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ statement "$\exists y \in \mathbb{R}\ \forall B \in \mathrm{uB}_\lambda\ (\mathrm{HC}; \in, B) \models \varphi[y]$" holds in some generic extension of M by a poset of size less than λ^M. For each $i < n$ take a poset $\mathbb{P}_i \in (V_\lambda)^M$ whose top condition forces this $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ statement to hold over M. Then any generic extension of (M, λ^M) by the product forcing $\mathbb{P}_0 \times \cdots \times \mathbb{P}_{n-1}$ satisfies the statements "$\exists y \in \mathbb{R}\ \forall B \in \mathrm{uB}_\lambda\ (\mathrm{HC}; \in, B) \models \varphi_i[y]$" for all $i < n$, and also it still satisfies ZFC + "λ is a limit of Woodin cardinals" because Woodin cardinals are preserved by small forcing, so it satisfies T_0. □

The next result is an equivalent condition for $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness to hold below λ in terms of a closure property of the pointclass uB_λ of λ-universally Baire sets of reals.

Proposition 2.5. *For any limit of Woodin cardinals λ and any real x, the following statements are equivalent.*

(1) $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ *generic absoluteness holds below* λ.
(2) Every $(\Delta^2_1)^{\mathrm{uB}_\lambda}$ *set of reals is* λ*-universally Baire.*

Proof. (1) $\Longrightarrow$ (2): The proof of this direction is analogous to [3, Theorem 3.1] with the pointclass $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ in place of the pointclass Σ^1_2. Let A be a $(\Delta^2_1)^{\mathrm{uB}_\lambda}$ set of reals and take formulas $\varphi(v)$ and $\psi(v)$ such that for all reals y we have

$$\begin{aligned} y \in A &\iff \exists B \in \mathrm{uB}_\lambda\,(\mathrm{HC};\in,B) \models \varphi[y] \\ &\iff \neg\exists B \in \mathrm{uB}_\lambda\,(\mathrm{HC};\in,B) \models \psi[y]. \end{aligned}$$

Let T_φ and T_ψ be trees such that in every generic extension $V[g]$ of V by a poset of size less than λ we have, for every real $y \in V[g]$,

$$\begin{aligned} y \in p[T_\varphi] &\iff \exists B \in \mathrm{uB}_\lambda\,(\mathrm{HC};\in,B) \models \varphi[y] \\ y \in p[T_\psi] &\iff \exists B \in \mathrm{uB}_\lambda\,(\mathrm{HC};\in,B) \models \psi[y]. \end{aligned}$$

In particular, in V we have $A = p[T_\varphi] = \mathbb{R} \setminus p[T_\psi]$. We claim that the trees T_φ and T_ψ are λ-absolutely complementing. Let $V[g]$ be a generic extension of V by a poset of size less than λ. As usual, the absoluteness of well-foundedness gives $p[T_\varphi] \cap p[T_\psi] = \emptyset$ in $V[g]$. On the other hand, the $\forall^{\mathbb{R}}(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ statement

$$\forall y \in \mathbb{R}\,\exists B \in \mathrm{uB}_\lambda\,(\mathrm{HC};\in,B) \models \varphi[y] \vee \psi[y]$$

holds in V, so by our hypothesis (1) it continues to hold in $V[g]$. Therefore we have

$$V[g] \models p[T_\varphi] \cup p[T_\psi] = \mathbb{R},$$

so the trees T_φ and T_ψ project to complements in $V[g]$.

(2) $\Longrightarrow$ (1): Suppose that the $\forall^{\mathbb{R}}(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ statement

$$\forall y \in \mathbb{R}\,\exists B \in \mathrm{uB}_\lambda\,(\mathrm{HC};\in,B) \models \varphi[y]$$

holds in V. We want to show that it continues to hold in generic extensions by posets of size less than λ. In this case Lemma 1.10 gives a total function $y \mapsto B(y)$ uniformly choosing $(\Delta^2_1(y))^{\mathrm{uB}_\lambda}$ sets of reals to witness our true $(\Sigma^2_1(y))^{\mathrm{uB}_\lambda}$ statements. That is, the relation

$$W = \{(y,z) \in \mathbb{R} \times \mathbb{R} : z \in B(y)\}$$

is $(\Delta^2_1)^{\mathrm{uB}_\lambda}$. By hypothesis (2), this relation W is λ-universally Baire. (The weaker fact that each of its sections $W_y = B(y)$ is λ-universally Baire is already given by the lemma.) The fact that

$$\forall y \in \mathbb{R}\,(\mathrm{HC};\in,W_y) \models \varphi[y]$$

holds can be expressed as a first-order property of the structure $(\mathrm{HC};\in,W)$. Letting $V[g]$ be any generic extension of V by a poset of size less than λ, and letting $W^{V[g]} \subset \mathbb{R}^{V[g]} \times \mathbb{R}^{V[g]}$ denote the canonical extension of W to $V[g]$, we have

$$(\mathrm{HC};\in,W) \prec (\mathrm{HC}^{V[g]};\in,W^{V[g]}).$$

Therefore we have

$$\forall y \in \mathbb{R}^{V[g]} \left(\mathrm{HC}^{V[g]}; \in, (W^{V[g]})_y\right) \models \varphi[y].$$

This shows that the sections $(W^{V[g]})_y$ for reals $y \in V[g]$ witness that our $\forall^{\mathbb{R}}(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ statement holds in $V[g]$, as desired. □

Remark 2.6. Recall that the canonical inner model M_ω for ω many Woodin cardinals does not satisfy $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below its limit of Woodin cardinals λ. This can now be seen to follow from the fact that M_ω has a $(\Delta^2_1)^{\mathrm{uB}_\lambda}$ well-ordering of its reals, and that no well-ordering of the reals can have the Baire property (which is a consequence of the universal Baire property for Cohen forcing.)

In general, the set of reals appearing in mice with uB_λ iteration strategies is a $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ set with a $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ well-ordering given by the comparison theorem for mice. Therefore if every real is in such a mouse (which is the case in M_ω—see [15, Remark 5.4]) then this well-ordering is a $(\Delta^2_1)^{\mathrm{uB}_\lambda}$ well-ordering of the reals.

Again this remark applies not only to M_ω but also to some other mice satisfying stronger large cardinal axioms. It is an open question whether every large cardinal hypothesis on λ is consistent with the existence of a $(\Delta^2_1)^{\mathrm{uB}_\lambda}$ well-ordering of the reals.

Next we consider the corresponding boldface generic absoluteness principle, which allows all reals in V as parameters. It is called "one-step" generic absoluteness to distinguish it from "two-step" generic absoluteness, which allows real parameters appearing in generic extensions of V and will be defined precisely in Section 4.

Definition 2.7. Let λ be a limit of Woodin cardinals. Then *one-step* $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ *generic absoluteness below* λ is the statement that for every formula $\varphi(v, v')$ in the language of set theory expanded by a new unary predicate symbol, every real parameter $x \in V$, and every generic extension $V[g]$ of V by a poset of size less than λ, we have

$$\begin{aligned} &\exists y \in \mathbb{R}^V \ \forall B \in \mathrm{uB}^V_\lambda \ (\mathrm{HC}^V; \in, B) \models \varphi[x, y] \\ \iff &\exists y \in \mathbb{R}^{V[g]} \ \forall B \in \mathrm{uB}^{V[g]}_\lambda \ (\mathrm{HC}^{V[g]}; \in, B) \models \varphi[x, y]. \end{aligned}$$

Generic absoluteness for $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ can also be recast as a closure property for the pointclass of λ-universally Baire sets via a straightforward relativization of Proposition 2.5 to arbitrary real parameters:

Proposition 2.8. *For any limit of Woodin cardinals λ, the following statements are equivalent.*

(1) One-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below λ.
(2) Every $(\utilde{\mathbf{\Delta}}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is λ-universally Baire.

