A model of set theory in which every set of reals is universally Baire

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[1] joint work with Paul Larson and Grigor Sargsyan

Outline

Aleph and beth numbers:

Definition

- $\aleph_0 = |\mathbb{N}|$
- $\aleph_{\alpha+1} = \aleph_\alpha^+$
- $\aleph_\lambda = \sup_{\alpha<\lambda} \aleph_\alpha$, if λ is a limit ordinal.

Definition

- $\beth_0 = |\mathbb{N}|$
- $\beth_{\alpha+1} = 2^{\beth_\alpha}$
- $\beth_\lambda = \sup_{\alpha<\lambda} \beth_\alpha$, if λ is a limit ordinal.

Definition

An infinite cardinal κ is:

- regular if $\left|\bigcup_{i\in I} A_i\right| < \kappa$ whenever $|I| < \kappa$ and $|A_i| < \kappa$ for all $i \in I$.
- strong limit if $|\mathcal{P}(A)| < \kappa$ whenever $|A| < \kappa$.

Example

- $\aleph_0$ is regular and strong limit.
- $\aleph_1$ is regular, but not strong limit.
- $\beth_\omega$ is strong limit, but not regular.

Definition

A cardinal κ is inaccessible if it is regular, strong limit, and uncountable.

Remark

Inaccessible cardinals cannot be proved to exist in ZFC:
If κ is inaccessible, then H_κ (sets of hereditary cardinality less than κ) is a model of ZFC:

- regular $\rightsquigarrow$ Replacement Schema
- strong limit $\rightsquigarrow$ Power Set Axiom
- uncountable $\rightsquigarrow$ Axiom of Infinity

Definition

A cardinal κ is measurable if it is uncountable and there is a κ-complete nonprincipal ultrafilter on $\mathcal{P}(\kappa)$.

Remark

A cardinal κ is measurable iff there is an elementary embedding $j : V \to M$ where M is a transitive model and

- $j \restriction \kappa = \mathrm{id}$
- $j(\kappa) > \kappa$

(κ is the *critical point* of j.)

Stronger large cardinal axioms require the target model M to be closer to V:

Definition

A measurable cardinal κ is strong if for any cardinal α we can get $j(\kappa) > \alpha$ and $V_\alpha \subset M$.

Definition

A measurable cardinal κ is supercompact if for any cardinal α we can get $j(\kappa) > \alpha$ and $M^\alpha \subset M$.

Large cardinal properties form a (mostly?) linear hierarchy under *consistency strength*:

supercompact

Woodin

strong

measurable

inaccessible

Large cardinals can influence smaller objects.

Example

Large cardinals give regularity properties for sets of reals:

- Lebesgue measurability
- the Baire property (BP)
- the perfect set property (PSP)

Remark

- A regularity property says that a set of reals is "nice."
- The Axiom of Choice (AC) can be used to build "pathological" sets without regularity properties.

Theorem (Solovay)

If the theory

$\mathrm{ZFC} +$ *"there is an inaccessible cardinal"*

is consistent, then so is the theory

$\mathrm{ZF} + \mathrm{DC} +$ *"every set of reals is Lebesgue measurable and has the BP and PSP."*

Remark

The converse holds also. (Specker)

Part of Solovay's argument:

- Use forcing to add generic surjections from $\mathbb{N}$ onto every ordinal less than κ.
- κ becomes $\aleph_1$.
- Some new reals $\mathbb{R}^*$ appeared.
- The model $L(\mathbb{R}^*)$ constructed from $\mathbb{R}^*$ satisfies $\mathsf{ZF} + \mathsf{DC} +$ "every set of reals is Lebesgue measurable and has the BP and PSP."

Remark

The larger model $V(\mathbb{R}^*)$ also satisfies this theory.

Question

What other regularity properties can we get from large cardinals?

A stronger regularity property:

Definition

A set of reals A is universally Baire if its preimage under any continuous function from a compact Hausdorff space has the property of Baire.

Remark

Every universally Baire set of reals:

- has the Baire property
- is Lebesgue measurable. (Feng–Magidor–Woodin)

Another strong regularity property:

Definition

For a set of reals A, consider a game between players I, II:

$$\begin{array}{c|ccccccc} \mathrm{I} & x_0 & & x_2 & & \dots & \\ \mathrm{II} & & x_1 & & x_3 & & \dots \end{array}$$

Rules:

- each move x_i is a natural number
- player I wins iff the sequence $\langle x_i : i \in \mathbb{N} \rangle$ is in A.[2]

A is determined if one of the players has a winning strategy.

[2] We use $\mathbb{N}^{\mathbb{N}}$ instead of $\mathbb{R}$ now, but still call the elements "reals"

Definition

The Axiom of Determinacy (AD) says that every set of reals is determined.

Remark

AD contradicts the Axiom of Choice:

- AD implies that every set of reals is Lebesgue measurable, has the BP, and has the PSP.
- One can also use the Axiom of Choice to build a non-determined game directly.

Theorem (Larson–Sargsyan–W.)

If the theory

ZFC + *"there is a cardinal that is a limit of strong cardinals and a limit of Woodin cardinals"*

is consistent, then so is the theory

ZF + AD + *"every set of reals is universally Baire."*

Remark

- Woodin got the conclusion from a stronger hypothesis: a proper class of Woodin limits of Woodin cardinals.
- Our hypothesis seems likely to be optimal.

To prove this we must consider the complexity hierarchy for sets of reals, unlike in Solovay's theorem.
In ZFC:

Theorem (Martin)

Borel sets of reals are determined.

Theorem (Feng–Magidor–Woodin[3])

Analytic sets (projections of Borel sets) are universally Baire.

[3]Note added on June 7, 2015: M. Magidor pointed out to me that the argument for this result predated the definition of "universally Baire" and is due to K. Schilling.

With measurable cardinals, we can go a step further in the projective hierarchy:

Theorem (Martin)

If there is a measurable cardinal, then analytic sets are determined.

Theorem (Martin–Solovay)

If κ is measurable, then PCA sets (projections of co-analytic sets) are κ-universally Baire.

Strong cardinals give more universal Baireness:

Theorem (Woodin)

If κ is a limit of strong cardinals, then in $V(\mathbb{R}^*)$, every projective set of reals is universally Baire.

Woodin cardinals give more determinacy:

Theorem (Martin–Steel)

If there are infinitely many Woodin cardinals, then every projective set of reals is determined.

We want to combine:

- universal Baireness from strong cardinals
- determinacy from Woodin cardinals.

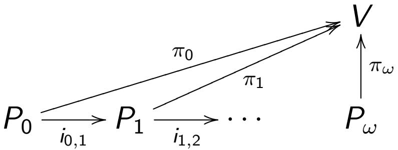
To go beyond the projective sets, we need:

Theorem (Woodin)

If κ is a limit of Woodin cardinals, then the model $L(\mathbb{R}^*)$ satisfies the Axiom of Determinacy.

Neeman's proof of AD in $L(\mathbb{R}^*)$:

- Take a countable sufficiently elementary hull P_0 of V
- Use genericity iterations to make $\mathbb{R}^V$ generic over P_ω:



- If $L(\mathbb{R}^*)$ has a counterexample then so does $L(\mathbb{R}^V)^{P_\omega(\mathbb{R}^V)}$
- $L(\mathbb{R}^V)^{P_\omega(\mathbb{R}^V)}$ is a level of $L(\mathbb{R})^V$
- Let A be the "least" counterexample in $L(\mathbb{R})^V$
- A is uniformly definable via genericity iterations
- By PD method this implies that A is determined after all.

Between AD and AD + "every set of reals is universally Baire":

Definition

$AD_{\mathbb{R}}$ says that for every "payoff" set $A \subset \mathbb{R}^{\mathbb{N}}$,
in the two-player game

$$\begin{array}{c|ccccccc} \mathrm{I} & x_0 & & x_2 & & \ldots & \\ \mathrm{II} & & x_1 & & x_3 & & \ldots \end{array}$$

where each move x_i is a real number,
and player I wins iff the sequence $\langle x_i : i \in \mathbb{N} \rangle$ is in A,
one of the players has a winning strategy.

For $AD_{\mathbb{R}}$ we need a model larger than $L(\mathbb{R}^*)$:

Definition

For a limit of Woodin cardinals κ, the derived model at κ is the $\subseteq$-maximal model contained in $V(\mathbb{R}^*)$ and satisfying $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$.[4]
(Exists by Woodin's *derived model theorem*.)

Theorem (Woodin)

If κ is a limit of Woodin cardinals and $<\kappa$-strong cardinals, (a weakening of strong cardinals,)
then the derived model at κ satisfies $AD_{\mathbb{R}}$.

[4] The difference between AD^+ and AD is not essential to this talk.

Let κ be a limit of strong cardinals and Woodin cardinals and let uB^* be the pointclass of universally Baire sets in $V(\mathbb{R}^*)$

- $L(\mathrm{uB}^*)$ is equal to the derived model at κ
- In particular, $L(\mathrm{uB}^*)$ satisfies AD.

Problem

$L(\mathrm{uB}^*)$ does not see that the sets in uB^* are universally Baire.

Solution

1. Add sufficient evidence of universal Baireness to $L(\mathrm{uB}^*)$.
2. But don't add any new set of reals.
   (Equivalently, don't add any nondetermined set of reals.)

"Evidence" of universal Baireness:

- When forcing, universally Baire sets extend canonically.
- Define a predicate F telling us how to extend:

$$F(p, Z, \dot{x}, A) \iff p \Vdash_{\mathrm{Col}(\omega, Z)} \dot{x} \in A^{V[G]}$$

The desired model is $L^F(\mathrm{uB}^*)$ as defined in $V(\mathbb{R}^*)$:

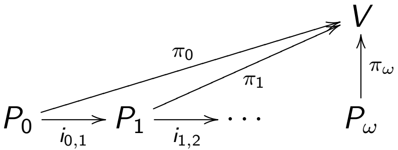
1. It sees that uB^* sets are universally Baire.
2. It satisfies AD.

Every uB* set is universally Baire in $L^F(\mathrm{uB}^*)$:

- every uB* set A has a uB*-scale (Steel)
- $L^F(\mathrm{uB}^*)$ knows how to extend this scale to generic extensions.
- $L^F(\mathrm{uB}^*)$ can build absolutely complementing trees witnessing that A is universally Baire.

$L^F(\mathrm{uB}^*)$ satisfies AD:

- Take a countable sufficiently elementary hull P_0 of V
- Use genericity iterations to make $\mathbb{R}^V$ generic over P_ω:



- uB^* sets collapse to uB^V sets under π_ω^{-1} (Steel)
- F collapses to F^V under π_ω^{-1}
- Let A be the "least" counterexample in $(L^F(\mathrm{uB}))^V$
- As before, we can show that A is determined after all.

Question

What else can we add to derived models?
(while preserving the Axiom of Determinacy)