Obtaining one-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness in the first place is not simply a matter of relativization, however. We may force to make an

$\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formula hold for all the real parameters in V for which it can be forced to hold, but this will add more reals that we must consider as parameters, so we must force again, *etc.* Fortunately, a mild large cardinal hypothesis is sufficient to show that this process eventually reaches a stopping point. Namely, the assumption that some cardinal $\delta < \lambda$ is Σ_2*-reflecting in* V_λ, meaning that it is inaccessible and $V_\delta \prec_{\Sigma_2} V_\lambda$.

We note a convenient reformulation: $V_\delta \prec_{\Sigma_2} V_\lambda$ if and only if for every set $x \in V_\delta$ and every formula ψ, if there is an ordinal $\beta < \lambda$ such that $V_\beta \models \varphi[x]$, then there is an ordinal $\bar{\beta} < \delta$ such that $V_{\bar{\beta}} \models \varphi[x]$. This reformulation is usually proved for $\lambda = \mathrm{Ord}$, but when λ is a limit of Woodin cardinals the model V_λ satisfies enough of ZFC for the proof.

It is convenient, although not necessary, to split the consistency proof of one-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness into two parts. First we prove a lemma using only the hypothesis $V_\delta \prec_{\Sigma_2} V_\lambda$ and then we use the inaccessibility of δ to get the full result.

Remark 2.9. One can also obtain one-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below a limit λ of Woodin cardinals as an application (in V_λ) of the boldface maximality principle $\utilde{\mathsf{MP}}$, which is shown in [5] to hold after the Levy collapse of a (fully-) reflecting cardinal. Our argument is also similar to that given in [3] to show that one-step $\utilde{\mathbf{\Sigma}}^1_3$ generic absoluteness holds after the Levy collapse of a Σ_2-reflecting cardinal.

Lemma 2.10. *Let λ be a limit of Woodin cardinals and let $\delta < \lambda$ be a cardinal such that $V_\delta \prec_{\Sigma_2} V_\lambda$. Let $\varphi(v, v')$ be a formula in the language of set theory expanded by a new unary predicate symbol and let x be a real parameter. Suppose that there is a poset $\mathbb{P} \in V_\lambda$ such that*

$$1 \Vdash_{\mathbb{P}} \exists y \in \mathbb{R}\, \forall B \in \mathrm{uB}_\lambda\ (\mathrm{HC}; \in, B) \models \varphi[x, y].$$

Then there is a poset $\bar{\mathbb{P}} \in V_\delta$ with the same property:

$$1 \Vdash_{\bar{\mathbb{P}}} \exists y \in \mathbb{R}\, \forall B \in \mathrm{uB}_\lambda\ (\mathrm{HC}; \in, B) \models \varphi[x, y].$$

Proof. Take a cardinal $\kappa < \lambda$ large enough that $\mathbb{P} \in V_\kappa$. We may assume that κ is inaccessible, which is more than sufficient to ensure that $(\mathrm{uB})^{V_\kappa} = \mathrm{uB}_\kappa$. After forcing with $\mathbb{P}$ we still have $(\mathrm{uB})^{V_\kappa} = \mathrm{uB}_\kappa$ because κ remains inaccessible. Also, by taking $\kappa < \lambda$ to be sufficiently large we may ensure that $\mathrm{uB}_\kappa = \mathrm{uB}_\lambda$ after forcing with $\mathbb{P}$ by an observation of Steel and Woodin; see [8, Theorem 3.3.5].[2] Therefore our assumption on $\mathbb{P}$ yields

$$V_\kappa \models 1 \Vdash_{\mathbb{P}} \exists y \in \mathbb{R}\, \forall B \in \mathrm{uB}\ (\mathrm{HC}; \in, B) \models \varphi[x, y].$$

Now because $V_\delta \prec_{\Sigma_2} V_\lambda$ we can take an inaccessible cardinal $\bar{\kappa} < \delta$ and a poset $\bar{\mathbb{P}}$ with the property that

$$V_{\bar{\kappa}} \models 1 \Vdash_{\bar{\mathbb{P}}} \exists y \in \mathbb{R}\, \forall B \in \mathrm{uB}\ (\mathrm{HC}; \in, B) \models \varphi[x, y].$$

[2]This observation is usually stated in terms of homogeneously Suslin sets: $\mathrm{Hom}_\eta = \mathrm{Hom}_{<\lambda}$ for sufficiently large $\eta < \lambda$. The present version is equivalent; one has only to let κ be greater than the second Woodin above η.

After forcing with $\bar{P}$ we have $(\mathrm{uB})^{V_{\bar{\kappa}}} = \mathrm{uB}_{\bar{\kappa}}$ because $\bar{\kappa}$ is inaccessible, and we trivially have $\mathrm{uB}_\lambda \subset \mathrm{uB}_{\bar{\kappa}}$ because $\bar{\kappa} < \lambda$, so the desired conclusion follows. □

Proposition 2.11. *Let λ be a limit of Woodin cardinals and let $\delta < \lambda$ be an inaccessible cardinal such that $V_\delta \prec_{\Sigma_2} V_\lambda$. Let $G \subset \mathrm{Col}(\omega, {<}\delta)$ be a V-generic filter. Then $V[G]$ satisfies one-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below λ.*

Proof. Let $x \in V[G]$ be a real parameter. We will show that $V[G]$ satisfies $\exists^{\mathbb{R}}(\Pi^2_1(x))^{\mathrm{uB}_\lambda}$ generic absoluteness below λ. Because δ is inaccessible, the real x is contained in the generic extension of V by some proper initial segment of the generic filter G. So because our large cardinal hypotheses on δ and λ are preserved by forcing with posets of size less than δ we may assume that $x \in V$. Now by Lemma 2.10 every $\exists^{\mathbb{R}}(\Pi^2_1(x))^{\mathrm{uB}_\lambda}$ statement that can be forced by a poset in V_λ (over $V[G]$, or equivalently over V) can also be forced by a poset in V_δ. This poset in V_δ can then be absorbed into $\mathrm{Col}(\omega, {<}\delta)$ by universality, so the desired statement holds in $V[G]$ by the upward direction of $\exists^{\mathbb{R}}(\Pi^2_1(x))^{\mathrm{uB}_\lambda}$ generic absoluteness, which is automatic. □

We remark that strong cardinals are Σ_2-reflecting, so the hypothesis of Proposition 2.11 follows from λ being a limit of Woodin cardinals and $\delta < \lambda$ being ${<}\lambda$-strong (the $\mathsf{AD} + \theta_0 < \Theta$ hypothesis.) However, it is much weaker than this because if λ is a Mahlo cardinal then there are many inaccessible cardinals $\delta < \lambda$ such that V_δ is a fully elementary substructure of V_λ. We have not proved any consistency strength lower bound, leading to the obvious question.

Question 2.12. What is the consistency strength of the statement "there is a limit λ of Woodin cardinals such that one-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below λ"?

3. Building absolute complements for trees

In this section, which may be read independently of the rest of the paper, we introduce a method for building an absolute complement to a given tree (Lemma 3.2 below.) First we prove the following lemma, Lemma 3.1, which is a strengthening of a well-known theorem of Woodin. Only Lemma 3.2 and its consequences (Lemmas 3.3 and 3.4) will be needed for the main results of this paper, but we prove Lemma 3.1 first in order to demonstrate the main ideas involved.

In Woodin's version of Lemma 3.1 the cardinal 2^κ is replaced with 2^{2^κ}. The usual proof (see [15]) uses a system of measures to construct an absolute complement via a Martin–Solovay construction, and 2^{2^κ} appears as an upper bound on the the number of measures on κ. In our proof, we form a semi-scale from norms corresponding to rank functions, and 2^κ appears as an upper bound on the number of inequivalent norms. One can show that

Lemma 3.1 is best possible in the sense that 2^κ cannot be reduced further to κ. For an introduction to semi-scales, see [7] or [10].

Lemma 3.1. *Let κ be a cardinal and let $\alpha \geq \kappa$ be a cardinal such that κ is α-strong as witnessed by an elementary embedding $j : V \to M$ with critical point κ, $j(\kappa) > \alpha$, and $V_\alpha \subset M$. Let T be a tree on $\omega \times \gamma$ for some ordinal γ. Then letting $g \subset \mathrm{Col}(\omega, 2^\kappa)$ be V-generic, in $V[g]$ there is an α-absolute complement for the tree $j(T)$.*

Proof. Note that for every real y that is generic over V for a poset of size less than κ, the map j extends to an elementary embedding $V[y] \to M[y]$, giving

$$y \in p[T] \iff y \in p[j(T)].$$

Furthermore, applying j to this statement and using $j(\kappa) > \alpha$ and $V_\alpha \subset M$, we see that for every real y that is generic over V (or equivalently over M) for a poset of size less than α we have

$$y \in p[j(T)] \iff y \in p[j(j(T))].$$

Therefore, for each $t \in j(j(\gamma))^{<\omega}$ we can define the norm φ_t on $\omega^\omega \setminus p[j(T)]$ in any generic extension by a poset of size less than α as follows:

$$\varphi_t(y) = \begin{cases} \mathrm{rank}_{j(j(T))_y}(t) & \text{if } t \in j(j(T))_y \\ 0 & \text{if } t \notin j(j(T))_y. \end{cases}$$

We claim that for every real x and every sequence of reals $(y_n : n < \omega)$ appearing in a generic extension of V (equivalently, of M) by a poset of size less than α such that

(1) $y_n \notin p[j(T)]$ for every $n < \omega$,
(2) $y_n \to x$ as $n \to \omega$, and
(3) for every $t \in j``j(\gamma)^{<\omega}$ the norm $\varphi_t(y_n)$ is eventually constant as $n \to \omega$,

we have $x \notin p[j(T)]$.

To prove the claim, suppose toward a contradiction that $x \in p[j(T)]$ as witnessed by $f \in j(\gamma)^\omega$. That is, for all $i < \omega$ we have $f \restriction i \in j(T)_x$. Then we have $j(f \restriction i) \in j(j(T))_x$, and because $y_n \to x$ as $n \to \omega$, we have $j(f \restriction i) \in j(j(T))_{y_n}$ for all sufficiently large n. Therefore for all sufficently large n the norm values $\varphi_{j(f\restriction i)}(y_n)$ and $\varphi_{j(f\restriction(i+1))}(y_n)$ are given by the rank of a node and its successor in a well-founded tree, so the limit values satisfy

$$\lim_{n\to\omega} \varphi_{j(f\restriction i)}(y_n) > \lim_{n\to\omega} \varphi_{j(f\restriction(i+1))}(y_n)$$

for all i and we get an infinite decreasing sequence of ordinals. This contradiction proves the claim.

The claim is giving us something like a semi-scale on $\omega^\omega \setminus p[j(T)]$ in generic extensions by posets of size less than α, except that the norms are indexed by $j``j(\gamma)^{<\omega}$ and not by ω. First we will show that a subset of norms of size $\leq 2^\kappa$ suffices, and then we will collapse 2^κ to get an actual semi-scale.

Indeed, there is a subset $\sigma \subset j(\gamma)^{<\omega}$ of size $\leq 2^\kappa$ such that for every node $t \in j(\gamma)^{<\omega}$ there is a node $t' \in \sigma$ with the property that, for any pair of reals y_0 and y_1 appearing in any generic extension of V by a poset of size less than κ, we have

$$\begin{aligned} &\operatorname{rank}_{j(T)_{y_0}}(t) \leq \operatorname{rank}_{j(T)_{y_1}}(t) \\ \iff &\operatorname{rank}_{j(T)_{y_0}}(t') \leq \operatorname{rank}_{j(T)_{y_1}}(t'). \end{aligned}$$

The reason for this is that, because κ is inaccessible, there are at most 2^κ many inequivalent $\operatorname{Col}(\omega, {<}\kappa)$-names for binary relations on the reals.

So applying j and using the fact that $j(\kappa) > \alpha$ and $V_\alpha \subset M$, we see that the two norms $\varphi_{j(t)}$ and $\varphi_{j(t')}$ are equivalent with respect to reals that are generic over V (or equivalently over M) for posets of size less than α, in the sense that they induce the same pre-wellordering of these reals. Therefore the claim still holds when condition (3) of its hypothesis is weakened to say

(3′) for every $t \in j``\sigma$ the norm $\varphi_t(y_n)$ is eventually constant as $n \to \omega$.

Working in a generic extension $V[g]$ by $\operatorname{Col}(\omega, 2^\kappa)$ we can take an enumeration $(t_i : i < \omega)$ of $j``\sigma$ and define the corresponding sequence of norms $\vec{\varphi} = (\varphi_i : i < \omega)$ by

$$\varphi_i(y) = \varphi_{t_i}(y).$$

What we have shown is that in every generic extension of $V[g]$ by a poset of size less than α, this sequence of norms $\vec{\varphi}$, interpreted in that extension, is a semi-scale on the set of reals $\omega^\omega \setminus p[j(T)]$.

Now let $V[g][h]$ be a generic extension of $V[g]$ by a poset of size less than α. By absorbing h into a Levy collapse we may assume it is a homogeneous extension. In $V[g][h]$ define

$$A = \omega^\omega \setminus p[j(T)]$$

and let $\tilde{T}$ be the tree of the semiscale $\vec{\varphi}$ on A. That is, for each $y \in A$ and each $i < \omega$ we put

$$\Big(\big((y(0), \ldots, y(i-1)\big), \big(\varphi_0(y), \ldots, \varphi_{i-1}(y)\big)\Big) \in \tilde{T}.$$

The result of this standard construction is that $p[\tilde{T}] = A$ in $V[g][h]$. The semi-scale $\vec{\varphi}$ and its associated tree $\tilde{T}$ are definable in $V[g][h]$ from the tree $j(j(T))$ and the sequence $(t_i : i < \omega)$ as parameters, so we have $\tilde{T} \in V[g]$ by homogeneity. Because it projects to the set $A = \omega^\omega \setminus p[j(T)]$ in the given generic extension $V[g][h]$ of $V[g]$, this tree $\tilde{T}$ is the desired absolute complement for T in $V[g]$. □

The next lemma is proved by the same technique. It is also related to [9, Theorem 3.2], which says that if κ is supercompact then any tree becomes weakly homogeneous in some small forcing extension. Woodin also showed that if κ is merely a Woodin cardinal then any tree becomes ${<}\kappa$-weakly homogeneous in some small forcing extension (see [8].) However, in our lemma we desire to weaken the hypothesis of supercompactness in a different direction, to say that κ is measurable rather than Woodin. In return we shall be content to prove κ-absolute complementation rather than $<\kappa$-weak homogeneity.

Crucially, our lemma will need a hypothesis saying that the tree in question does not construct too many sets.

Lemma 3.2. *Let T be a tree on $\omega \times \gamma$ for some ordinal γ. Let κ be a measurable cardinal and suppose there is a normal measure μ on κ concentrating on the set of $\alpha < \kappa$ such that*

$$|\wp(V_\alpha) \cap L(T, V_\alpha)| = \alpha.$$

Then there is a generic extension $V[g]$ of V by a poset of size less then κ in which T is κ-absolutely complemented.

Proof. Denote the μ-ultrapower map by $j : V \to M$. Then by Łoš's Theorem we have

$$|\wp(V_\kappa) \cap L(j(T), V_\kappa)| = \kappa$$

in M, and because M contains all subsets of κ, this holds in V as well. Note that for every real y that is generic over V for a poset of size less than κ, we have

$$y \in p[T] \iff y \in p[j(T)]$$

because j extends to an elementary embedding $V[y] \to M[y]$. Applying j to this statement we see that for every real y that is generic over M for a poset of size less than $j(\kappa)$, we have

$$y \in p[j(T)] \iff y \in p[j(j(T))].$$

Therefore, for each $t \in j(j(\gamma))^{<\omega}$ we can define the norm φ_t on $\omega^\omega \setminus p[j(T)]$ in any generic extension of M by a poset of size less than $j(\kappa)$ as follows:

$$\varphi_t(y) = \begin{cases} \mathrm{rank}_{j(j(T))_y}(t) & \text{if } t \in j(j(T))_y \\ 0 & \text{if } t \notin j(j(T))_y. \end{cases}$$

Just as before, we can prove that for every real x and every sequence of reals $(y_n : n < \omega)$ appearing in a generic extension of M by a poset of size less than $j(\kappa)$ such that

(1) $y_n \notin p[j(T)]$ for every $n < \omega$,
(2) $y_n \to x$ as $n \to \omega$, and
(3) for every $t \in j``j(\gamma)^{<\omega}$ the norm $\varphi_t(y_n)$ is eventually constant as $n \to \omega$,

we have $x \notin p[j(T)]$. This gives us something like a semi-scale on $\omega^\omega \setminus p[j(T)]$ in generic extensions of M by posets of size less than $j(\kappa)$, except that the norms are indexed by $j``j(\gamma)^{<\omega}$ and not by ω, and the set of norms may fail to be an element of M.

However, observe that there is a subset $\sigma \subset j(\gamma)^{<\omega}$ of size $\leq \kappa$ such that for every node $t \in j(\gamma)^{<\omega}$ there is a node $t' \in \sigma$ with the property that, for any pair of reals y_0 and y_1 appearing in some generic extension of V by a poset of size less than κ, we have

$$\begin{aligned} &\operatorname{rank}_{j(T)_{y_0}}(t) \leq \operatorname{rank}_{j(T)_{y_1}}(t) \\ \iff &\operatorname{rank}_{j(T)_{y_0}}(t') \leq \operatorname{rank}_{j(T)_{y_1}}(t'). \end{aligned}$$

The reason for this is that the model $L(j(T), V_\kappa)$ contains only κ many subsets of V_κ by assumption, so it contains at only κ many inequivalent $\mathrm{Col}(\omega, {<}\kappa)$-names for binary relations on the reals.

So applying j we see that the two norms $\varphi_{j(t)}$ and $\varphi_{j(t')}$ are equivalent with respect to reals that are generic over M by a poset of size less than $j(\kappa)$ in the sense that they induce the same pre-wellordering of these reals. Therefore the claim still holds when condition (3) of its hypothesis is weakened to say

(3′) for every $t \in j``\sigma$ the norm $\varphi_t(y_n)$ is eventually constant as $n \to \omega$.

Because $|\sigma| \leq \kappa$ and κ is the critical point of j, the set of ordinals $j``\sigma$ is in M and has size κ there. So working in a generic extension $M[g]$ of M by $\mathrm{Col}(\omega, \kappa)$ we can take an enumeration $(t_i : i < \omega)$ of σ and define the corresponding sequence of norms $\vec{\varphi} = (\varphi_i : i < \omega)$ by

$$\varphi_i(y) = \varphi_{t_i}(y).$$

What we have shown is that in every generic extension of $M[g]$ by a poset of size less than $j(\kappa)$, this sequence of norms $\vec{\varphi}$, interpreted in that extension, is a semi-scale on the set of reals $\omega^\omega \setminus p[j(T)]$. Now as before, the "tree of a semi-scale" construction shows that in $M[g]$, which is a generic extension of M by a poset of size less than $j(\kappa)$, the tree $j(T)$ is $j(\kappa)$-absolutely complemented. The desired conclusion now follows from the elementarity of the ultrapower map j. □

The consequences of Lemma 3.2 that we will use in our applications to generic absoluteness are stated below.

Lemma 3.3. *Let T be a tree on $\omega \times \gamma$ for some ordinal γ. Let κ be a measurable cardinal. Then there is a generic extension of V by a poset of size less than κ in which at least one of the following statements holds.*

(1) T is κ-absolutely complemented.
(2) $\mathbb{R} \cap L[T, x]$ is uncountable for some real x.

Proof. Suppose that (1) fails in every generic extension of V by a poset of size less than κ. Then by Lemma 3.2 we may take an inaccessible cardinal $\alpha < \kappa$ such that the model $L(T, V_\alpha)$ has more than α many subsets of V_α. Then after forcing with $\mathrm{Col}(\omega, \alpha)$ we get a real x coding V_α, so any α^+ many distinct subsets of V_α in $L(T, V_\alpha)$ are coded by uncountably many reals in $L[T, x]$, and (2) holds. □

We can strengthen case (2) of Lemma 3.3 by a standard argument using Solovay's almost disjoint coding method.

Lemma 3.4. *Let T be a tree on $\omega \times \gamma$ for some ordinal γ. Let κ be a measurable cardinal. Then there is a generic extension of V by a poset of size less than κ in which at least one of the following statements holds.*

(1) T is κ-absolutely complemented.
(2) $\mathbb{R} \subset L[T, x]$ for some real x.

Proof. Take a generic extension $V[g_0]$ of V by a poset of size less than κ as in Lemma 3.3. If case (1) holds then we are done. If case (2) of Lemma 3.3 holds, this is witnessed by some real $x_0 \in V[g_0]$. By forcing with $\mathrm{Col}(\omega_1, \mathbb{R})$ if necessary to get CH we may assume that

$$V[g_0] \models |\mathbb{R}| = |\mathbb{R} \cap L[T, x_0]| = \omega_1.$$

In $V[g_0]$ we have a subset X_1 of ω_1 coding HC. Forcing over $V[g_0]$, we will use Solovay's almost disjoint coding to code our subset X_1 of ω_1 by a real x_1. This is a standard argument, which we include for the convenience of readers to whom it is not familiar. We let

$$\vec{a} = (a_\xi : \xi < \omega_1^{V[g_0]}) \in L[T, x_0]$$

be a family of almost disjoint subsets of ω and let $\mathbb{P}_{\vec{a}, X_1}$ denote the forcing notion consisting of partial functions $p : \omega \to 2$ such that $p^{-1}(\{1\})$ is finite and $\mathrm{dom}(p) \cap a_\xi$ is finite for every $\xi \in X_1$.

Let g_1 be a $V[g_0]$-generic filter for $\mathbb{P}_{\vec{a}, X_1}$ and let $x_1 \subset \omega$ be the corresponding generic real, meaning that for every $n < \omega$ we have $n \in x_1$ if and only if $\left(\bigcup g_1\right)(n) = 1$. Then x_1 codes X_1 relative to $\vec{a}$ in the sense that for every $\xi < \omega_1$, we have $\xi \in X_1$ if and only if $a_\xi \cap x_1$ is infinite. We have $x_1, \vec{a} \in L[T, x_0, x_1]$ so we have $X_1, \mathbb{P}_{\vec{a}, X_1} \in L[T, x_0, x_1]$ as well. Moreover, in the model $V[g_0]$ the forcing $\mathbb{P}_{\vec{a}, X_1}$ is a subset of HC and has the countable chain condition, so every real $y \in V[g_0][g_1]$ is the interpretation of a hereditarily countable $\mathbb{P}_{\vec{a}, X_1}$-name $\dot{y} \in V[g_0]$ by the generic filter g_1. The set X_1 codes the name $\dot{y}$, among other elements of $\mathrm{HC}^{V[g_0]}$, so we have $\dot{y} \in L[T, x_0, x_1]$. But the model $L[T, x_0, x_1]$ contains the generic filter g_1, so we have $y \in L[T, x_0, x_1]$. This shows that, letting $V[g] = V[g_0][g_1]$ and $x = \langle x_0, x_1 \rangle$, we have

$$V[g] \models \mathbb{R} \subset L[T, x].$$

Therefore (2) holds in the generic extension $V[g]$. □

4. Two-step $\exists^{\mathbb{R}}(\boldsymbol{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness

In this section we consider the following generic absoluteness principle, which is a strengthening of one-step $\exists^{\mathbb{R}}(\boldsymbol{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness.

Definition 4.1. Let λ be a limit of Woodin cardinals. *Two-step $\exists^{\mathbb{R}}(\boldsymbol{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below* λ is the statement that every generic extension $V[g]$ of V by a poset of size less than λ satisfies one-step $\exists^{\mathbb{R}}(\boldsymbol{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below λ.

The essential difference between one-step and two-step generic absoluteness is that in the definition of two-step generic absoluteness we allow real parameters x from $V[g]$ and not just from V. Indeed the natural notion of two-step generic absoluteness for the lightface pointclass $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ would simply be equal to $\exists^{\mathbb{R}}(\Pi^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness as it was already defined in Definition 2.1, and the same is true for its relativization to any particular real $x \in V$. The distinction between one-step and two-step generic absoluteness only exists for the boldface versions.

Applying Proposition 2.8 in generic extensions, we get a characterization of two-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness in terms of a closure property of uB_λ.

Proposition 4.2. *For any limit of Woodin cardinals λ, the following statements are equivalent.*

1. *Two-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below λ.*
2. *In every generic extension $V[g]$ of V by a poset of size less than λ, every $(\utilde{\mathbf{\Delta}}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is λ-universally Baire.*

One can obtain two-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness from trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas by a standard argument using the absoluteness of well-foundedness. By "trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas" we mean trees $\tilde{T}_\varphi$ that are analogous to the trees T_φ for $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ formulas given by Corollary 1.7. To be precise:

Definition 4.3. We say *there are trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas* if, for every formula $\varphi(v)$ there is a tree $\tilde{T}_\varphi$ such that for every generic extension $V[g]$ of V by a poset of size less than λ and every real $x \in V[g]$ we have

$$x \in p[\tilde{T}_\varphi] \iff \forall B \in \mathrm{uB}_\lambda^{V[g]}\, (\mathrm{HC}^{V[g]}; \in, B) \models \neg\varphi[x].$$

(We negate the formula φ so that the tree $\tilde{T}_\varphi$ will be a λ-absolute complement of the tree T_φ as from Corollary 1.7.)

In the case that there is a $<\lambda$-strong cardinal δ below our limit λ of Woodin cardinals, Woodin showed that trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas appear after forcing with $\mathrm{Col}\left(\omega, 2^{2^\delta}\right)$. This is implicit in his proof that, assuming this large cardinal hypothesis, the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies $\mathsf{AD}+$ "every $\utilde{\mathbf{\Pi}}^2_1$ set of reals is Suslin" (see [15, §8] for the proof.) We can use Lemma 3.1 to obtain a slight strengthening: trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas appear after forcing with $\mathrm{Col}(\omega, 2^\delta)$. The argument follows the one given in [15, §8], only using our strengthened Lemma 3.1 in the appropriate place. We give the full argument below for the convenience of the reader.

Proposition 4.4. *Let λ be a limit of Woodin cardinals and let $\delta < \lambda$ be a $<\lambda$-strong cardinal. Let $g \subset \mathrm{Col}(\omega, 2^\delta)$ be a V-generic filter. Then in $V[g]$ there are trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas.*

Proof. Let $\alpha < \lambda$ be a cardinal. We want to show that the tree T_φ for a Σ^2_1 formula from Corollary 1.7 is α-absolutely complemented in $V[g]$. Without loss of generality we may assume that $\alpha > 2^\delta$. As in the proof of Proposition 2.11 we may take a cardinal $\gamma < \lambda$ which is sufficiently large that $\mathrm{uB}_\gamma = \mathrm{uB}_\lambda$ in every generic extension of V by a poset of size less than α. We may also assume that γ is inaccessible.

Take an elementary embedding $j : V \to M$ witnessing that δ is γ-strong. That is, $\mathrm{crit}(j) = \delta$ and $V_\gamma \subset M$. Then by Lemma 3.1 the tree $j(T_\varphi)$ is γ-absolutely complemented in $V[g]$. To show that the original tree T_φ is also α-absolutely complemented in $V[g]$ it suffices to show that $p[T_\varphi] = p[j(T_\varphi)]$ in every generic extension of V by a poset of size less than α.

We have $p[T_\varphi] \subset p[j(T_\varphi)]$ by considering pointwise images of branches. Conversely, suppose that in some generic extension $V[h]$ by a poset of size less than α we have a real $x \in p[j(T_\varphi)]$. Then we have $M[h] \models x \in p[j(T_\varphi)]$, so

$$M[h] \models \exists B \in \mathrm{uB}_{j(\lambda)}\, (\mathrm{HC}; \in, B) \models \varphi[x].$$

(We will have $j(\lambda) = \lambda$ if our elementary embedding j comes from an extender of length less than λ, but this doesn't matter.) Because $\gamma < j(\lambda)$ and $V_\gamma \subset M$, the same set of reals M witnesses the statement

$$V[h] \models \exists B \in \mathrm{uB}_\gamma\, (\mathrm{HC}; \in, B) \models \varphi[x].$$

Because γ was chosen to make $\mathrm{uB}_\gamma = \mathrm{uB}_\lambda$ in $V[h]$, we have

$$V[h] \models \exists B \in \mathrm{uB}_\lambda\, (\mathrm{HC}; \in, B) \models \varphi[x],$$

so $x \in p[T_\varphi]$. Therefore $p[j(T_\varphi)] \subset p[T_\varphi]$ in $V[h]$ as desired. $\square$

As an immediate consequence of Proposition 4.4 and the absoluteness of well-foundedness we obtain the following result.

Corollary 4.5. *Let λ be a limit of Woodin cardinals and let $\delta < \lambda$ be a $<\lambda$-strong cardinal. Let $g \subset \mathrm{Col}(\omega, 2^\delta)$ be a V-generic filter. Then in $V[g]$ two-step $\exists^{\mathbb{R}}(\boldsymbol{\Pi}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below λ.*

The existence of trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas is equivalent to some of its obvious consequences.

Proposition 4.6. *For a limit λ of Woodin cardinals, the following statements are equivalent.*

(1) There are trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas.
(2) In every generic extension $V[g]$ of V by a poset of size less than λ, every $(\boldsymbol{\Sigma}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is λ-universally Baire.
(3) every $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ set of reals is λ-universally Baire.

Proof. (1) $\implies$ (2): The trees $\tilde{T}_\varphi$ are λ-absolute complements of the trees T_φ for $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ formulas given by Corollary 1.7. Therefore in V and in every generic extension by a poset of size less than λ, the pair $(T_\varphi, \tilde{T}_\varphi)$ witnesses that the corresponding $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ set of reals is λ-universally Baire.

Every $(\underset{\sim}{\mathbf{\Sigma}}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is a section of a $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ set of reals, making it λ-universally Baire as well. The implication (2) $\implies$ (3) is trivial.

(3) $\implies$ (1): Let $\varphi(v)$ be a formula in the language of set theory expanded by a new unary predicate symbol. By Corollary 1.7 there is a tree $T_\varphi \in V$ that projects to the $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ set of reals defined by φ in every generic extension by a poset of size less than λ. We will show that the tree T_φ is λ-absolutely complemented. The $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ set

$$A = p[T_\varphi] = \{y \in \mathbb{R} : \exists B \in \mathrm{uB}_\lambda \ (\mathrm{HC}; \in, B) \models \varphi[y]\}$$

is λ-universally Baire by our hypothesis, so there is some λ-absolutely complementing pair of trees $(T, \tilde{T})$ such that $p[T] = p[T_\varphi]$.

We claim that the pair $(T_\varphi, \tilde{T})$ is also λ-absolutely complementing, or equivalently that for every generic extension $V[g]$ of V by a poset of size less than λ we have

$$V[g] \models p[T_\varphi] = p[T].$$

We have $p[T_\varphi] \cap p[\tilde{T}] = \emptyset$ in V, so by the usual argument this holds in $V[g]$ as well, giving $V[g] \models p[T_\varphi] \subset p[T]$. For the reverse inclusion let $y \mapsto B(y)$ be the partial function given by Lemma 1.10 that chooses $(\Delta^2_1(y))^{\mathrm{uB}_\lambda}$ witnesses $B(y)$ uniformly for reals $y \in A$. Then the relation

$$W = \{(y, z) \in \mathbb{R} \times \mathbb{R} : y \in A \ \& \ z \in B(y)\}$$

is $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ and we have

$$\forall y \in A \ (\mathrm{HC}; \in, W_y) \models \varphi[y]$$

where $W_y = B(y)$ is the corresponding section of W. By our hypothesis, the relation W is λ-universally Baire. Let $A^{V[g]}$ and $W^{V[g]}$ denote the canonical extensions of A and W respectively to $V[g]$. Then we have

$$(\mathrm{HC}; \in, A, W) \prec (\mathrm{HC}^{V[g]}; \in, A^{V[g]}, W^{V[g]}),$$

and it follows that

$$\forall y \in A^{V[g]} \left(\mathrm{HC}^{V[g]}; \in, (W_y)^{V[g]}\right) \models \varphi[y],$$

which shows that $V[g] \models p[T] \subset p[T_\varphi]$. This completes the proof that $\tilde{T}$ is a λ-absolute complement of the tree T_φ. Accordingly, we write $\tilde{T}_\varphi = \tilde{T}$. $\square$

A natural question is whether two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below λ can be added to the list of equivalences in 4.6 (rather than being strictly weaker.) A positive answer to this question could be seen as an explanation of this generic absoluteness principle in terms of a continuous reduction to the absoluteness of well-foundedness.

Question 4.7. Let λ be a limit of Woodin cardinals and assume that two-step $\exists^{\mathbb{R}}(\underset{\sim}{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below λ. Must there be trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas?

If there are trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas then the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies (in addition to AD) the statement "every $\utilde{\mathbf{\Pi}}^2_1$ set of reals is Suslin," as noted in [15]. This conclusion follows even if the trees for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas do not appear in V but only in small generic extensions of V. Accordingly, one might ask the weaker question:

Question 4.8. Let λ be a limit of Woodin cardinals and assume that two-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below λ. Must the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfy "every $\utilde{\mathbf{\Pi}}^2_1$ set of reals is Suslin?"

In the case where λ is measurable we may apply Lemma 3.3 to get a positive answer, yielding a proof of Theorem 0.1. The proof is similar to Woodin's proof in [16] of "every set has a sharp" from two-step $\utilde{\mathbf{\Sigma}}^1_3$ generic absoluteness except that it uses Lemma 3.3 in place of Jensen's covering lemma for L.

Proof of Theorem 0.1. Assume that λ is a measurable cardinal and a limit of Woodin cardinals, and that two-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness holds below λ. We want to show that the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies "every $\utilde{\mathbf{\Pi}}^2_1$ set of reals is Suslin." As noted in [15], it suffices to show that there are trees $\tilde{T}_\varphi$ for $(\Pi^2_1)^{\mathrm{uB}_\lambda}$ formulas, and moreover to find these trees we may pass to generic extensions of V by a poset of size less than λ because these generic extensions can be absorbed into $\mathrm{Col}(\omega, {<}\lambda)$ and therefore do not affect the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$.

Fix a formula $\varphi(v)$ and let T_φ be the tree for the corresponding $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ formula as given by Corollary 1.7. By Lemma 3.3 we may take a generic extension $V[g]$ of V by a poset of size less than λ in which one of the following cases holds: $\mathbb{R} \cap L[T_\varphi, x]$ is uncountable for some real x, or T_φ is λ-absolutely complemented. In the latter case, a λ-absolute complement for T_φ is a tree $\tilde{T}_\varphi$ as desired, so we are done. We must use our generic absoluteness hypothesis to rule out the former case.

Working in $V[g]$, assume that there is a real x such that $\mathbb{R} \cap L[T_\varphi, x]$ is uncountable. The tree T_φ, as obtained from Theorem 1.6, is definable in the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$. Therefore every real $y \in L[T_\varphi, x]$ is ordinal-definable from x in $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$. On the other hand, every real y that is OD from x in $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ is in V by the homogeneity of the forcing $\mathrm{Col}(\omega, {<}\lambda)$ used to obtain the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$.

The statement "the set of reals that are OD_x in the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ is uncountable" is a $(\Pi^2_1(x))^{\mathrm{uB}_\lambda}$ statement that is true in $V[g]$ but becomes false after any forcing that makes the reals countable, violating our generic absoluteness hypothesis. We will show that this statement is indeed a $(\Pi^2_1(x))^{\mathrm{uB}_\lambda}$ statement.

It is a well-known consequence of $\mathsf{AD}^+ + V = L(\wp(\mathbb{R}))$ that the set of OD_x reals is a $\Sigma^2_1(x)$ set. If we replace "$y \in \mathrm{OD}_x$" with "$\exists A \subset \mathbb{R}\ y \in \mathrm{OD}_x^{L(A,\mathbb{R})}$" then this is not hard to prove, and then the stated version follows from Woodin's Σ_1-reflection theorem (see [13] for a proof.) Alternatively, one can simply replace "$y \in \mathrm{OD}_x$" with "$\exists A \subset \mathbb{R}\ y \in \mathrm{OD}_x^{L(A,\mathbb{R})}$" everywhere in the paper and check that our arguments can be adapted accordingly.

Therefore by Woodin's generic absoluteness theorem 1.5 the set of reals that are OD_x in $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ is $(\Sigma^2_1(x))^{\mathrm{uB}_\lambda}$, uniformly in all generic extensions by posets of size less than λ. The statement that this set of reals is countable is equivalent to the statement that there is a single real y coding a sequence of reals $(y_n : n < \omega)$ such that every real that is OD_x in $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ is equal to y_n for some $n < \omega$. It is therefore an $\exists^{\mathbb{R}}(\Pi^2_1(x))^{\mathrm{uB}_\lambda}$ statement as desired. $\square$

We can weaken the hypothesis of Theorem 0.1 by using Lemma 3.4 instead of Lemma 3.3. A similar argument is used in [2] to construct projective well-orderings of the reals from an anti-large-cardinal hypothesis (for $\utilde{\mathbf{\Delta}}^1_2$ well-orderings the argument is probably folklore.) Recall that two-step $\exists^{\mathbb{R}}(\utilde{\mathbf{\Pi}}^2_1)^{\mathrm{uB}_\lambda}$ generic absoluteness below λ is equivalent to the statement that, in every generic extension by a poset of size less than λ, every $(\utilde{\mathbf{\Delta}}^2_1)^{\mathrm{uB}_\lambda}$ set of reals is λ-universally Baire. Note that in particular this statement rules out the existence of a $(\utilde{\mathbf{\Delta}}^2_1)^{\mathrm{uB}_\lambda}$ well-ordering of the reals in a small forcing extension.

Proposition 4.9. *If λ is a measurable cardinal and a limit of Woodin cardinals, and every generic extension by a poset of size less than λ satisfies "there is no $(\utilde{\mathbf{\Delta}}^2_1)^{\mathrm{uB}_\lambda}$ well-ordering of the reals," then the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies AD + "every $\utilde{\mathbf{\Pi}}^2_1$ set of reals is Suslin."*

Proof. We follow the proof of Theorem 0.1. Fix a formula $\varphi(v)$ and let T_φ be the tree for the corresponding $(\Sigma^2_1)^{\mathrm{uB}_\lambda}$ formula as given by Corollary 1.7. By Lemma 3.4 we may take a generic extension $V[g]$ of V by a poset of size less than λ in which one of the following cases holds: $\mathbb{R} \subset L[T_\varphi, x]$ for some real x, or T_φ is λ-absolutely complemented. Again in the latter case we are done. We must show that the former case gives rise to a $(\utilde{\mathbf{\Delta}}^2_1)^{\mathrm{uB}_\lambda}$ well-ordering of the reals in $V[g]$.

Working in $V[g]$, assume that there is a real x such that $\mathbb{R} \subset L[T_\varphi, x]$. Our tree T_φ is definable in the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$, so every real $y \in L[T_\varphi, x]$ is ordinal-definable from x in $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$. In $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ as in any model of AD^+ the set of OD_x reals is $\Sigma^2_1(x)$ and has a $\Sigma^2_1(x)$ well-ordering. So in $V[g]$ the set this set of reals is $(\Sigma^2_1(x))^{\mathrm{uB}_\lambda}$ and has a $(\Sigma^2_1(x))^{\mathrm{uB}_\lambda}$ well-ordering. But by our choice of x the domain of the well-ordering contains all reals of $V[g]$, so $V[g]$ has a $(\Delta^2_1(x))^{\mathrm{uB}_\lambda}$ well-ordering of its reals. $\square$

5. The theory of $L(\mathrm{uB}_\lambda, \mathbb{R})$

In this section we consider the generic absoluteness of the theory of $L(\mathrm{uB}_\lambda, \mathbb{R})$. The following theorem (see [8, Theorems 3.4.17–19]) gives an upper bound in terms of large cardinals for the consistency strength of this generic absoluteness hypothesis, and it also says something about what the generically absolute theory is in this situation.

Theorem 5.1 (Woodin). *If λ is a limit of Woodin cardinals, $|V_\lambda| = \lambda$, and $\delta < \lambda$ is $<\lambda$-supercompact, then letting $g \subset \mathrm{Col}(\omega, 2^\delta)$ be a V-generic filter we have*

$$L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]} \equiv L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g][h]}$$

for every generic extension $V[g][h]$ of $V[g]$ by a poset of size less than λ, and moreover the model $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ satisfies $\mathsf{AD} + \mathsf{DC} +$ "every set of reals is Suslin."

To prove Theorem 0.2, which gives a lower bound for the consistency strength of generic absoluteness of the theory of $L(\mathrm{uB}_\lambda, \mathbb{R})$ in the case that λ is measurable as well as being a limit of Woodin cardinals, we will use a version of Proposition 4.9 relativized to an arbitrary set of reals $A \in \mathrm{uB}_\lambda$. The modification is straightforward and gives the following result.

Proposition 5.2. *If λ is a measurable cardinal and a limit of Woodin cardinals, A is a λ-universally Baire set of reals, and every generic extension $V[g]$ by a poset of size less than λ satisfies "there is no $(\undertilde{\mathbf{\Delta}}^2_1(A^{V[g]}))^{\mathrm{uB}_\lambda}$ well-ordering of the reals" then the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies AD + "every $\undertilde{\mathbf{\Pi}}^2_1(A^*)$ set of reals is Suslin."*

Applying Proposition 5.2 to all λ-universally Baire sets appearing in all small generic extensions of V yields the following result.

Proposition 5.3. *If λ is a measurable cardinal and a limit of Woodin cardinals, and for every generic extension $V[g]$ by a poset of size less than λ we have*

$$L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]} \models \text{"there is no well-ordering of the reals,"}$$

then the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies AD+DC+ "every set of reals is Suslin."

Proof. In the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$, to show that every set of reals is Suslin, arguing as in [15, §8] it suffices to show that every $\undertilde{\mathbf{\Pi}}^2_1(A^*)$ set of reals is Suslin for every set of reals $A^* \in \mathrm{Hom}^*_\lambda$. Let $A^* \in \mathrm{Hom}^*_\lambda$. By passing to a small forcing extension we may assume that A^* is the canonical extension of some set of reals $A \in \mathrm{uB}_\lambda$. For every generic extension $V[g]$ by a poset of size less than λ, every $(\undertilde{\mathbf{\Delta}}^2_1(A^{V[g]}))^{\mathrm{uB}_\lambda}$ set of reals is an element of the model $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$, so by our hypothesis there cannot be a $(\undertilde{\mathbf{\Delta}}^2_1(A^{V[g]}))^{\mathrm{uB}_\lambda}$ well-ordering of the reals. Applying Proposition 5.2, every $\undertilde{\mathbf{\Pi}}^2_1(A^*)$ set of reals is Suslin in $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ as desired.

Now we get DC by a standard argument using the inaccessibility of λ. Because $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies "every set of reals is Suslin" we have $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda) \cap \wp(\mathbb{R}) = \mathrm{Hom}^*_\lambda$, and because λ is inaccessible every ω-sequence from Hom^*_λ is coded by a set in Hom^*_λ. So in the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ we have $\mathrm{cof}(\Theta) > \omega$ and DC follows by [12, Theorem 1.3]. $\square$

We can prove the theorem now.

Proof of Theorem 0.2. Assume that λ is a measurable cardinal and a limit of Woodin cardinals, and that $L(\mathrm{uB}_\lambda, \mathbb{R}) \equiv L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ for every generic extension $V[g]$ by a poset of size less than λ. We want to show that the model $L(\mathrm{Hom}^*_\lambda, \mathbb{R}^*_\lambda)$ satisfies $\mathsf{AD} + \mathsf{DC} +$ "every set of reals is Suslin." By Proposition 5.3 and our generic absoluteness hypothesis it suffices to find *some* generic extension $V[g]$ by a poset of size less than λ such that the model $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ has no well-ordering of its reals.

Let $\delta < \lambda$ be an inaccessible cardinal and let $g \subset \mathrm{Col}(\omega, {<}\delta)$ be a V-generic filter. In $V[g]$ every element of the model $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ is ordinal-definable from a set of reals $A \in \mathrm{uB}_\lambda^{V[g]}$ and a real in $\mathbb{R}^{V[g]}$. Because λ is a limit of Woodin cardinals we have $\mathrm{uB}_\lambda = \mathrm{Hom}_{<\lambda}$ in $V[g]$, so in particular $A \in \mathrm{Hom}_{<\lambda}^{V[g]}$. The measures in a δ-homogeneity system for A are the canonical extensions of measures in V by the Levy–Solovay theorem, and A is definable from the system of these measures, which is countable. Therefore every element of the model $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ is definable from a countable sequence of elements of V. Because δ is inaccessible there can be no well-ordering of the reals in $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ by Solovay's theorem. □

Theorem 0.2 gives us a lower bound on the consistency strength of the generic absoluteness of the theory of $L(\mathrm{uB}_\lambda, \mathbb{R})$. It also gives us information about what this generically absolute theory is, just as in the situation of Woodin's theorem 5.1.

Corollary 5.4. *Assume that λ is a measurable cardinal and a limit of Woodin cardinals, and $L(\mathrm{uB}_\lambda, \mathbb{R}) \equiv L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ for every generic extension $V[g]$ by a poset of size less than λ. Then $L(\mathrm{uB}_\lambda, \mathbb{R})$ satisfies $\mathsf{AD}+\mathsf{DC}+$ "every set of reals is Suslin."*

Proof. Let $j : V \to M$ be an elementary embedding with critical point λ witnessing the measurability of λ. Let $\mathbb{R}^*_G$ and Hom^*_G come from some V-generic filter $G \subset \mathrm{Col}(\omega, {<}\lambda)$. Because λ is inaccessible we have $\mathbb{R}^*_G = \mathbb{R}^{V[G]} = \mathbb{R}^{M[G]}$. By Theorem 0.2 it suffices to show that

$$L(\mathrm{Hom}^*_G, \mathbb{R}^*_G) = L(\mathrm{uB}_{j(\lambda)}, \mathbb{R})^{M[G]} \equiv L(\mathrm{uB}_{j(\lambda)}, \mathbb{R})^M \equiv L(\mathrm{uB}_\lambda, \mathbb{R}).$$

To show that $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G) = L(\mathrm{uB}_{j(\lambda)}, \mathbb{R})^{M[G]}$ we will show that $\mathrm{Hom}^*_G = \mathrm{uB}_{j(\lambda)}^{M[G]}$. Every set $A^* \in \mathrm{Hom}^*_G$ is given as a projection $p[T]$ for some λ-absolutely complementing pair of trees $(T, \tilde{T}) \in V[g]$ where g is a proper initial segment of the generic filter G. We can extend j to a map $\hat{\jmath} : V[g] \to M[g]$ and by the elementarity of this map the trees $\hat{\jmath}(T)$ and $\hat{\jmath}(\tilde{T})$ are $j(\lambda)$-absolutely complementing in $M[g]$. Considering pointwise images of branches we have the inclusions $p[T] \subset p[\hat{\jmath}(T)]$ and $p[\tilde{T}] \subset p[\hat{\jmath}(\tilde{T})]$, so in fact both inclusions are equalities and the trees $\hat{\jmath}(T)$ and $\hat{\jmath}(\tilde{T})$ witness that A^* is $j(\lambda)$-universally Baire in $M[G]$.

Conversely, given a set of reals $A^* \in \mathrm{uB}_{j(\lambda)}^{M[G]}$ we can write $A \in \mathrm{Hom}_{<j(\lambda)}^{M[G]}$ because $j(\lambda)$ is a limit of Woodin cardinals in $M[G]$. The measures in a

λ-homogeneity system for A^* are the canonical extensions of measures in M by the Levy–Solovay theorem, and this homogeneity system appears in $M[g]$ for some proper initial segment g of the generic filter G because it is a countable sequence and λ is inaccessible. Therefore A^* is the canonical extension of a set of reals $A \in \mathrm{uB}^{M[g]}_{j(\lambda)}$. We have $\mathrm{uB}^{M[g]}_{j(\lambda)} \subset \mathrm{uB}^{M[g]}_{\lambda} = \mathrm{uB}^{V[g]}_{\lambda}$, so $A^* \in \mathrm{Hom}^*_G$ as desired.

Now it remains to note that by our generic absoluteness hypothesis applied in M, the model $L(\mathrm{uB}_{j(\lambda)}, \mathbb{R})^{M[G]}$ is elementarily equivalent to the model $L(\mathrm{uB}_{j(\lambda)}, \mathbb{R})^M$, which in turn is elementarily equivalent to the model $L(\mathrm{uB}_\lambda, \mathbb{R})$ by the elementarity of j. $\square$

One might hope that the consistency strength lower bound given by Theorem 0.2 can be improved. The next natural target would be the theory $\mathsf{AD}_\mathbb{R}$ + "Θ is regular." (Here we yield to convention and write $\mathsf{AD}_\mathbb{R}$ instead of the equivalent statement "every set of reals is Suslin.")

Question 5.5. Assume that λ is a measurable cardinal and a limit of Woodin cardinals, and $L(\mathrm{uB}_\lambda, \mathbb{R}) \equiv L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$ for every generic extension $V[g]$ by a poset of size less than λ.

(1) Does $L(\mathrm{uB}_\lambda, \mathbb{R})$ satisfy $\mathsf{AD}_\mathbb{R}$ + "Θ is regular"?
(2) Does $L(\Gamma, \mathbb{R})$ satisfy $\mathsf{AD}_\mathbb{R}$ + "Θ is regular" for some pointclass Γ contained in uB_λ?

In unpublished work, Woodin has strengthened the conclusion of Theorem 5.1 to get $\mathsf{AD}_\mathbb{R}$ + "Θ is regular" in the model $L(\mathrm{uB}_\lambda, \mathbb{R})^{V[g]}$, making a positive answer to Question 5.5(1) plausible. However, Question 5.5(2) might be a more reasonable target for current inner model theoretic techniques such as those developed by Sargsyan in [11]. One might also hope to dispense with the hypothesis that λ is measurable.

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